Modern Hadron Spectroscopy : Challenges and Opportunities

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Lecture 1: Hadrons as laboratory for QCD:

- **Introduction to QCD**
- **Bare vs effective effective quarks and gluons**
- **Phenomenology of Hadrons**

Lecture 2: Phenomenology of hadron reactions

- **Kinematics and observables**
- **Space time picture of Parton interactions and Regge phenomena**
- **Properties of reaction amplitudes**

Lecture 3: Complex analysis

Lecture 4: How to extract resonance information from the data

- **Partial waves and resonance properties**
- **Amplitude analysis methods (spin complications)**

Probing QCD resonances (using physical states)

- When (color neutral) mesons and baryons a smashed, their quarks overlap, “stick together” to form resonances (quasi QCD eigenstates). They are short lived and decay to lowest energy, asymptotic states (pions, K’s, proton,…)

- Resonances are fundamental to our understanding of QCD dynamics since they appear beyond perturbation theory.

- (QCD) Resonances challenge QFT practitioners to develop all orders calculations (still ways to go).

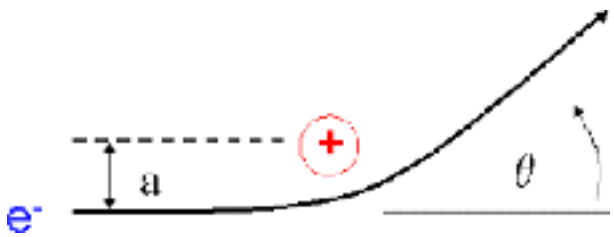
- (QCD) Resonance lead to extremely rich phenomenology (e.g. XYZ states).

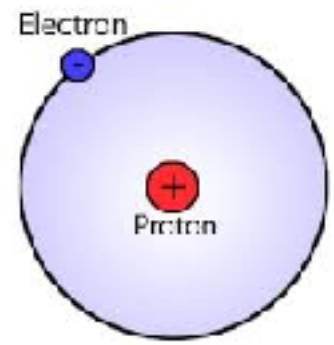
- In practice, one requires tools that relate asymptotic states before collision to asymptotic states after collision that include flexible parametrization of microscopic dynamics. This is often referred to as amplitude analysis. The rest of these lectures will focus on this topic.

Bound states/Resonances/Asymptotic states

$$\left[\frac{p^2}{2m_e} - \frac{\alpha}{r}\right]\psi(r) = E\psi(r)$$

$$\alpha = \alpha_{QED} = \frac{1}{137}$$





$$\psi(r) = \frac{e^{-ikr}}{r} - S(E,\theta)\frac{e^{ikr}}{r}$$

$$S(E,\theta) = 1 + O(\alpha)$$

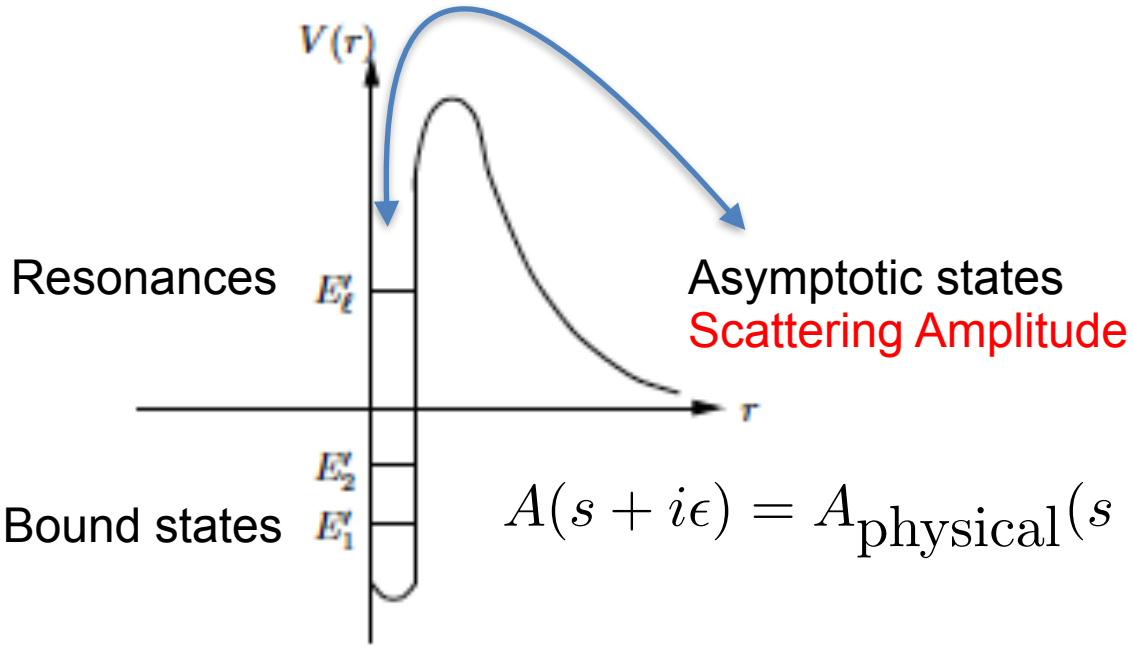
Born approximation : lowest order perturbation on free motion

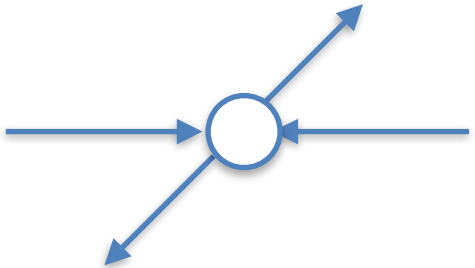
$$\psi(r) \propto e^{-\alpha m_e r}$$

Bound states: compact wave function contains interaction to all orders.

Resonances: particles interact to all orders (like bound states) but eventually decay (connect with asymptotically free states). Their effect appears in the S-matrix

Amplitude analyticity: it is much about complex functions



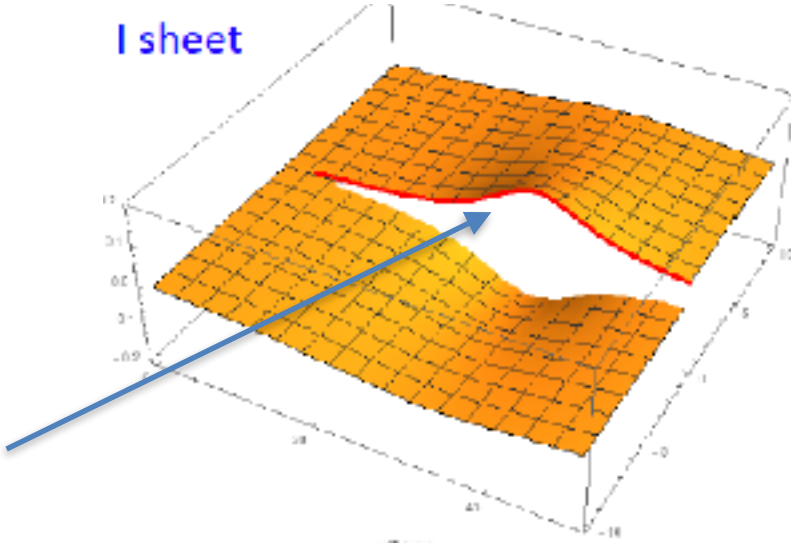


$$s = E_{c.m}^2$$

$$A(s + i\epsilon) = A_{\text{physical}}(s = \text{real and above threshold})$$

Scattering amplitude describes evolution between asymptotic states. The information related to formation about resonances is "hidden" in unphysical domains (sheets) of the kinematical variables.

This "bump" is an indication of a "hidden" phenomenon. To uncover it one needs to analytically continue outset the physical sheet



Introduction to Scattering

$$H = H_{kin} + V \rightarrow H_0 + V(t) \qquad V \rightarrow V(t) = Ve^{-\epsilon|t|}$$

Interaction is switched on adiabatically at t=0

- Time evolution pictures: Schrodinger, Heisenberg, Interaction

$$O_I(t) = e^{iH_0}O(0)e^{-iH_0t} \longrightarrow \begin{aligned} H_{0,I}(t) &= H_0 \\ V_I(t) &= e^{iH_0}Ve^{-iH_0t}e^{-\epsilon|t|} \end{aligned}$$

$$|t\rangle_I = e^{iH_0t}|t\rangle_S \longrightarrow i\frac{d}{dt}|t\rangle_I = V_I(t)|t\rangle_I$$

- As $t \rightarrow \pm\infty$ interaction picture states evolve to eigenstates of H_{kin}, i.e. to free particles

- At t=0 interactions picture states are solution of the full Hamiltonian

S-matrix and T-matrix

$$i\frac{d}{dt}|t\rangle_I = V_I(t)|t\rangle_I \longrightarrow |t\rangle_I = U(t,-\infty)|initial\rangle$$

Evolution operator

- S-matrix

$$S_{fi} = \langle f(t=+\infty)|i(t=-\infty\rangle = \langle f,(out)|i,(in)\rangle$$
$$= \langle f|U(+\infty,-\infty)|i\rangle$$

$$U(+\infty,-\infty) = \mathcal{P}\exp\left(-i\int_{-\infty}^{+\infty} dt V_I(t)\right) = I - 2\pi i\delta(E_f - E_i)T$$

- T-matrix

$$T = V + VG_0V + \cdots \qquad G_0 = \frac{1}{E - H_0}$$

$$E = E_i = E_f$$

T matrix : Example

$$T = V + VG_0V + \cdots$$

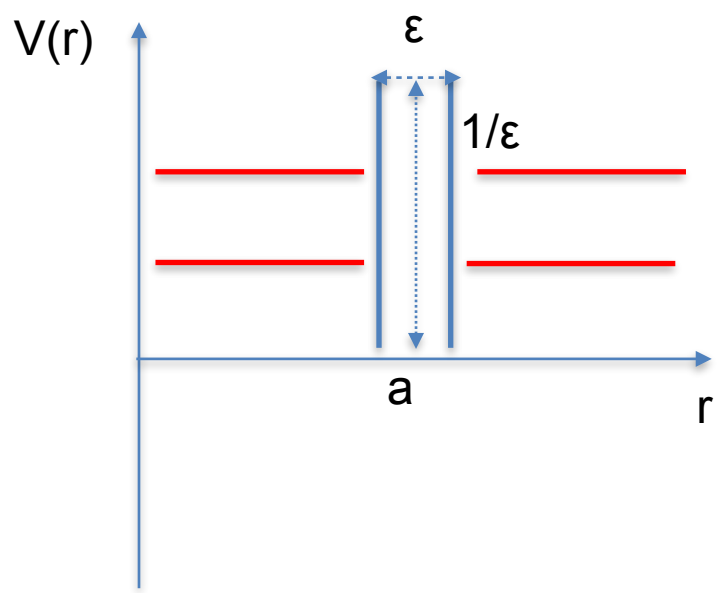
Example

$$\dim\lambda = -1$$

Nonrelativistic particle scarring in external potential

$$V = \frac{\lambda}{2\mu a^2}\delta(r-a)$$

$$H = \frac{p^2}{2\mu} + V$$



Method 1: In coordinate space
Method 2: Lippmann-Schwinger (see above)

- It has ∞ number of zeros (this is related to ∞ number of poles when calculated to all

Solution

From method 1

$$f(k) = \frac{\left[-\lambda \frac{\sin^2(ka)}{(ka)^2}\right]}{\left[1 + \frac{\lambda}{a}\frac{\sin(ka)\cos(ka)}{ka}\right] - ik\left[-\lambda \frac{\sin^2(ka)}{(ka)^2}\right]}$$

$E = k^2/2\mu$

$$f(k) = \frac{K(E)}{1 - iK(E)k} = \frac{1}{K^{-1}(E) - ik}$$

$$= \frac{P(k)}{Q(k)}$$

∞ of zeros

∞ of zeros → Poles

Solution

From method 2

$$f(k) = \frac{-\lambda \frac{\sin^2(ka)}{(ka)^2}}{1 - \frac{1}{\pi}\int_0^\infty dE' k' \frac{-\lambda \frac{\sin^2(k'a)}{(k'a)^2}}{E' - E(k)}}$$

(∞) zeros of K !

$$k' = k(E') = \sqrt{2\mu E'}$$

$$K(E) = \frac{-\lambda \frac{\sin^2(ka)}{(ka)^2}}{1 - \frac{1}{\pi}\Re\int \cdots}$$

Analyticity

$$K(k = k_R + ik_I) \to const. + O(e^{-2k_I a}) = \frac{1}{ik} + O(e^{-2k_I a})$$

$$f(k) = \frac{K(E)}{1 - iK(E)k} = O(e^{+2ik_I a})$$

Essential singularity at infinity in the physical sheet !

"Conspiracy" between zeros and poles !!!
E.g. ∞ number of zeros of K(s) are "fixed" by geometry of the sphere ("dynamics") and this specific "physics" fixes all the poles.

In more general case (no fixed scattering radius) correlation between zeroes and poles persist", an infinite number poles requires infinite number of zeroes (and vice versa)

Simple model for how external particles (e.g. e^-'s) can access "quick model" states

$$V(r) = \frac{\lambda}{2\mu}\delta(r-a) \qquad -\frac{d^2u}{dr^2} + V(r) = Eu(r)$$

$\mu E = k^2$

$\lambda = \frac{\hat{\lambda}}{a^2}$

$u = \sin kr$ ← regular at $r=0$, $r<a$

$u = A\sin kr + B\cos kr \quad r>a$ free wave.

at a: $A\sin ka + B\cos ka = \sin ka$

$akA\cos ka - akB\sin ka = ak\cos ka + \lambda^a \sin ka$

$\Rightarrow f = \ldots$ Bound states $= \ldots$

$-u_+ + u_- + V = 0$

$u_+ = u_- + v$

$A \sin ka + B \cos ka = \sin ka$

$+ A k \cos ka - B k \sin ka = k \cos ka + \lambda s$

$$\begin{pmatrix} -ks & -c \\ -kc & s \end{pmatrix} \begin{pmatrix} s \\ kc + \lambda s \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \qquad A = -k_a - \lambda c s \qquad B = \lambda s^2 a$$

$$u(r>0) = A\left[\frac{e^{ikr} - e^{-ikr}}{2i}\right] + B\left[\frac{e^{ikr} + e^{-ikr}}{2}\right]$$ (large distance away from scatterer)

$$= -\frac{1}{2i} e^{-ikr}\left[A - iB\right] + \frac{1}{2i} e^{+ikr}\left[A + iB\right]$$

$$= -\frac{1}{2i}\left[A - iB\right]\left[e^{-ikr} - \frac{A+iB}{A-iB} e^{+ikr}\right]$$

outgoing $-i\xi k$ λe

incoming

compare with free: $e^{-ikr} - e^{+ikr}$

$-S$

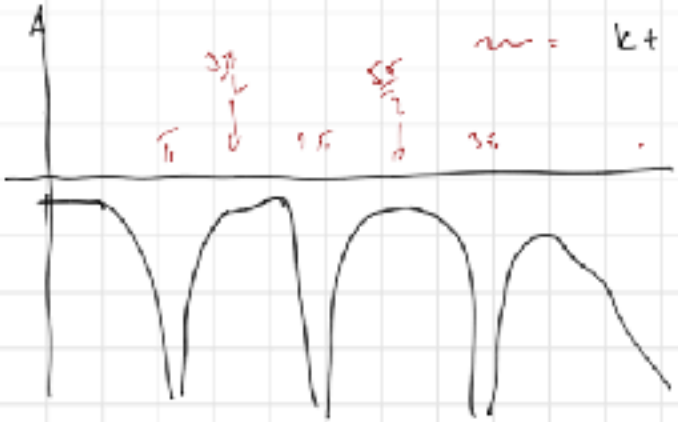
$$S = \frac{A+iB}{A-iB} \qquad f = \frac{S-1}{2i} = \frac{B}{A-iB} = \frac{a\lambda s^2}{-k_a - \lambda_a c s - i\lambda_a s^2}$$

$$f = \frac{a\lambda \sin^2 ka}{-ak - \lambda a \sin ka \cos ka - i a \lambda \sin^2 ka} \qquad f = \frac{1}{-\left[k + \lambda s c\right]/\lambda s^2 - i}$$

$$\tilde{f} = \frac{f}{k} = \frac{1}{k^{-1} - ik} \qquad k^{-1} = -\left[1 + \lambda\left(\frac{1}{k}\right)c\right]\left(\frac{k}{s}\right)^2 \frac{1}{\lambda}$$

$$K^{-1} = -\left[1+\lambda\left(\frac{s}{k}\right)c\right]\left(\frac{k}{s}\right)^{\frac{1}{\lambda}} \qquad \lambda = 1 \quad a = 1$$

$\sin ak$ $\quad k = \frac{\pi n}{a}$



$\sim = k + \pi/2 \sin 2ka$

oscillatory maxima

$k \sim \frac{\pi(n+1/2)}{a}$

$K^{-1} - iK$ will have zeros in the strip

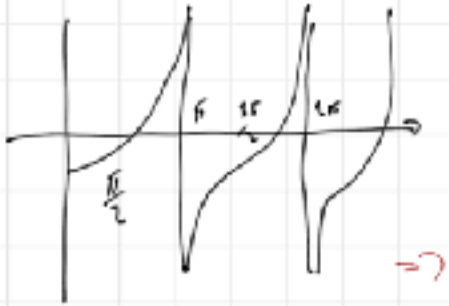
$n\frac{\pi}{a} < \mathrm{Re}\, k < (n+1)\frac{\pi}{a} \quad n = 0, 1, \ldots$

at $k = 0 \Rightarrow f^{-1} = -1 - \lambda$ is finite.

(for $\lambda < 0$, $\lambda \approx -1$ there is an additional state, near zero energy bound state)

① Poles of $K^{-1} \Rightarrow$ zeros of amplitude $\Rightarrow$ CDD poles.
near zeros of f^{-1} there are poles $\Rightarrow$ resonances

In the limit of $A \to \infty \Rightarrow$



$\Rightarrow$ poles (resonances) come to the real axis $k = (n+\frac{1}{2})\frac{\pi}{a}$

$\Rightarrow$ this violates unitarity!

It is interesting to study asymptotic behavior of $f(k)$

take $k = iR$ $R \to \infty$ (upper half plane)

$$f = \frac{x \sin^2 ka \to e^{2R}}{-k \to \sin 2ka - i x \sin^2 ka}$$

$\sin k \to \frac{1}{2i}(-e^{R})$

$\cos k \to \frac{1}{i}(e^{R})$

$$+ \frac{x}{4}\frac{1}{i}e^{2R} + \frac{ix}{4}e^{2R} \to \infty\, e^{2R}$$

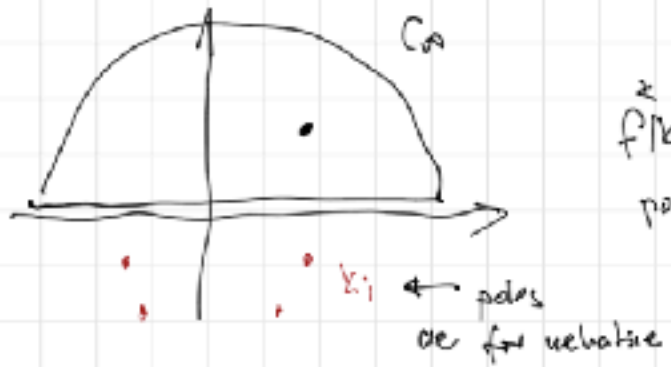
$f \to$ blows up exponentially $\sim e^{R}$ in the upper plane

$k \to -iR$ $R \to \infty$ $\sin k \to \frac{1}{2i}e^{R}$ $\cos k \to \frac{1}{i}e^{R}$

$$\sim -\frac{x}{4}\frac{1}{i}e^{2R} + i\frac{x}{4}e^{2R} \sim e^{2R} \quad f \to \text{const}$$

$\Rightarrow$ the blow up is related to existence of ∞ at poly/zero

Often one employs dispersion relation.



$\hat{f}^*(k) = f(-k^*)$

pole at k
has a mirror
at $-k^*$

Consider $\hat{f} = \tilde{f} e^{+2ika}$ ⇒ convergent in the upper plane.

there is ∞ of poles in the upper plane.

$$\hat{f}(k) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dk' \frac{\hat{f}(k')}{k'-k} + \int_{C_\infty} \to 0$$

to write this one needs the know the assymptotic behavior ⇒ this is unavoidable at the (∞) zeros/poles.

Alternatively one could use the lower half plane
⇒ there one needs the resonances explicitly

(as we'll see the data is on upper plane)

Use every trick

$k = \sqrt{E 2\mu}$ use the following determination $\sqrt{z} = \sqrt{|z|}e^{i\phi} = \sqrt{|z|}e^{i\phi/2}$

$0 < \phi < 2\pi$

$k>0$

$f^*(k+i\varepsilon) = f(-k+i\varepsilon)$ $\qquad$ $f^*(E+i\varepsilon) = f(E-i\varepsilon)$

$\sqrt{E} = \sqrt{|E|}e^{i(\pi-\varepsilon)} = \sqrt{|E|}e^{i\pi}e^{-i\varepsilon}$

$= -\sqrt{|E|}(1-i\varepsilon) = -\sqrt{|E|} + i\varepsilon$

one can write dispersion for $f(z)$

$$\frac{1}{2\pi i}\left[\int_{+\infty}^{0} dE' \frac{\hat{f}(E'-i\varepsilon)}{E'-E} + \int_{0}^{\infty} dE' \frac{\hat{f}(E'+i\varepsilon)}{E'-E}\right] = \frac{1}{2\pi i}\int_{0}^{\infty} \frac{2i\,\mathrm{Im}\,f(E+i\varepsilon)}{E'-E}$$

$$\rightarrow f(E) = \frac{1}{\pi}\int_{0}^{\infty} dE' \frac{\mathrm{Im}\,f(E')}{E'-E}$$

$$\hat{f} = \frac{\times \sin ka \, e^{2ika}}{-k - \times \sin 2ak \cos ka - i \times \sin^2 ka} \times \frac{1}{k} \quad \Big| \quad \hat{f} = e^{2ika} \tilde{f}$$

$\operatorname{Im} \tilde{f} = k |\tilde{f}|^2 \Leftarrow$ unitarity. $\Rightarrow$ dispersion relation would involve f only!

but we wanted $\hat{f}$ or some information from dispersion relation:

What about $\frac{1}{\tilde{f}} \Rightarrow \operatorname{Im} \frac{1}{\tilde{f}} = 2k$, It converges asymptotically

but needs ∞ of poles (CDD).

Supposed the subtracted were not used.

$\operatorname{Im} f = k |f|^2 \Rightarrow$ reason

$$f(t) = \frac{1}{\pi} \int_c dt' \frac{\operatorname{Im} f(t')}{t' - t} \Rightarrow \text{continuation}$$

if $\operatorname{Im} f$ is bounded

$$\hat{f}_B = -\hat{x} \frac{\sin kc}{(ka)^2}$$

$$\hat{f} = \frac{\gamma \sin^2 ka \, e^{2ika}}{-k - \gamma \sin^2 ka \cos ka - i\gamma \sin^2 ka} \times \frac{1}{k}$$

$\gamma \to$ unitarity Born: $f_B = \gamma \sin^2 ka$

$$f = \frac{-\gamma \sin^2 ka/k}{1 + \int \frac{\gamma \sin^2 k/k}{E' - E}} \frac{1}{k}$$

$$f = \frac{(-\gamma \sin^2 ka/k)}{\left(1 - \int \frac{(-\gamma \sin^2 ka/k')}{E' - E}\right)} \frac{1}{k} = \frac{N}{1 - \int \rho \frac{N}{E' - E}} \qquad \rho = k$$

$$\frac{1}{f} = \frac{1}{N} - \frac{1}{N} \int \rho \frac{N}{E' - E} \qquad \operatorname{Im} \frac{1}{f} = -i\rho$$

$\frac{1}{f}$ = disp. rel. $\qquad \frac{1}{f}$ convergent

$$\frac{k}{\sin k} + \frac{\gamma \cos ka}{\sin ka}$$

$$\langle \vec{k}'|\vec{k}\rangle = (2\pi)^3 \delta^3(k'-k) = (2\pi)^3 \frac{1}{k^2}\delta(k'-k)\,\delta^2(\Omega'-\Omega) =$$

$$= (2\pi)^3 \frac{1}{k^2}\left(\frac{dk}{dE}\right)^{-1}\delta(E'-E)\,\delta^2(\Omega'-\Omega) \qquad E = \frac{k^2}{2\mu}$$

$$= (2\pi)^3 \frac{1}{\nu k}\,\delta(E'-E)\,\delta^2(\Omega'-\Omega) = (2\pi)^3 \frac{1}{k\mu}\,\delta(E'-E)\sum_{\ell m} Y^*_{\ell m}(\Omega')\,Y_{\ell m}(\Omega)$$

$\leftarrow V + V G_0 V + \dots$

$$\langle \vec{k}'|S|\vec{k}\rangle = \text{nur} - (2\pi)\,\delta(E'-E)\,i\,\langle \vec{k}'|T|\vec{k}\rangle$$

$$\langle \vec{k}'|T|\vec{k}\rangle = \sum T_\ell\, Y_{\ell m}\, Y_{\ell m}$$

$$\langle \vec{k}'|S|\vec{k}\rangle = (2\pi)^3 \frac{1}{k\mu}\,\delta(E'-E)\sum_{\ell m} Y_{\ell m}\, Y_{\ell m} \times$$

$$\Big[\, 1 - \underbrace{\frac{1}{(2\pi)^2}\frac{(2\pi)}{4}\,(2\mu)\,k\,T_\ell}_{S_\ell} \,\Big]$$

$$f_\ell = \frac{(S-1)}{2i}$$

$$S_\ell S_\ell^* = 1 \qquad S_\ell = e^{2i\delta} \qquad [\dots] = 1 + 2ikf_\ell = e^{2i\delta}$$

$$\Rightarrow f_\ell = \frac{(e^{2i\delta}-1)}{2ik} = \frac{e^{i\delta}\sin\delta}{k}$$

$$f_\ell = -\frac{1}{(4\pi)^2}\,\underbrace{(2\mu T_\ell)}_{-1} \qquad \mu \text{ weg} \quad T_\ell = \langle \ell|v'|\ell\rangle = -3+1+\dots 2$$

$$\langle \vec{k}'|T|\vec{k}\rangle = \sum T_\ell\, Y_{\ell m}\, Y_{\ell m} = \sum_\ell \frac{(2\ell+1)}{2\pi}\, T_\ell\, P_\ell(\cos\theta)$$

$$T_\ell = \frac{1}{2}\int d\vartheta\, P_\ell(\cos\theta)\, 4\pi\, \langle \vec{k}'|T|\vec{k}\rangle \Rightarrow f_\ell = -\frac{1}{2}\left(\frac{1}{4\pi}\right)\int d\vartheta\, P_\ell(\vartheta)\,\langle \vec{k}'|2\mu V|\vec{k}\rangle$$

$$f_\ell = -\frac{1}{2}\left(\frac{1}{4\pi}\right)\int dz\, P_\ell(z)\, \langle k'|\, 2\mu T\, |k\rangle$$

$$f_\ell = -\frac{1}{2}\left(\frac{1}{4\pi}\right)\int dz\, P_\ell(z)\, \langle k'|\, 2\mu T\, |k\rangle$$

$$2\mu T = V + V G_0 V + \dots \qquad G_0 = \frac{1}{2\mu H - 2\mu E}$$

Take $V = \frac{A}{2\mu a^2}\delta(r-a)$ $\qquad [A]_L = -1$

$$\langle k'|2\mu V|k\rangle = \int d^3r\, e^{ik'r}\, e^{-ikr}\, \frac{A}{a^2}\delta(r-a) \left\{ e^{ikr} = \sum_\ell j_\ell(ka)\, i^\ell 4\pi Y Y \right.$$

$$= \sum_\ell \left[j_\ell(ka)\right]^2 A\,(2\ell+1)\, P_\ell(z_{ka})\, 4\pi$$

$$-\frac{1}{2}\frac{1}{4\pi}\int dz\, P_\ell(z) = -j_\ell^2(ka)\, A = -\frac{\sin^2 ka}{(ka)^2} A$$

$$f_0 = -\frac{i}{4\pi}\,\frac{\sin^2 ka}{(ka)^2}\,A$$

$$f_e = -\frac{1}{2}\left(\frac{1}{4\pi}\right)\int d^3k\, B(k)\,\langle k'|\,2\mu V\,|k\rangle$$

compute in 2nd order:

$$\langle k'|\,2\mu V\,G_0\,2\mu V\,|k\rangle = \int d^3x\, e^{i\vec{k}'\cdot x}\,\frac{A}{a^2}\,\delta(r-a)\,G_0\,\frac{A}{a^2}\,\delta(r'-a)\,e^{-ik\cdot x'}$$

$$G_0(x,x') = \frac{1}{(2\pi)^3}\,e^{-ik''\cdot(x-x')}\,\frac{1}{k''^2 - 2\mu E} \quad (-)$$

$$\langle k'|\,2\mu V\,|k''\rangle = \sum_\ell j_\ell^2\,A\,(2\ell+1)\,P_\ell(x)\,4\pi = \sum_\ell j_\ell^2\,A\,(4\pi)^2\,Y Y$$

$$\int \frac{k''^2 dk''}{(2\pi)^3}\, j_\ell^4 A^2 (4\pi)^4\, \underbrace{Y_\ell(k)\,Y(k')}_{\eta}\,\frac{1}{k''^2 - 2\mu E} \quad (-)$$

$$-D^2 \to i^2\, i$$
$$-i^2$$
$$\frac{1}{r}\,j$$

$$\int \frac{k''^2 dk''}{(2\pi)^3}\,(4\pi)^3\, j_\ell^4 A^2\,(2\ell+1)\,\frac{1}{k''^2 - 2\mu E} = \mathrm{Re}\left(-\frac{1}{4\pi}\right)\frac{1}{j^2\, j^2} \quad (-)$$

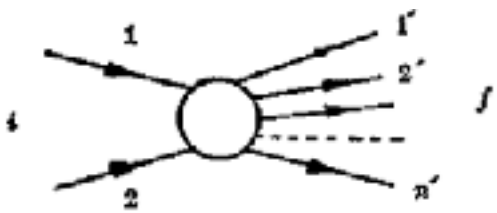
$$\ominus = \ominus\,\frac{(4\pi)^2}{(2\pi)^3}\int k''^2 dk''\, A^2\,\frac{j_\ell^4}{k''^2 - 2\mu E} = \oplus\,\frac{1}{\pi}\int k''^2 dk''\,\frac{j_\ell^4 A^2}{k''^2 - 2\mu E}$$

$$\frac{dE}{\hbar k} = \frac{k}{\mu} \qquad dk = \frac{d\mu}{\hbar k}\,dE = \frac{A}{k}\,dE = \frac{2\mu k\,dE}{k'^2 - k^2} = \frac{2}{k}\,\frac{k\,dE}{E}$$

S-matrix properties (in relativistic theory)

- Related to transition probability

$$P_{fi} = |\langle f|S|i\rangle|^2 = \langle i|S^\dagger|f\rangle\langle f|S|i\rangle$$



- Conservation of Probability = Unitarity

$$\sum_f P_{fi} = 1$$

$$S^\dagger S = I$$

$$2ImT_{ft} = \sum_n 2\pi\delta(E_i - E_n)T^*_{fn}T_{ni}$$

- Lorentz symmetry: T is a product of Lorentz scalars and covariant factors representing wave functions of external states, e.g for $\pi(k_1) + N(p_1, \lambda_1) \to \pi(k_2) + N(p_2, \lambda_2)$

$$\bar{u}(p_1, \lambda_1)[A(s,t) + (k_1 + k_2)_\mu \gamma^\mu B(s,t)]u(p_2, \lambda_2)$$

- Crossing symmetry: the same scalar functions describe all process related by permutation of legs between initial and final states (only the wave function change)

$$\pi(k_1) + \pi(-k_2) \to \bar{N}(-p_1, \mu_1) + N(p_2, \mu_2)$$

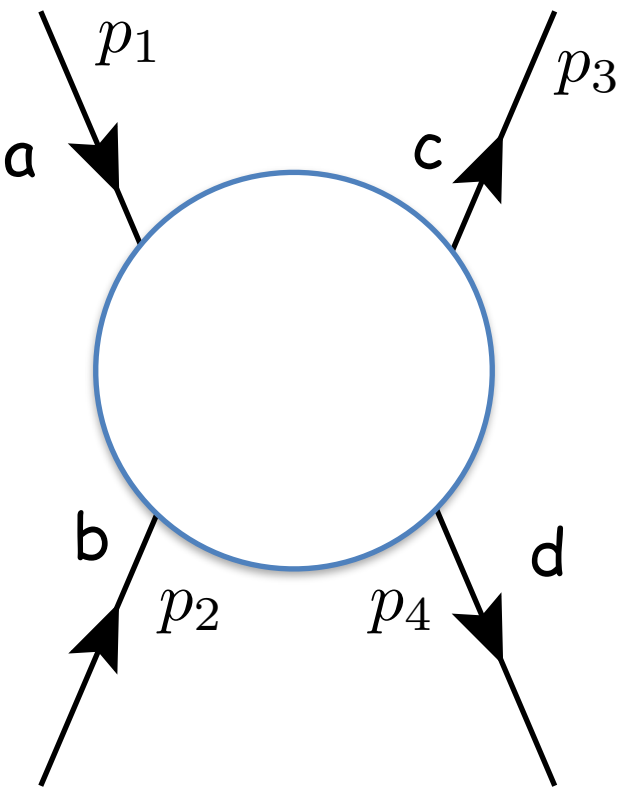
$$\bar{v}(p_1, \mu_1)[A(s,t) + (k_1 + k_2)_\mu \gamma^\mu B(s,t)]u(p_2, \mu_2)$$

- Analyticity: The scalar functions are analytical functions of invariants

Lorentz symmetry

N-to-M scattering depends on 4(N+M)-4-10 = 3(N+M)-10 invariants

e.g for 2-to-2: 2 invariants related to the c.m. energy and scattering angle



$$s = (p_1 + p_2)^2 = (E_{1,cm} + E_{2,cm})^2$$

$$t = (p_1 - p_3)^2$$

$$t = m_1^2 + m_2^2 - 2E_{1,cm}E_{2,cm} + 2|p_{1,cm}||p_{2,cm}|z_s$$

$$u = (p_1 - p_4)^2 \qquad s + t + u = \sum_i m_i^2$$

$$u = m_1^2 + m_4^2 - 2E_{1,cm}E_{4,cm} - 2|p_{1,cm}||p_{4,cm}|z_s$$

$$2\pi\delta(E_f - E_i)iT = \langle c, d|(S-1)|a, b\rangle$$

Dimensions $\langle p', \beta|p, \alpha\rangle = 2E(\mathbf{p})\delta(\mathbf{p}_f - \mathbf{p}_i)\delta_{\alpha,\beta}$

r.h.s has dim = -4

$$T = (2\pi)^3\delta(\mathbf{p}_f - \mathbf{p}_i)A(s,t,u)$$

A(s,t,u) is a scalar function of mass dimension =0

Question

How many independent variables describe

- Decay proces $A \rightarrow a + b + c$
- Three particle production $A + B \rightarrow a + b + c$

Helicity amplitudes

We work in the c.m. frame $$\frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}|p,\lambda\rangle = \lambda|p,\lambda\rangle$$

$$\langle p_3,\lambda_3;p_4,\lambda_4|A|p_1,\lambda_1;p_2,\lambda_2\rangle = A_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(s,t,u)$$

Helicity states vs canonical spin states: $$S_z|p,m\rangle_z = m|p,m\rangle_z$$

$$|p,m\rangle_z = \Lambda(\vec{p}\leftarrow 0)|0,m\rangle_z$$

$$|p,\lambda\rangle = R(\hat{p})\Lambda(|\vec{p}|\hat{z}\leftarrow 0)|0,m\rangle_z$$

Exercise show this: $$|p,\lambda\rangle_z = \sum_{m=-S}^{S}|p,m\rangle_z D^S_{m,\lambda}(\hat{p})$$

- Even though this looks non relativistic it is relativistic. Notion of LS amplitudes, LS vs. helicity relations are relativistic

Parity $$A_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(s,t,u) = \eta A_{-\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4}(s,t,u)$$

Question

How many independent scalar functions describe

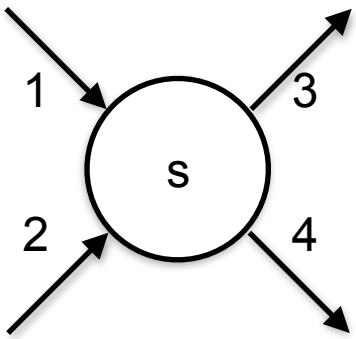
$J/\psi \rightarrow \pi^+ \pi^- \pi^0$

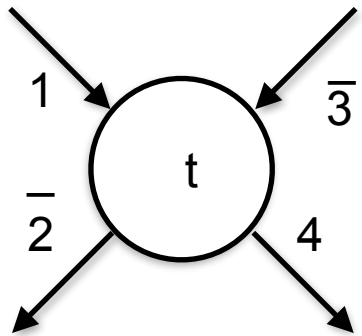
$\gamma p \rightarrow \pi^0 p$

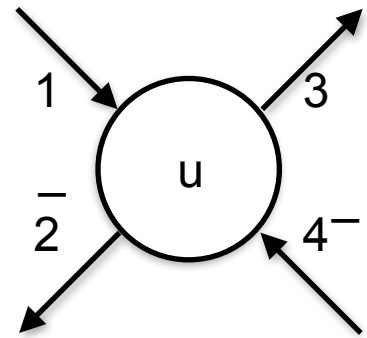
Crossing symmetry

$\bar{p}_i = -p_i = (-\vec{p}_i, -E_i)$

$a(p_1) + \bar{c}(\bar{p}_3) \rightarrow \bar{b}(\bar{p}_2) + d(p_4)$







$a(p_1) + b(p_2) \rightarrow c(p_3) + d(p_4)$

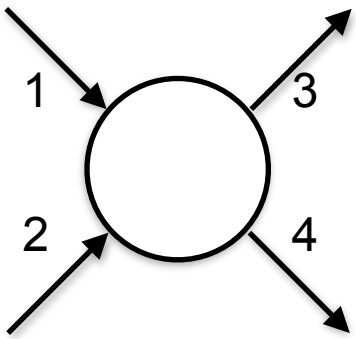
$a(p_1) + \bar{d}(\bar{p}_4) \rightarrow c(p_3) + \bar{b}(\bar{p}_2)$

$E_{c.m}$	$s = (p_1 + p_2)^2$	$t = (p_1 + p_{\bar{3}})^2$	$u = (p_1 + p_{\bar{4}})^2$
Cos(θ)	$t = (p_1 - p_3)^2$	$s = (p_1 - p_{\bar{2}})^2$	$t = (p_1 - p_3)^2$
Cos(θ)	$u = (p_1 - p_4)^2$	$u = (p_1 - p_4)^2$	$s = (p_1 - p_{\bar{2}})^2$

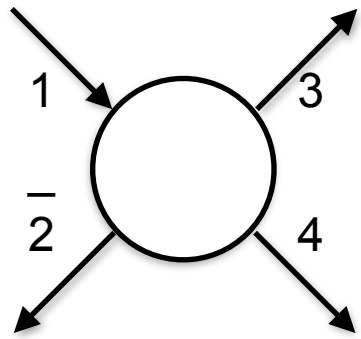
$$A^{(s)}_{\lambda_1,\ldots}(s + i\epsilon, t, u) \rightarrow \sum_{\lambda'_1,\ldots} [D^{S_1}_{\lambda_1,\lambda'_1} \cdots] A^{(t)}_{\lambda'_1,\ldots}(s, t + i\epsilon, u) \rightarrow \cdots$$

- The iε is important. Function values at, e.g. s + iε vs s - iε are different !

Crossing Symmetry : Decays $M_1 > m_2 + m_3 + m_4$



$a(p_1) + b(p_2) \rightarrow c(p_3) + d(p_4)$



$a(p_1) \rightarrow \bar{b}(\bar{p}_2) + c(p_3) + d(p_4)$

$$A(s,t,u) \rightarrow A(M_1^2 + i\epsilon, s + i\epsilon, t + i\epsilon, u + i\epsilon)$$

- In decay kinematics, the decaying mass becomes a dynamical variable, (iε important)
- Crossing from one kinematical region (e.g. s-channel) to another (e.g. t-channel) requires taking the corresponding variables off the real axis and to the complex plane : analytical continuation.

Analyticity

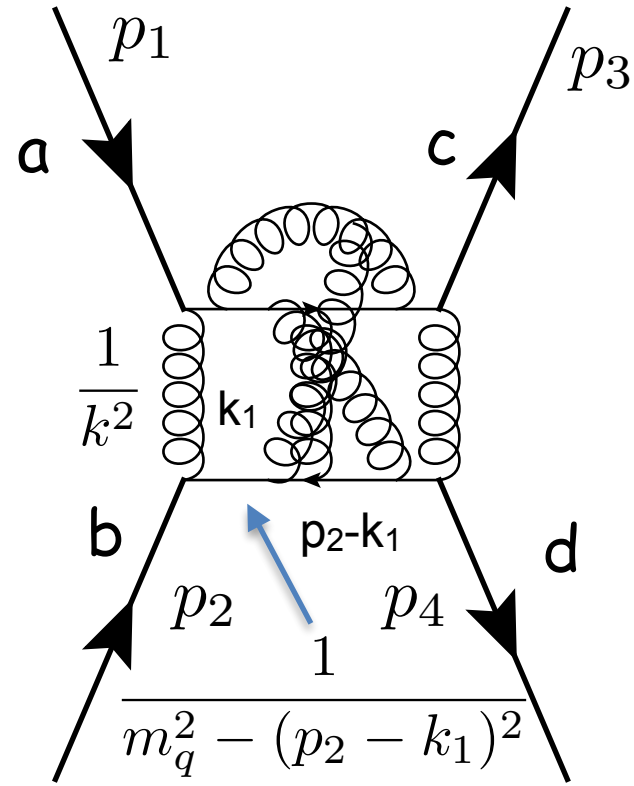
Feynman diagrams

$$A(p_1,\cdots) \propto \int [\Pi_j d^4k_j] \frac{\text{polynomial in } \mathrm{k}_j}{(m_q^2-(p_i-k_j)^2-i\epsilon)((k_i-k_j)^2-i\epsilon)\cdots}$$

$$m^2-p^2=[m^2+\mathbf{p}^2]-(p^0)^2$$

$$m^2-p^2=0 \rightarrow p^0=\pm(m^2+\mathbf{p}^2)^{1/2}$$

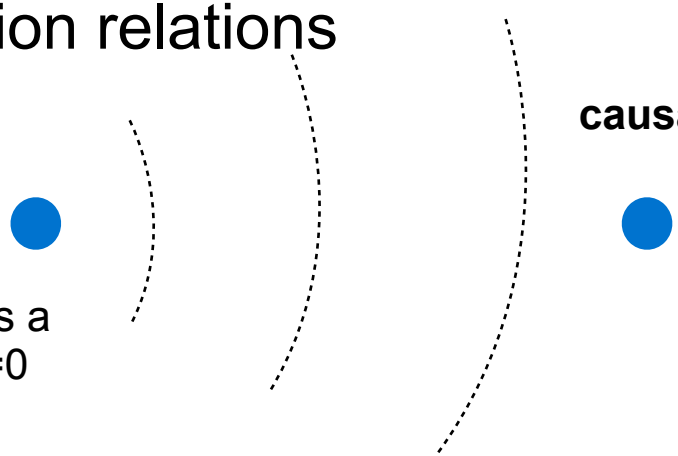
- Integrand becomes singular when intermediate states go on shell.
- Thresholds for producing physical intermediate are the only reason why amplitudes are singular.
- Production of intermediate states is related to unitarity. Thus we expect unitarity to determine singularities of the amplitudes.



On the role of iε $\quad \mathrm{Im}\left[\frac{1}{\sqrt{m^2+\mathbf{p}^2}\mp i\epsilon-p^0}\right]=\pm\pi\delta(p_0-\sqrt{m^2+\mathbf{p}^2})$

Analyticity and Causality

Dispersion relations



causality: receiver receives at t>0 and not at t<0

source emits a signal at t=0

amplitude of the signal

$$f(t) \propto \theta(t)$$

consider the Fourier transform (E → energy)

$$f(E) \equiv \int dt e^{iEt} f(t)$$

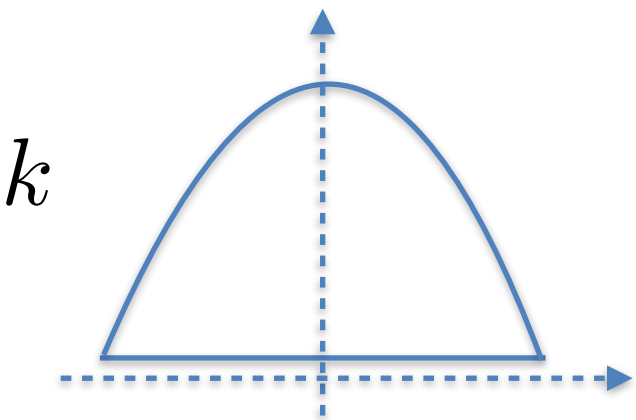
and extend definition to complex plane E → z, then f(z) is holomorphic for Im E > 0

Causality: The outgoing wave cannot appear before the incoming one. Causality determines analytical properties of the scattering amplitude as function on energy/ momenta/scattering angle. The specific from of these conditions depend on the type of interactions and kinematics (e.g. relativistic vs non relativistic)

momentum vs energy planes

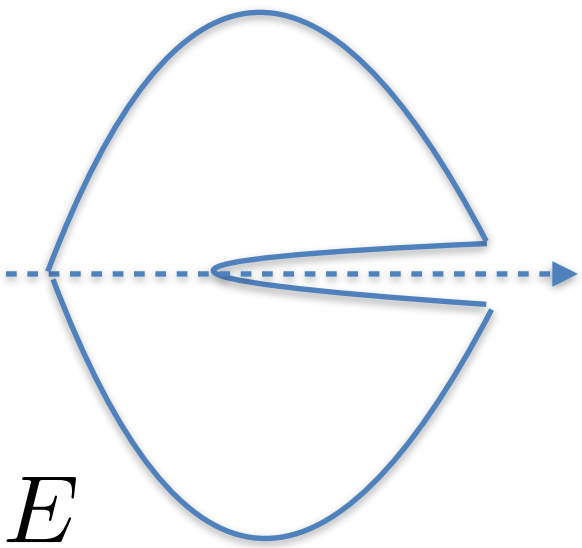
$$f^*(k) = f(-k^*)$$

$$f^*(E) = f(E^*)$$



$$E = \frac{k^2}{2\mu}$$

$$k = \sqrt{2\mu E}$$



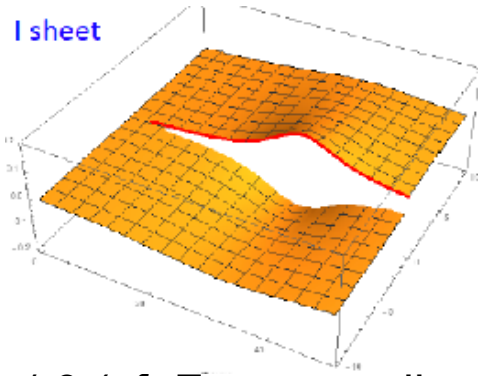
The function is analytical in the whole E-plane not only the upper half

How unitarity constrains singularities

- Unitarity "operates" in the physical domain, i.e. s real and above threshold and |Cos(θ)|<1. This domain is the boundary of the complex plane where analytical amplitude are defined

$$A(s+i\epsilon) = A_{\text{physical}}(s = \text{real and above threshold})$$

sign fixed by "arrow of time V(t) = V exp(-t |ε|)



- The difference (discontinuity) A(s + iε) - A(s - iε) ≠ 0 (cf. Feynman diagrams), comes from particle production this we expect it being determined by unitarity.

$$2ImT_{ft} = \sum_n 2\pi\delta(E_i - E_n)T^*_{fn}T_{ni}$$

- Cauchy theorem : singularities determine the amplitude !!!

Relativistic S-matrix fundamentals

Causality: Determines domain of analyticity of reaction amplitudes as function of kinematical variables.

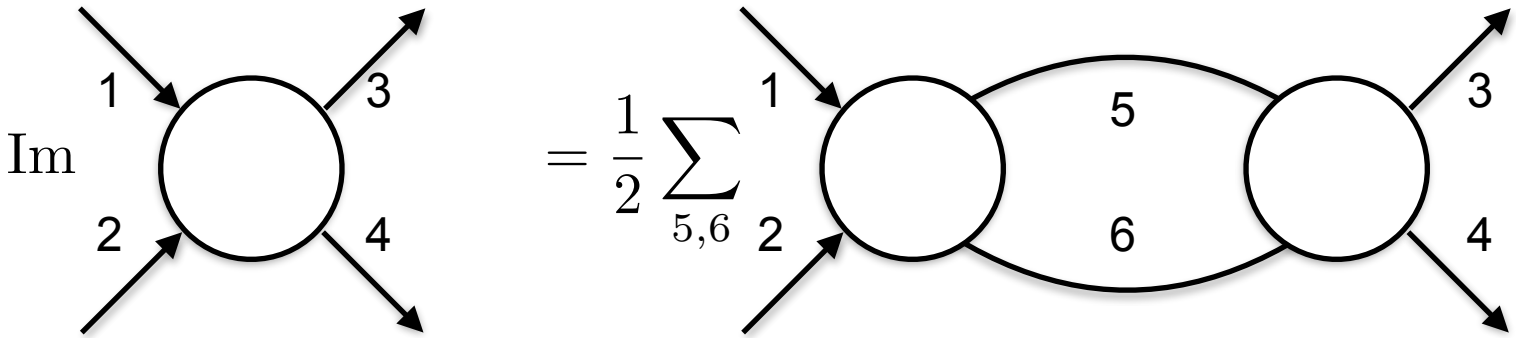
Unitarity: Determines singularities.

Crossing: Dynamical relation, aka reaction amplitudes in the exchange channel (forces) are analogous to amplitude in the direct channel (resonance)

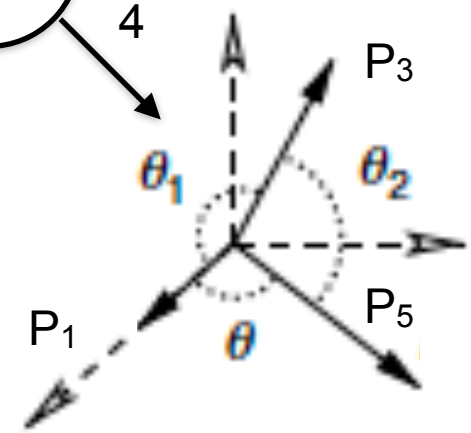
These defined the Bootstrap program of the 60's. It is equivalent to non-relativistic QM, but not to QFT, i.e. "bootstrap equations" do not have unique solutions. For example it failed to reproduce the QCD resonance spectrum, which needs "external parameters". (aka. K-matrix poles, CDD -poles, etc.)

How unitarity constrains singularities: simple example

$$2ImT_{ft} = \sum_n 2\pi\delta(E_i - E_n)T^*_{fn}T_{ni}$$



Consider elastic scattering of spineless particles



$$ImA(s,t) = \frac{\rho(s)}{16\pi}\int \frac{d\Omega}{4\pi} A(s,\cos\theta_1)A^*(s,\cos\theta_2)$$

$$\rho(s) = 2k_{cm}(s)/\sqrt{s}$$

At fixed s, this is a complicated, integral relation w.r.t momentum transfer, t
It is simplified (diagonalized) by expanding A(s,t) in partial waves

$$A(s,t) = 16\pi\sum_{l=0}^{\infty}(2l+1)f_l(s)P_l(\cos\theta) \qquad Imf_l(s) = \rho(s)|f_l(s)|^2$$

How unitarity constrains singularities

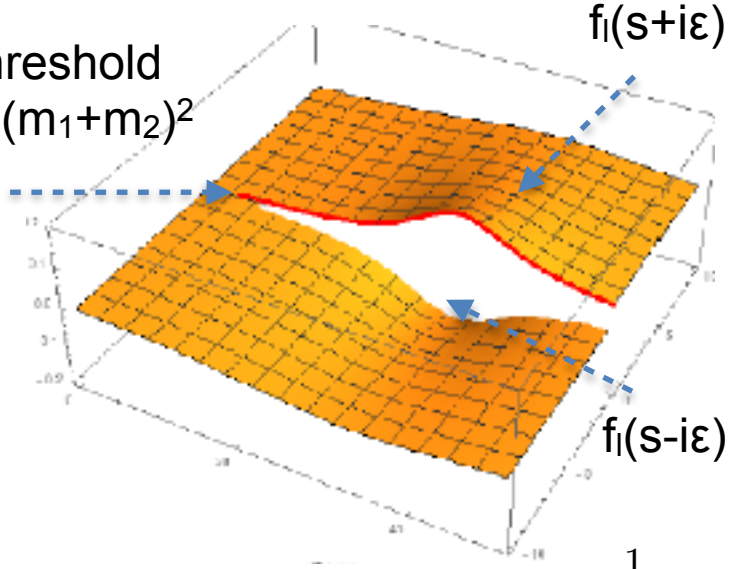
Properties of the partial wave, $f_l(s)$ (for fixed l as function of s):

$$f_l(s) = \frac{1}{32\pi}\int_{-1}^{1} d\cos\theta P_l(\cos\theta) A(s, t(s,\cos\theta))$$

- $f_l(s)$ is real for s below threshold
- Im $f_l(s)$ is finite above threshold.
- $f_l(s)$ is complex for diffidently negative s
- $f_l(s)$ is analytical (since A(s,t) is)

→ Reflection theorem (Calculus 101): $f_l(s^*) = f_l(s^*)$

Threshold
$s=(m_1+m_2)^2$



- Even though $f_l(s)$ has physical meaning for s real and above threshold, there is a unique function in the complex plane which reduces to $f_l(s)$ on the real axis (+iε).
- Furthermore, unitarity which is a condition for physical s-values, becomes a restriction on the complex function, $f_l(s)$.

$$\frac{1}{2i}[f_l(s+i\epsilon) - f_l(s-i\epsilon)] = \rho(s) f_l(s+i\epsilon) f_l(s-i\epsilon)$$

Second sheet

$$f(s+i\epsilon) = \frac{f(s-i\epsilon)}{1-2i\rho(s)f(s-i\epsilon)}$$

Singularity = Resonance at complex s when

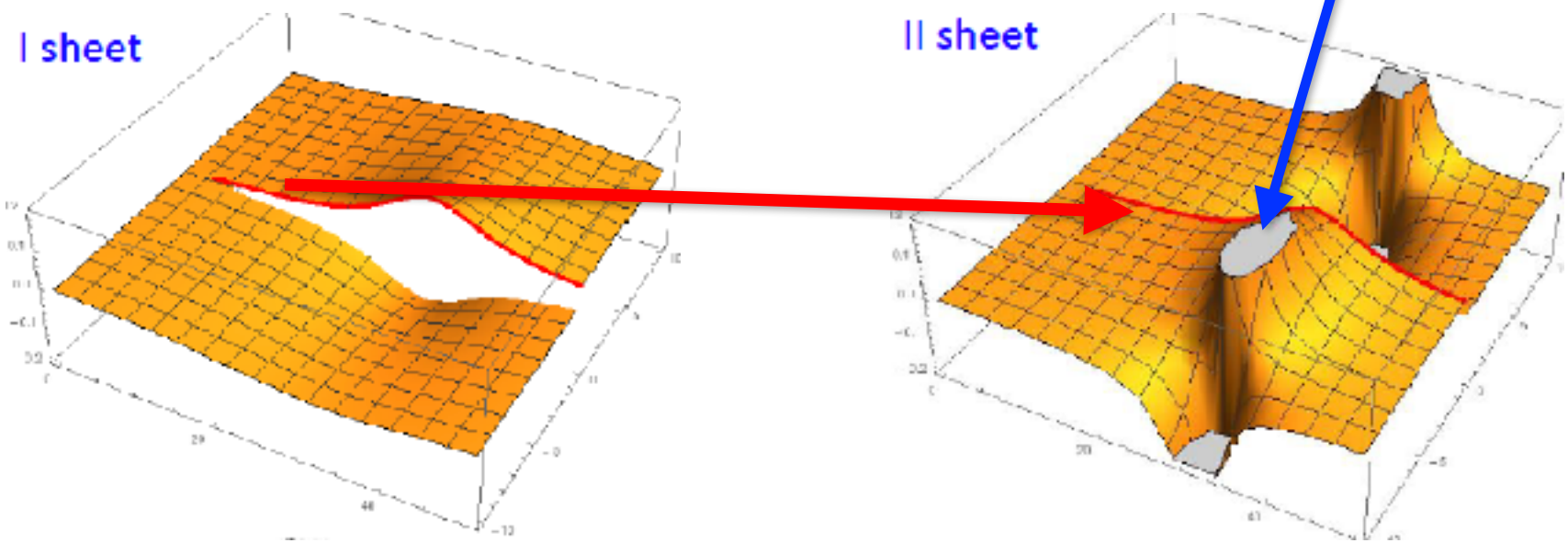
$$f(s) = \frac{1}{2i\rho(s)}$$

Define for Im s < 0

$$f_{II}(s) = \frac{f(s)}{1-2i\rho(s)f(s)}$$

$$f_{II}(s-i\epsilon) = f(s+i\epsilon)$$

This is analytical continuation of f(s) below the real axis



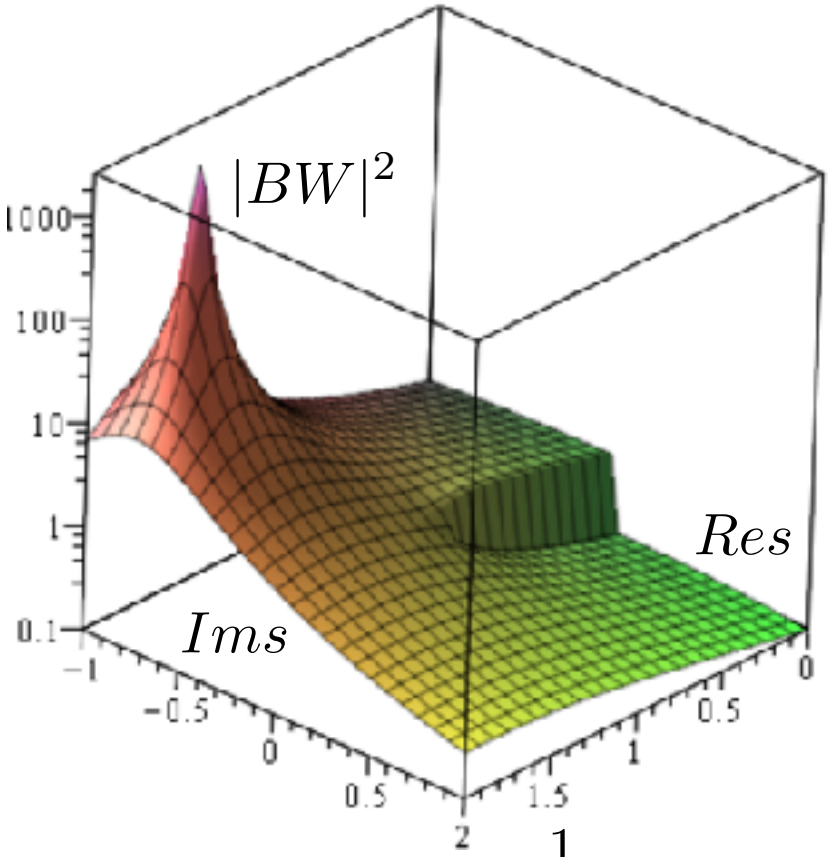
Breit-Wigner Formula

$$BW(s) = \frac{1}{m_r^2 - s - i m_r \Gamma(s)}$$

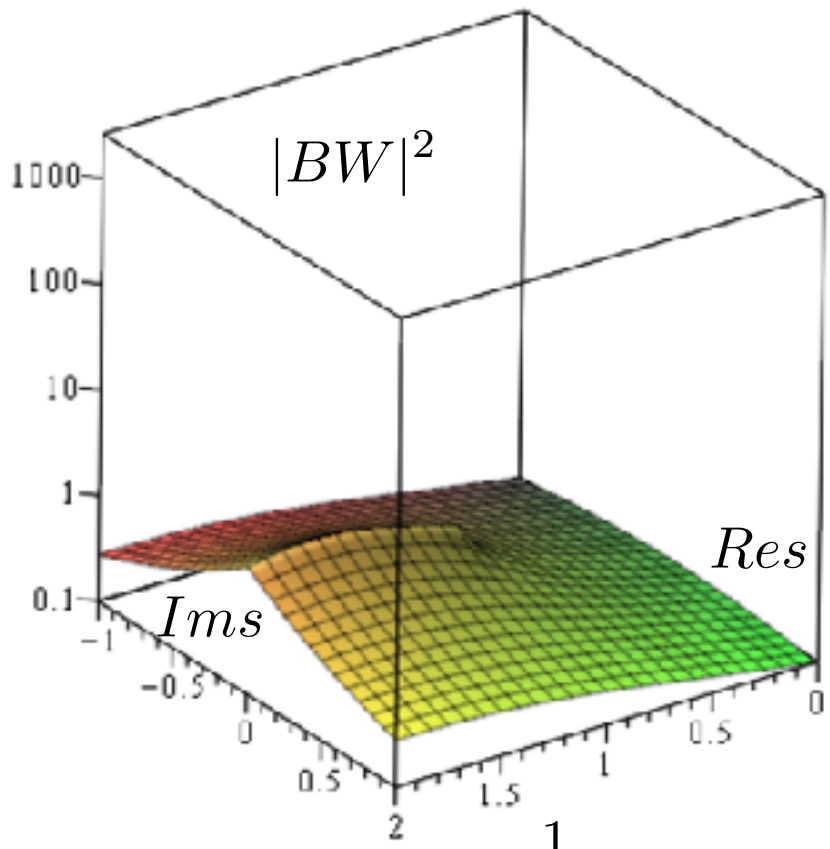
Threshold factor

$$k_{cm} \sim \sqrt{s - s_{th}}$$

$$\Gamma(s) = k_{cm}(s)\gamma(s)$$ **"rest"**



$$BW(s) \sim \frac{1}{2 - s - 0.8i\sqrt{s-1}}$$



$$BW(s) \sim \frac{1}{2 - s + 0.8\sqrt{1-s}}$$

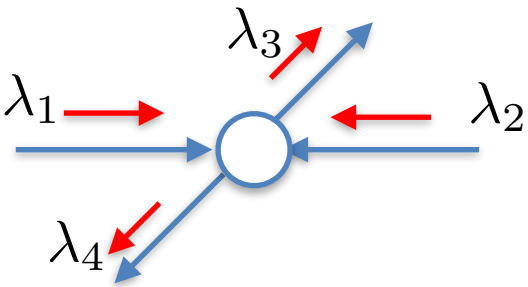
Kinematical vs Dynamical Singularities

For particles with spin

$$A_{\lambda_i}(s,t) = 16\pi \sum_{J=-M}^{M} (2J+1) f^J_{\lambda_i}(s) d^J_{\lambda,\lambda'}(\theta)$$

$$M = \max(|\lambda|, |\lambda'|)$$

$$\lambda' = \lambda_3 - \lambda_4 \quad \lambda = \lambda_1 - \lambda_2$$



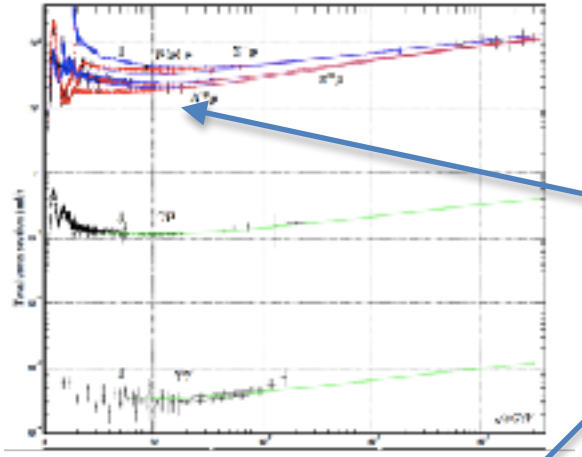
$$f^J_{\lambda_i}(s) = \frac{1}{32\pi} \int_{-1}^{1} dz_s A_{\lambda_i}(s, t(s,\theta)) d^J_{\lambda,\lambda'}(\theta)$$

- Wigner d-functions lead to kinematical singularities
- Threshold (barrier factors) originate from kinematical factors in relation between t and cos(θ) (through dependence of A_λ on t)
- Unequal masses give lead to “daughter poles”
- Dynamical singularities : from dynamical (unitary cuts) in A(s,t).

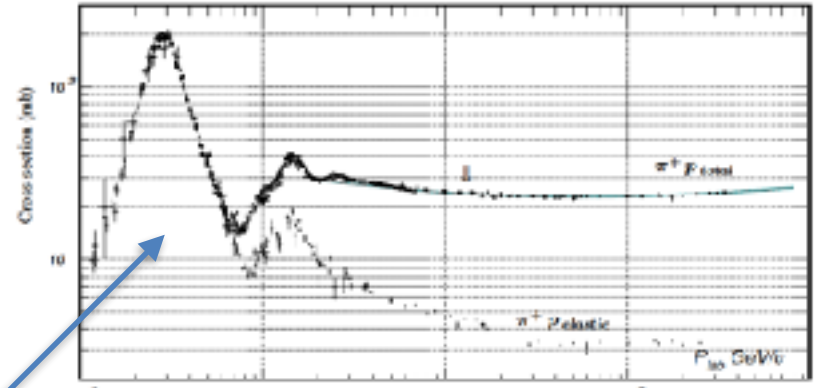
Phenomenology of hadron interaction

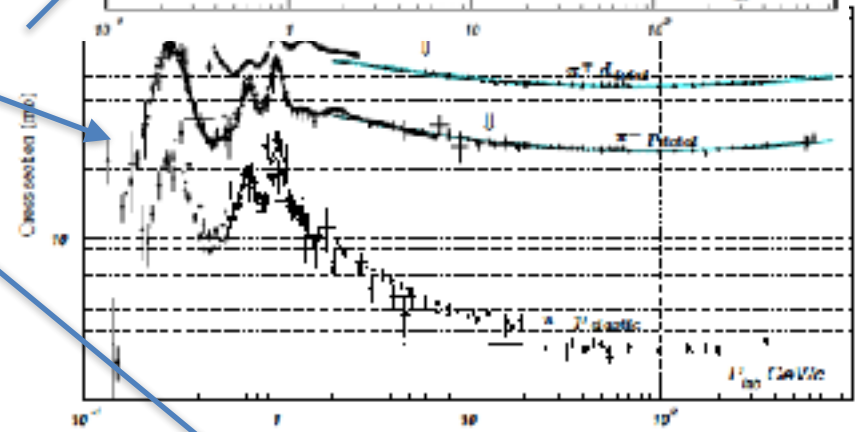
$$\sigma_{a+b\to a+b} \propto \int \frac{dt}{s^2}|A(s,t)|^2$$

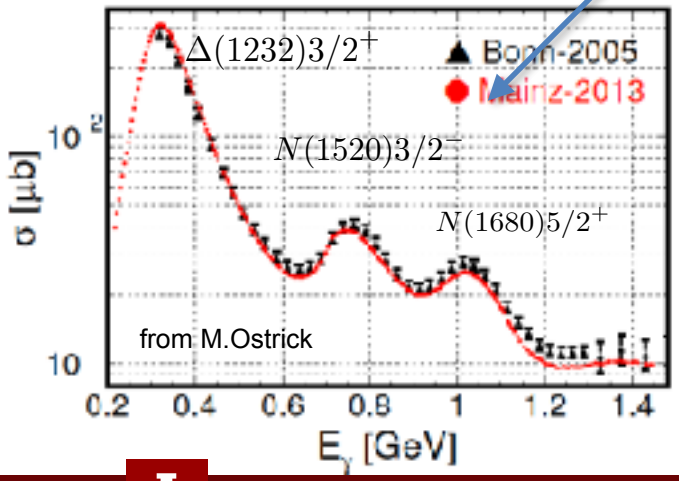
$$\sigma_{a+b\to X} \propto \frac{ImA(s,0)}{s}$$ from unitarily

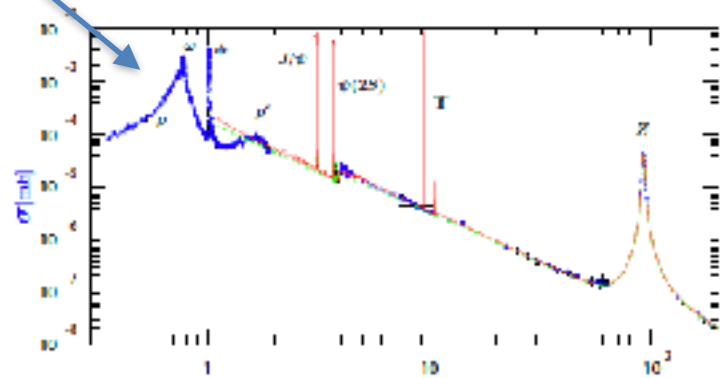


Resonance scattering





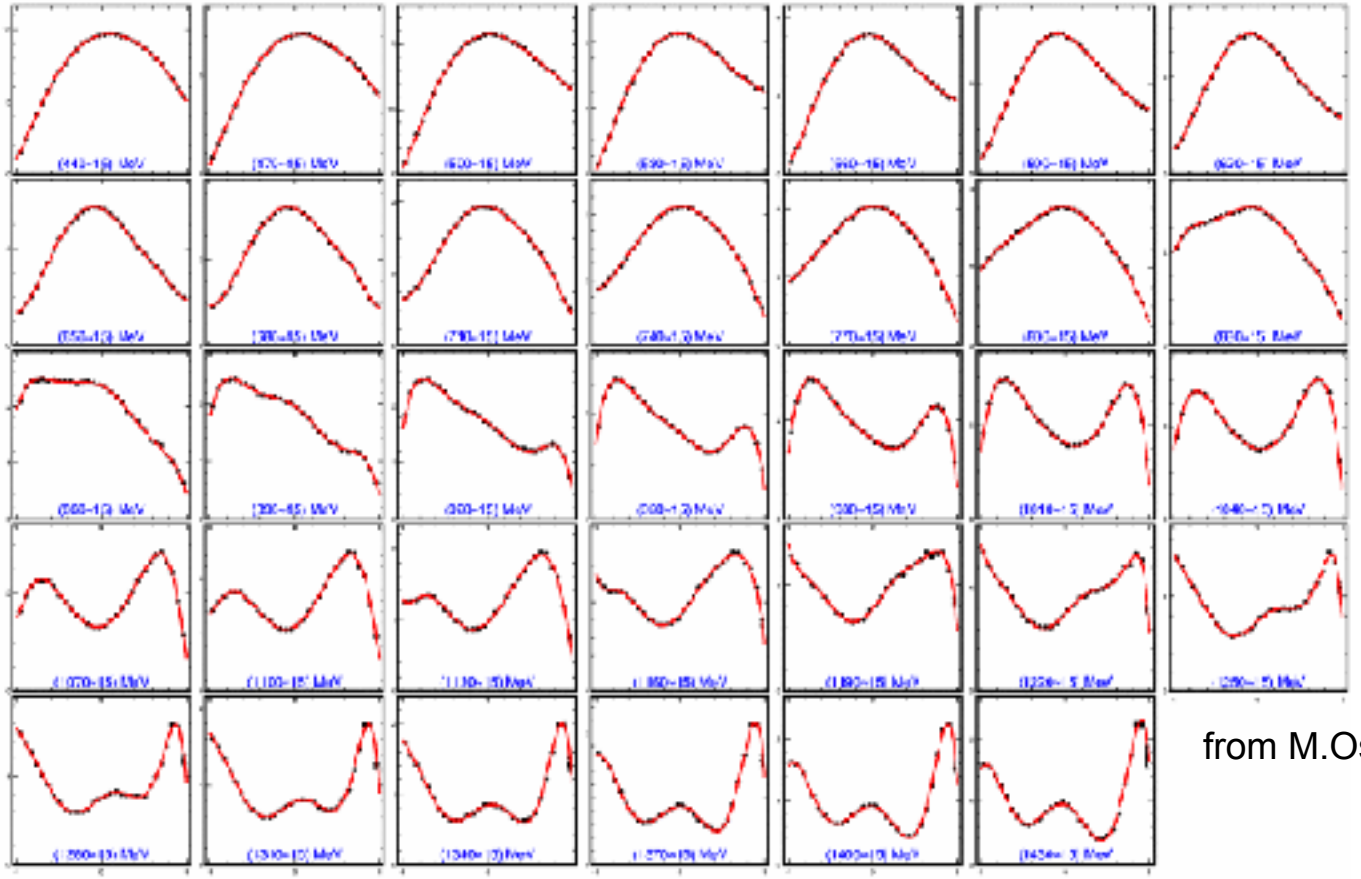




Resonance Scattering

$$\frac{d\sigma}{dt} \propto \frac{|A(s,t)|^2}{s^2}$$

Angular distribution: a few “wiggles”

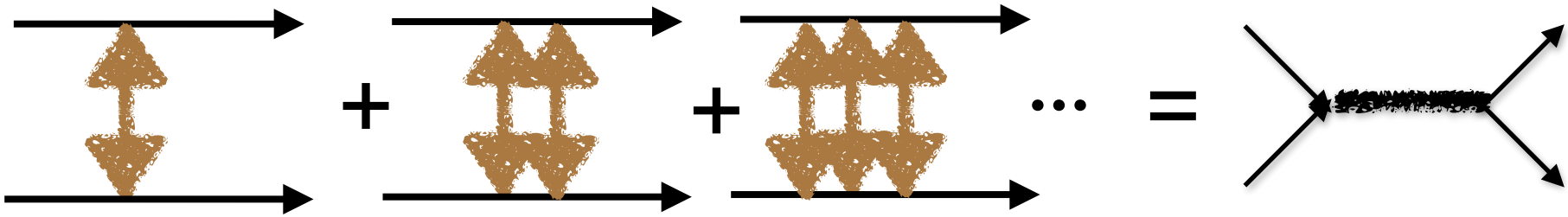


from M.Ostrick

more pronounced forward/backward peaks as energy increases

Resonance scattering

Multiple quark/gluon exchanges



- If QCD was confining resonance would appear at all energy and angular momenta (infinitely rising Regge trajectories).

- String/flux tube breaking leads to screening of color charge and resonance seem to appear with finite angular momentum.

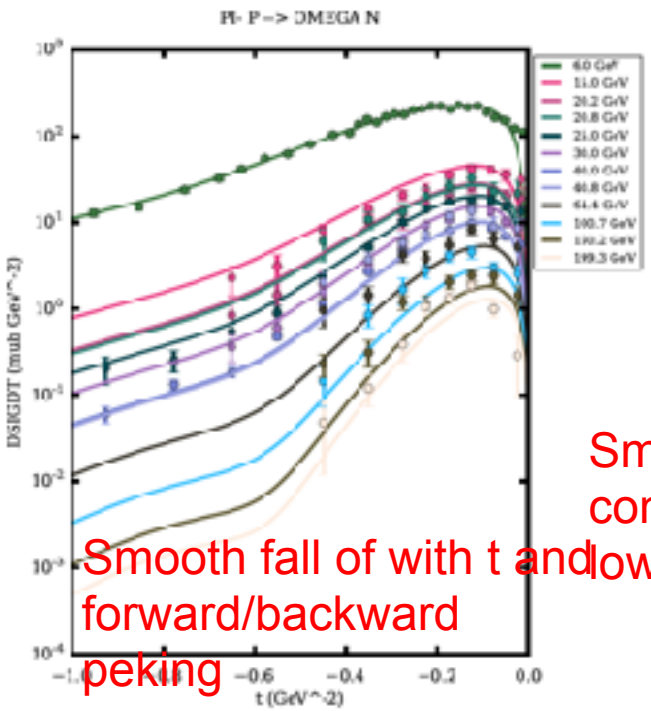
- For l_{max} ~ 5 and nteraction range r_0 ~0.5fm this gives p_{lab} <~ 10/fm ~ 2GeV, [or W ~ $(2\, P_{lab}\, m_p)^{1/2}$ ~ 2GeV]

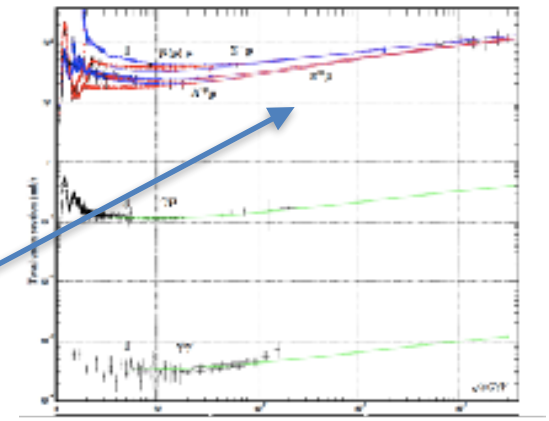
- For resonance scattering

$$A(s,t) = \sum_l (2l+1) f_l(s) P_l(z_s(t)) \longrightarrow A(s,t) \sim \frac{P_{l_R}(z_s(t))}{s - s_R}$$

Scattering at High energies

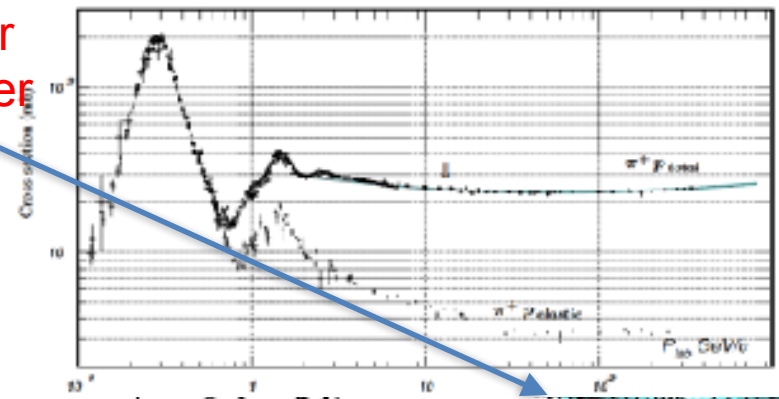
$$\sigma_{a+b\to X} = \frac{1}{s} Im A_{ab\to ab}(s,0)$$

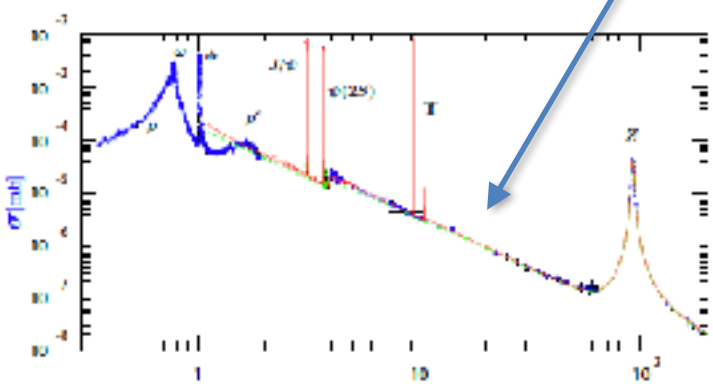


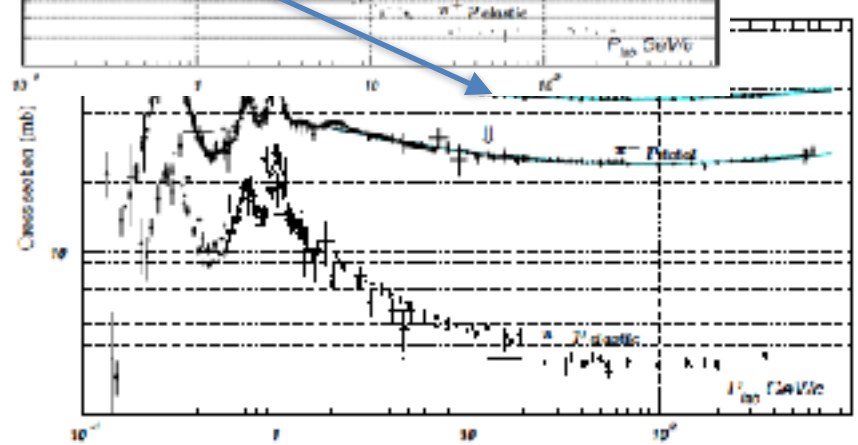


Smooth behavior
constant or power
low fall off

Smooth fall of with t and
forward/backward
peking





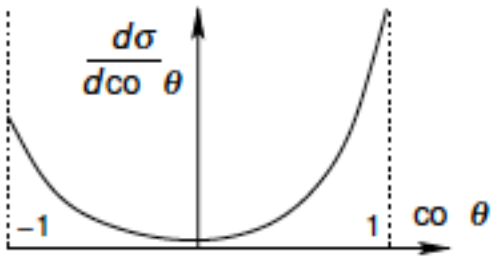


Scattering at high energies

- s-dependence:
  - •many intermediate particles can be produced, unitarity becomes complicated and less useful.

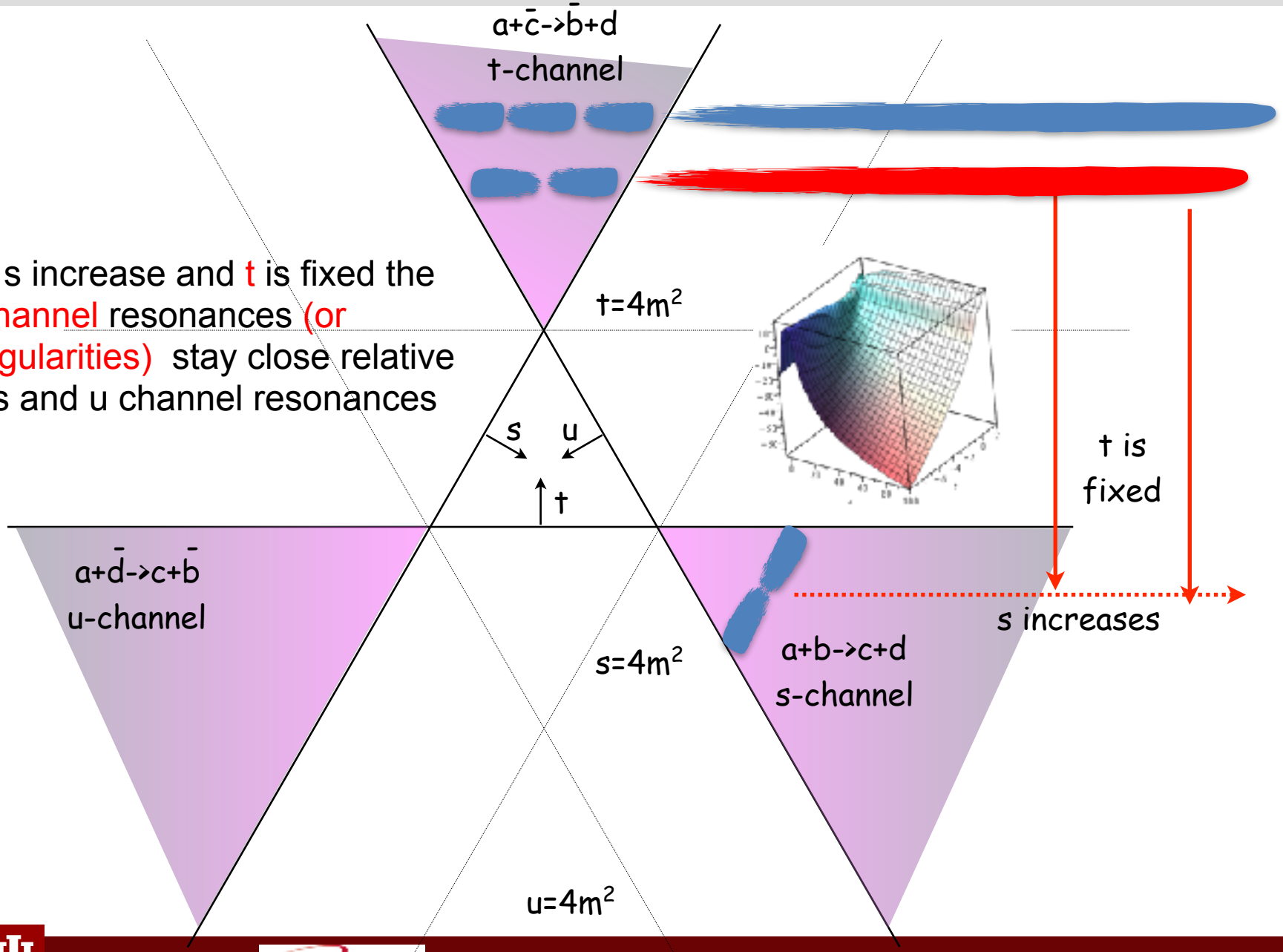
- t-dependence:
  - •high partial waves become important, several Legendre functions are needed.

- There is universality in both s and t-dependencies: smooth (constant or falling s-dependence), and forward/(backward) peaking in t. The universality hints into importance of t/(u) channel singularities.

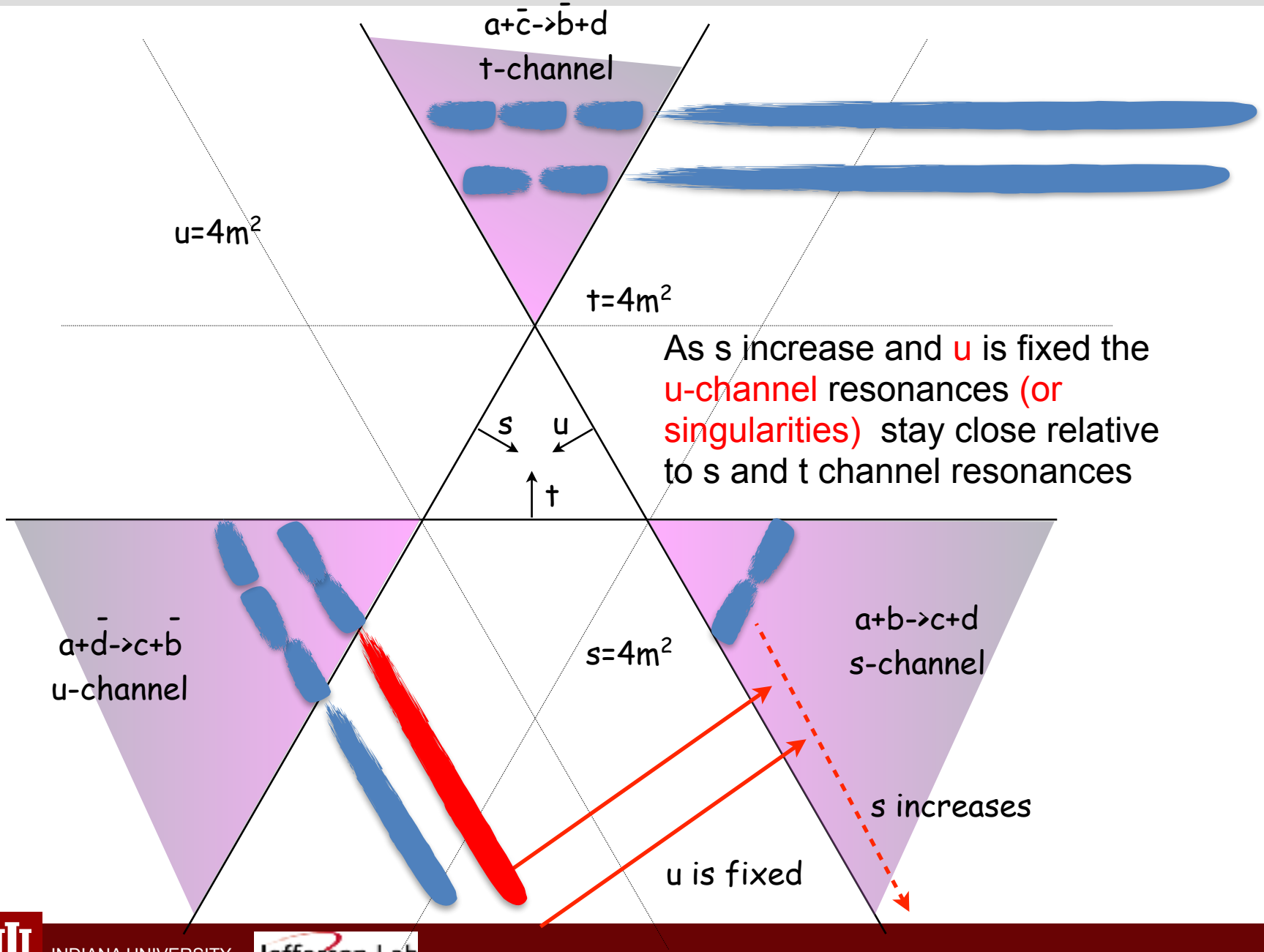


From t-channel to s-channel (high energy forward scattering)

As s increase and t is fixed the t-channel resonances (or singularities) stay close relative to s and u channel resonances



From u-channel to s-channel (high energy backward scattering)



As s increase and u is fixed the u-channel resonances (or singularities) stay close relative to s and t channel resonances

Regge phenomena

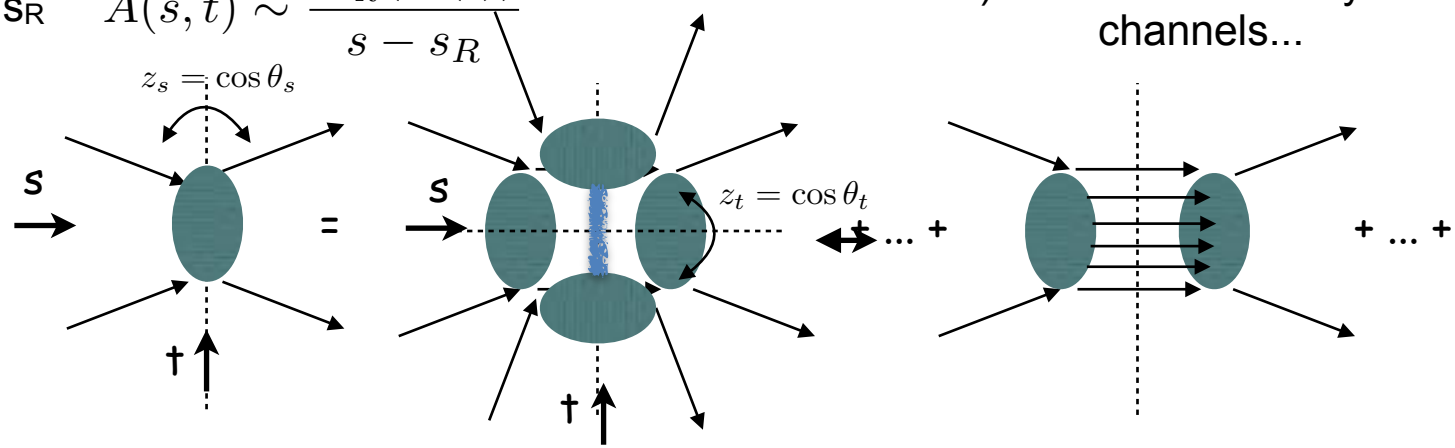
* Low energies: resonance dominance

$$A(s,t) = \sum_l (2l+1) f_l(s) P_l(z_s(t))$$

for s ~ s_R $\quad A(s,t) \sim \dfrac{P_{l_R}(z_s(t))}{s - s_R}$

* How to extend to high energies

in principle A(s,t) determined from s-channel unitarity (s-channel dispersion relations) but there are many intervening channels...



* ...looks like resonance in t-channel: hint explore unitarity in t-channel

$$A(s,t) = \sum_l (2l+1) f_l(t) P_l(z_t(s))$$

resonance poles in $f_l(t)$ at $t=t_R$ can be considered as poles in l: $t_R=t(l)$ --> $l=\alpha(t_R)$

thus for large s

$$A(s,t) \sim s^l = s^{\alpha(t)}$$

to rigorously establish the large-s behavior one needs to make sense out of the divergent sum.
(Gribov-Froissart projection + Somerfeld-Watson transformation)

Example of analytical continuation

$$A(s,t) = \sum_l (2l+1) f_l(t) P_l(z_t) \qquad s = -\frac{t-4m^2}{2}(1-z_t)$$

The series converges for $|z_t|<1$ (cosine of scattering angle in the t-channel), i.e. in the t-channel physical region. We want to know A(s,t) for in the s-channel physical region, in particular for large s, with corresponds to $|z_t| >> 1$.

For example, assume $f_l(t) = \dfrac{1}{l-\alpha(t)}$ i.e. it has a pole (resonance) where α(t)=l

$$A(s,t) \sim J(z_t) = \sum_l \frac{z_t^l}{l-\alpha(t)}$$ for α<0 and $|z_t| < 1$ use $$\frac{1}{l-\alpha} = \int_0^\infty dx e^{-x(l-\alpha)}$$

to obtain $$J(z) = \int_0^\infty dx \left[\frac{e^{x\alpha}}{1+ze^{-x}}\right] = z^\alpha \int_0^z \frac{dy}{y^{\alpha+1}(1+y)} \qquad y = ze^{-x}$$

provides analytical continuation for α>0 for large z =z(s) ~ s

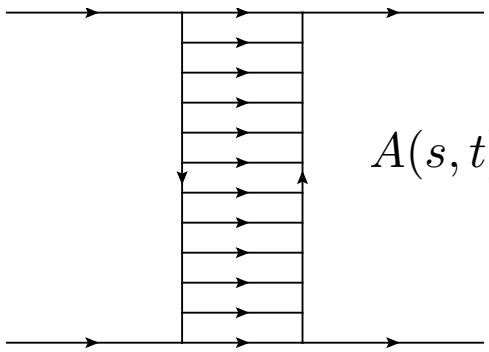
$$J(z) = -\frac{z^\alpha \pi}{\sin \pi\alpha} + z^\alpha \int_z^\infty \frac{dy}{y^{\alpha+1}(1+y)} \to -\frac{z^\alpha \pi}{\sin \pi\alpha} \qquad z \to \infty$$

In general use Sommefeld-Watson transformation to sum a series

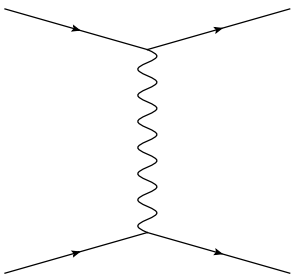
Pomeron vs Reggeons

s-channel: multi-particle production

t-channel: collection of resonances: “Regge” exchanges



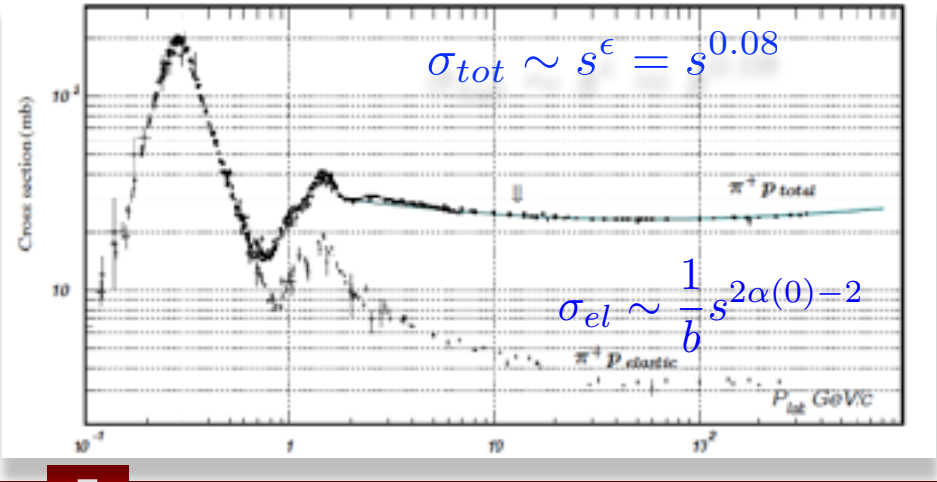
$$A(s,t) \propto r(t) s^{\alpha(t)} \quad \text{<--} \quad f_l(t) = \frac{r(t)}{l - \alpha(t)}$$

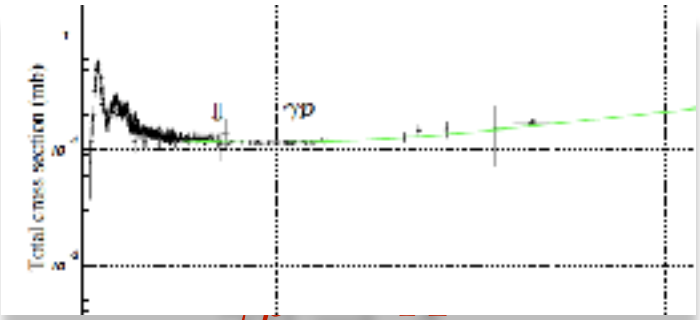


$$A(s, t \sim 0) \sim i s^{\alpha(0)} \sim s\sigma_{tot}$$

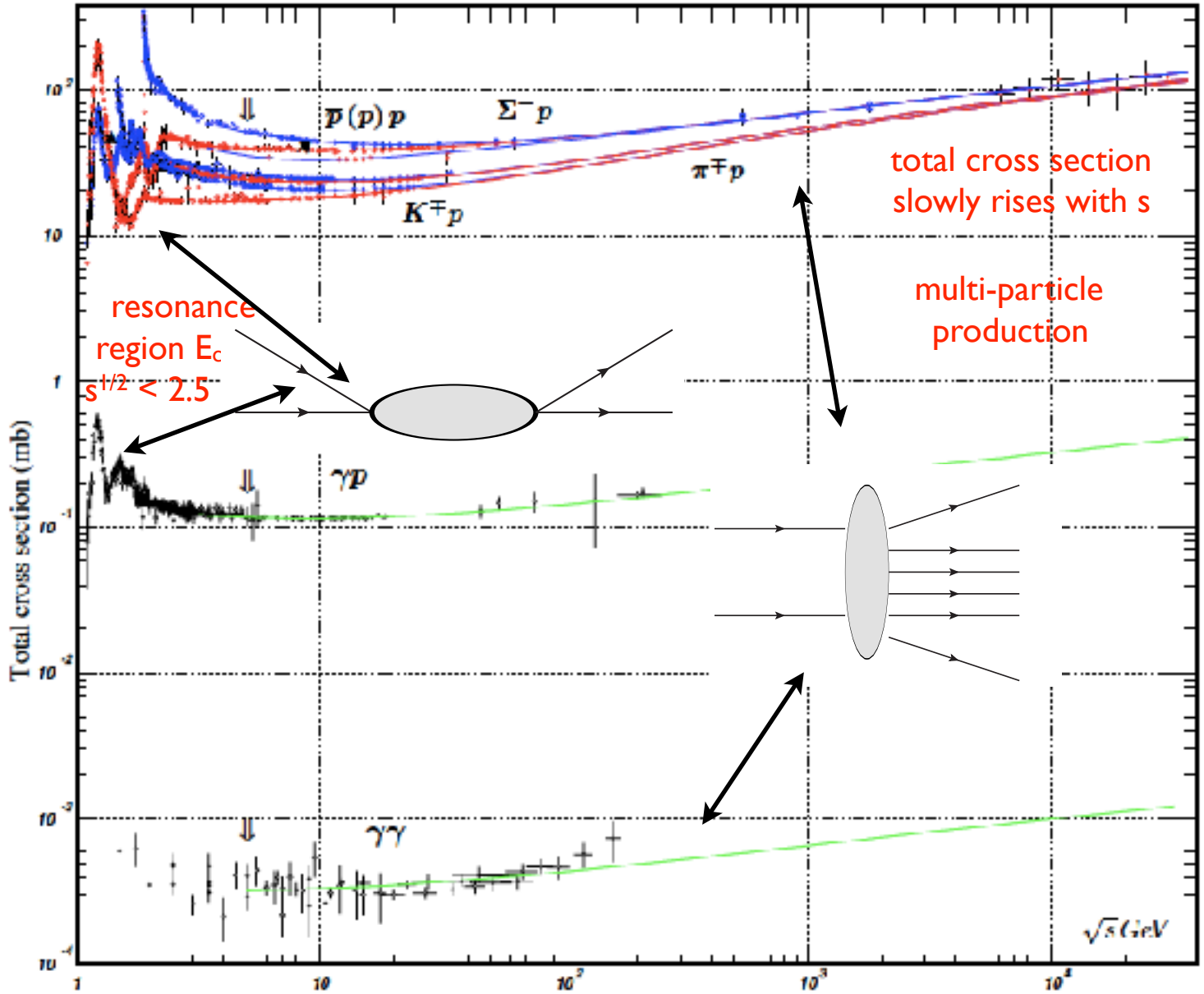
* Rightmost singularity in l-plane dominates large-s limit of the amplitude and forward cross section (it has vacuum quantum numbers: Pomeronα(0) = 1 + ε)

(exchange of non-vacuum q.n. falls with energy)





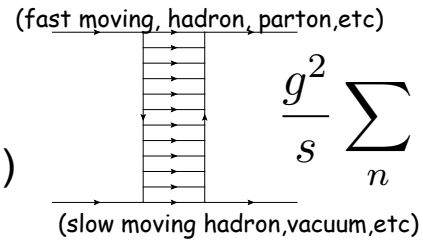
Comparing with Experiment



Growing Radius, partons, saturation,…

* Where does to parton model come from

adding correlated partons is beneficial (expansion not in g^2 but in $g^2 \log s$)



$$\frac{g^2}{s}\sum_n \frac{\beta^{n-1}(t)}{(n-1)!}\log^{n-1} s \to s^{\alpha(-k_\perp^2)}$$

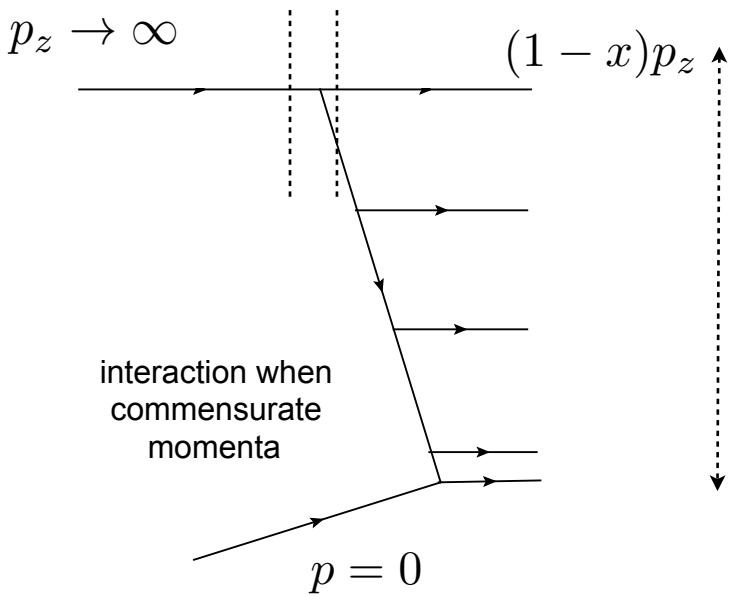
$$\alpha(t) = -1 + \beta(t)$$

... and in space-time assuming Pomeron α(0)=1

$$A(s,r_\perp) \sim \int d^2k_\perp e^{ik_\perp r_\perp} e^{\alpha(-k_\perp^2)\log s} \sim \frac{1}{\log(s)} e^{-r_\perp^2/\log(s)}$$

hadron swells

$$\Delta E \sim \frac{\mu_\perp^2}{x(1-x)p_z}$$



* long lived fluctuations finite <x>

$$\langle x\rangle^{\langle n\rangle} = \frac{p_z}{\mu} \quad \langle n\rangle \sim \log(s)$$

random walk in transverse space

* $$\langle r_\perp\rangle \sim \sqrt{\langle n\rangle \frac{1}{\mu_\perp}} \sim \log^{1/2}(s)$$

* large-s behavior universal (Pomeron = vacuum pole, universal mid-rapidity)

* it takes "a long time" to develop a low-x parton out of a fast one

Summary of Lecture 2

S-matrix principles : Crossing symmetry, Analyticity, Unitarity provide important constraints/insights into reaction dynamics.

For example: low energy scaring is dominated by a few direct channel partial waves, resonance appear as poles on the IInd sheet with widths constrained by unitarity, large-s scattering is given by t/u channel exchanges, etc.

In QCD resonances are not predicted by exchange forces (Bootstrap idea), they have to be “inserted by hand”.