# Modern Hadron Spectroscopy : Challenges and Opportunities

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Lecture 1: Hadrons as laboratory for QCD:

- Introduction to QCD
- Bare vs effective effective quarks and gluons
- Elements of hadron spectroscopy and structure

Lecture 2: Phenomenology of hadron reactions

- Kinematics and observables
- Space time picture of Parton interactions and Regge phenomena
- Properties of reaction amplitudes

Lecture 3: Complex analysis

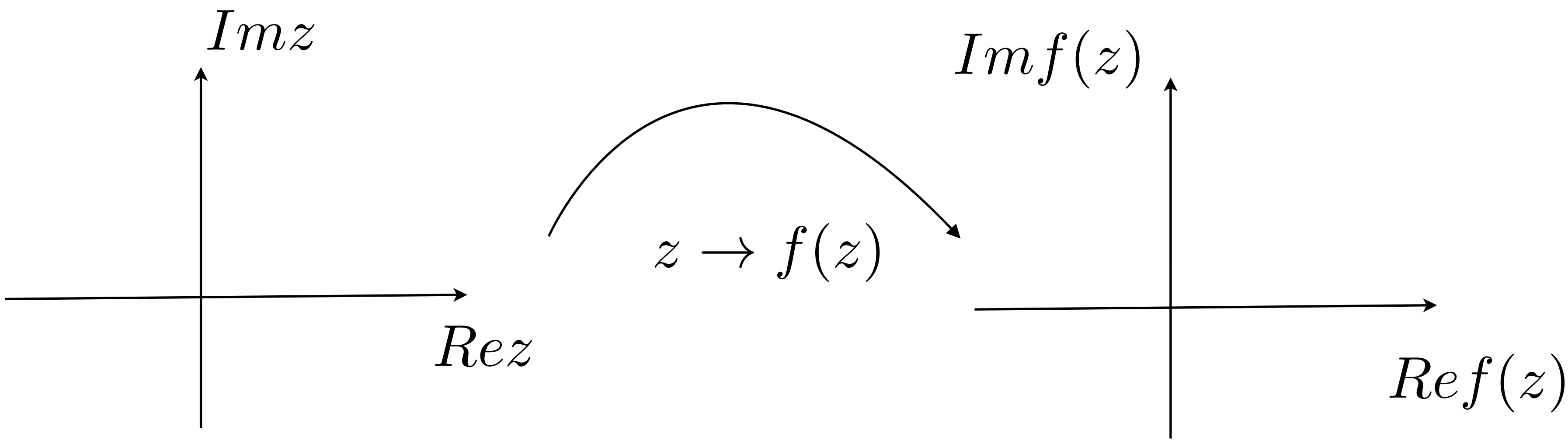
Lecture 4: How to extract resonance information from the data

- Partial waves and resonance properties
- Amplitude analysis methods (spin complications)

# Complex functions

$$z = a + bi \to f(z) = Ref(z) + iImf(z)$$

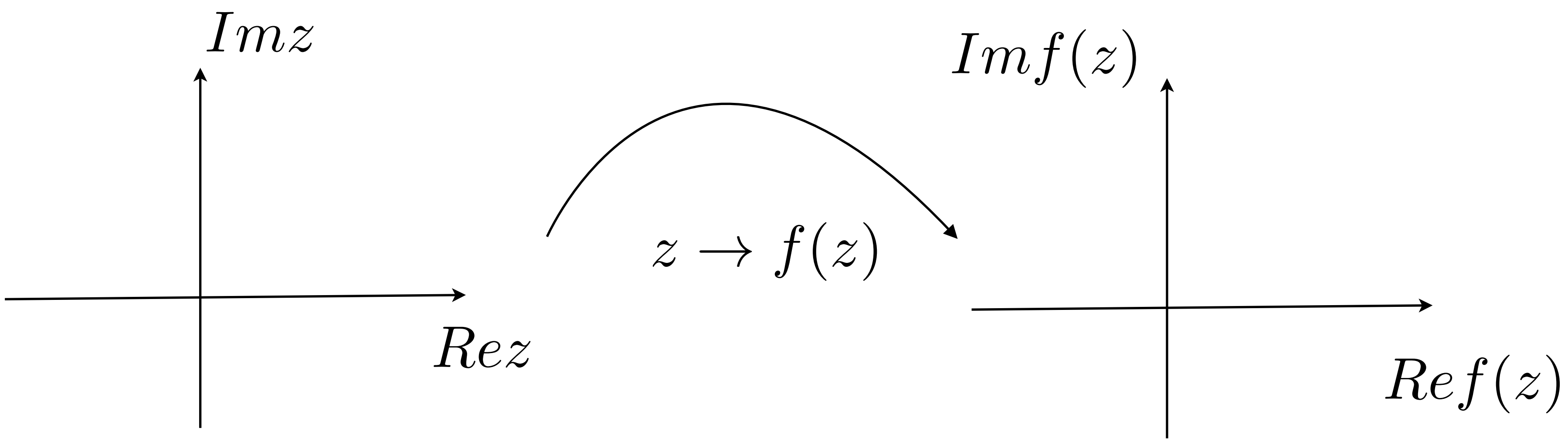
Elementary functions: you can also think of them as maps of one complex plane (z) to another (f(z)): z→ f(z)




To define a function we can use the algebraic relations e.g

$f(z) = \sqrt{z}$ is such that $z = f(z) \times f(z)$

# Complex functions




Often the mapping is not "one-to-one" and one needs to be careful in defining domains which give a unique value for the function, e.g. is

$$\sqrt{-25} = +5I \text{ or } -5I \ ?$$

# Complex functions



Complex functions (and complex calculus) : Continuity imposes very strong conditions of functions (much stronger than in the case of real variables)

"Smooth" (holomorphic, analytic) functions are "boring" all "action" is in the singularities.

Singularities determine functions "far away" from location of the singularity (e.g. charge determines potentials)

Physical observables are functions of real parameters, however physics law can be generalized to complex domains and become "smooth" but any "constraint" results in singularities.

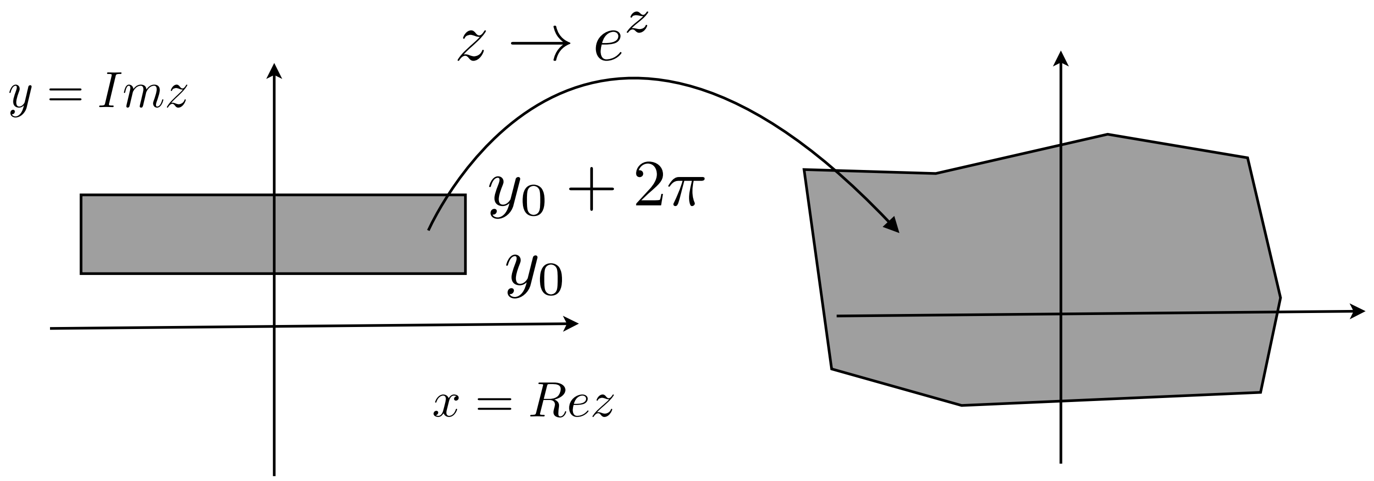
# Example exp(z) is periodic!

(exp of complex argument has the same algebraic properties as exp of real arg., e.g. exp($z_1 z_2$) = exp($z_1$) exp($z_2$) )

$$e^{i\phi} = [1 - \frac{\phi^2}{2} + \cdots] + i[\phi - \frac{\phi^3}{3!} + \cdots] \quad = \cos\phi + i\sin\phi$$

$$z \to e^z = e^{Rez + iImz} = e^{Rez}(\cos Imz + i \sin Imz)$$

$$e^{z+2\pi i} = e^z$$




one needs to be careful when defining its inverse i.e. logarithm*:*
*the z-plane can be mapped back in many different ways*

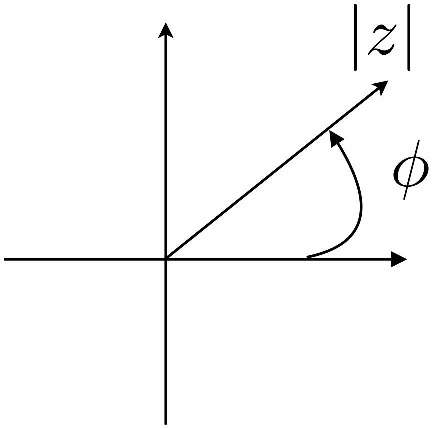
# Example $\sqrt{z}$

similar issue with the $\sqrt{z}$

$$z = |z|e^{i\phi}$$

$$\sqrt{z} \equiv \sqrt{|z|}e^{i\frac{\phi}{2}}$$

$$\sqrt{z}\sqrt{z} = \sqrt{|z|}e^{i\frac{\phi}{2}}\sqrt{|z|}e^{i\frac{\phi}{2}} = |z|e^{i\phi}$$




using $\phi = [-\pi, \pi)$

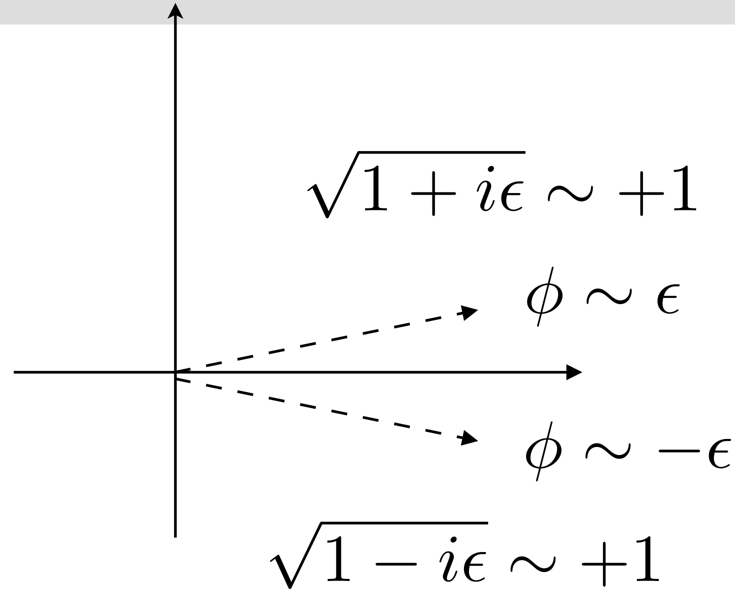
or $\phi = [0, 2\pi)$

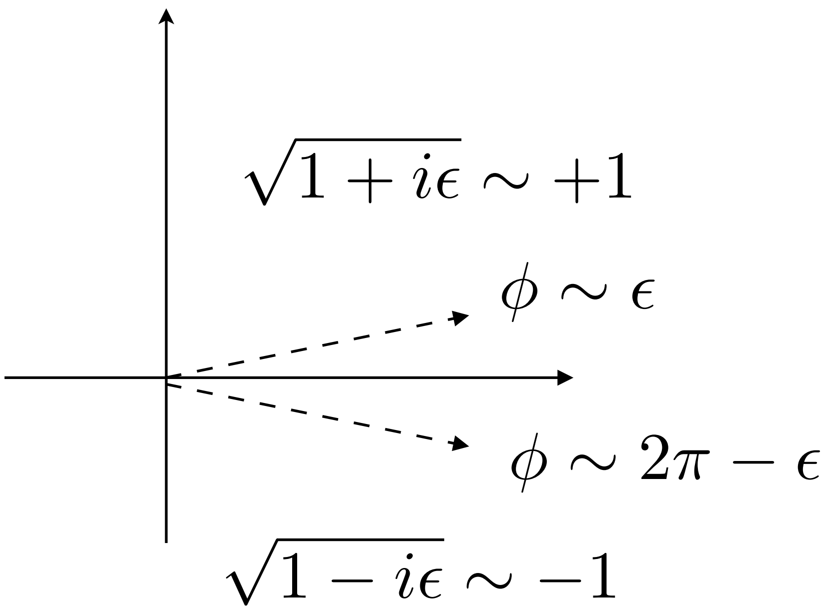
gives different results for $\sqrt{z}$

$$\sqrt{z} \equiv \sqrt{|z|}e^{i\frac{\phi}{2}}$$

using $\phi = [-\pi, \pi)$

gives square root that is continuous near the positive real axis



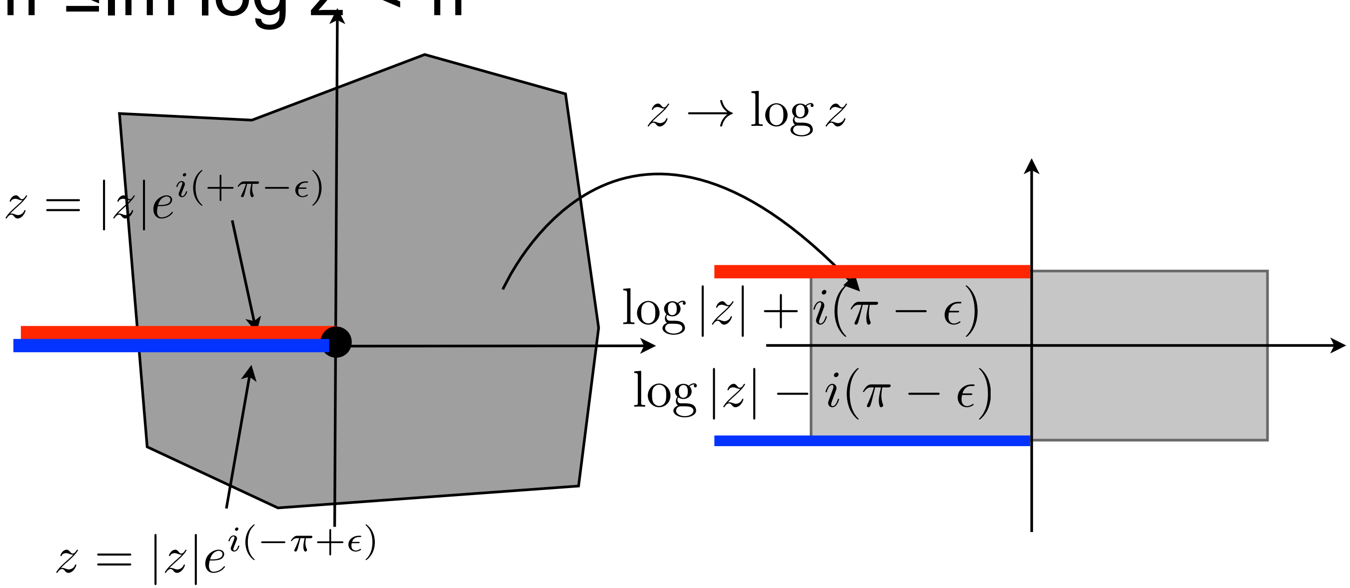

using $\phi = [0, 2\pi)$

gives square root that is discontinuous near the positive real axis

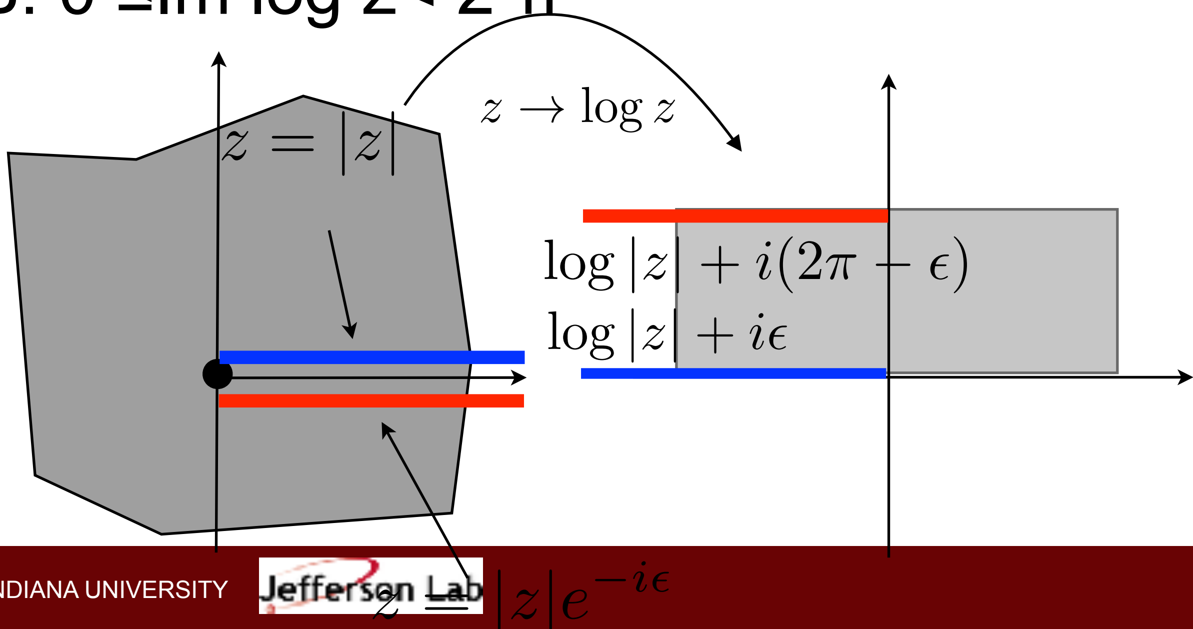
In both case it has the same value when approaching the positive real axis rom above

# log(z)

## Case A: -π ≤Im log z < π




## Case B: 0 ≤Im log z< 2 π

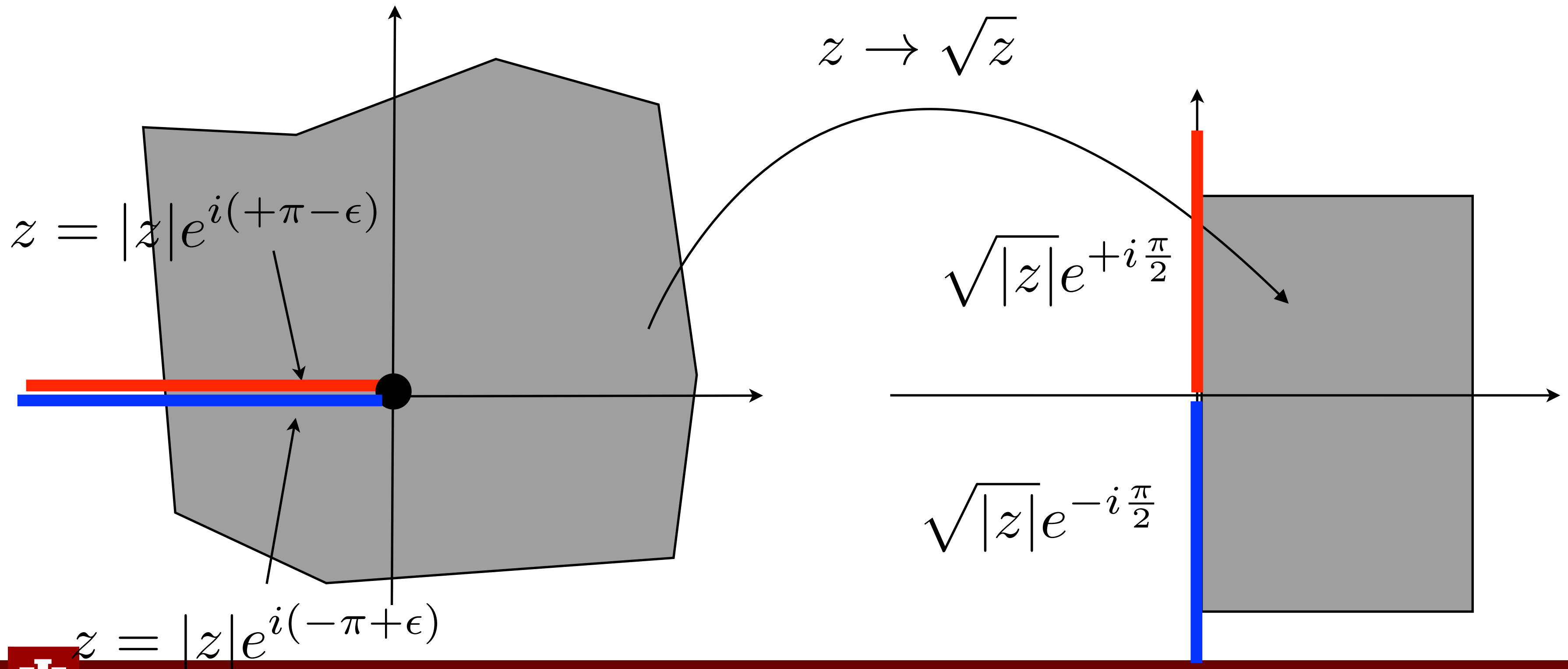


# Powers

Powers: $a^b = e^{b \log(a)}$ (for chosen branch of log)

$$\sqrt{z} = e^{\frac{1}{2}\log(z)} = \sqrt{|z|}e^{\left[i\frac{arg\,z}{2} + (mod\, i\pi)\right]}$$

for example: using the principal branch (- π ≤ arg z < π)

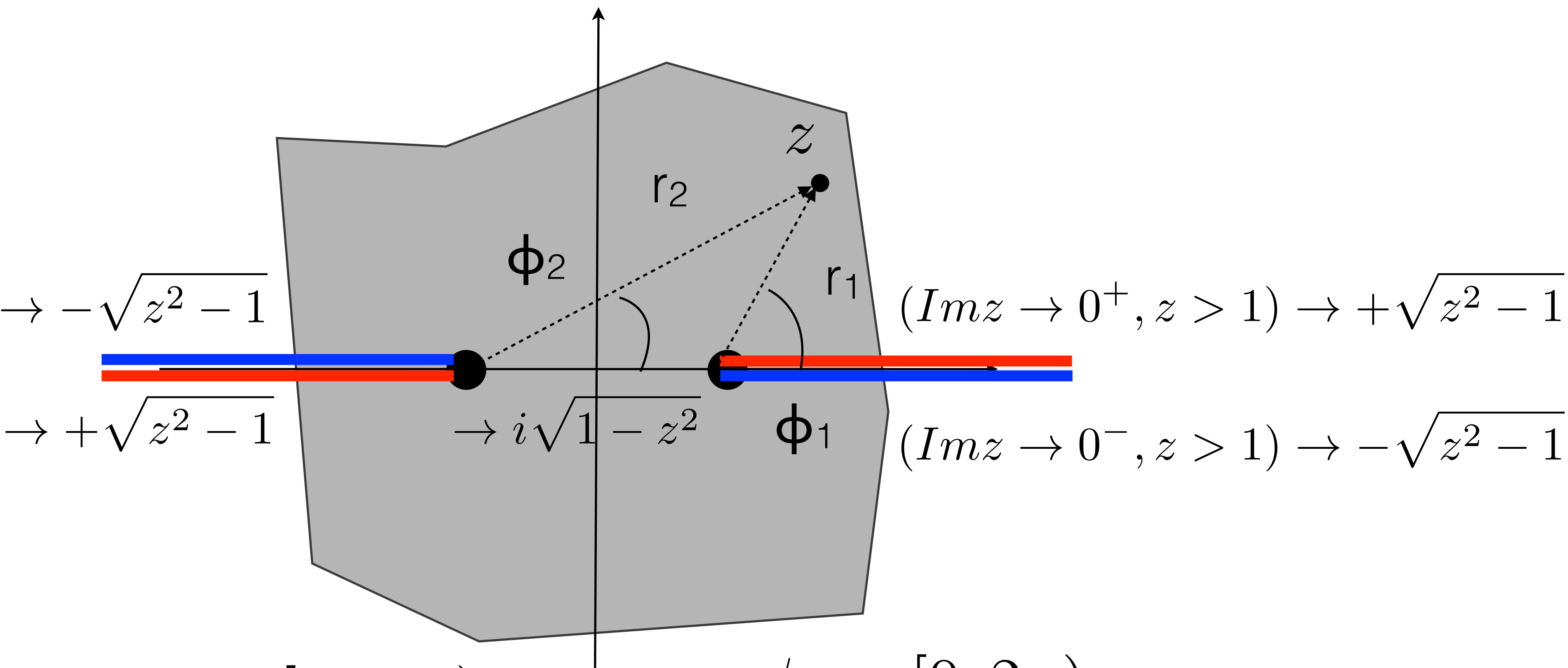


# More complicated functions



A. $z \to \sqrt{z^2 - 1}$

$$\sqrt{z^2 - 1} = \sqrt{r_1 r_2} e^{i \frac{\phi_1 + \phi_2}{2}}$$




$\phi_2 \in [-\pi, \pi)$ $\phi_1 \in [0, 2\pi)$

# or

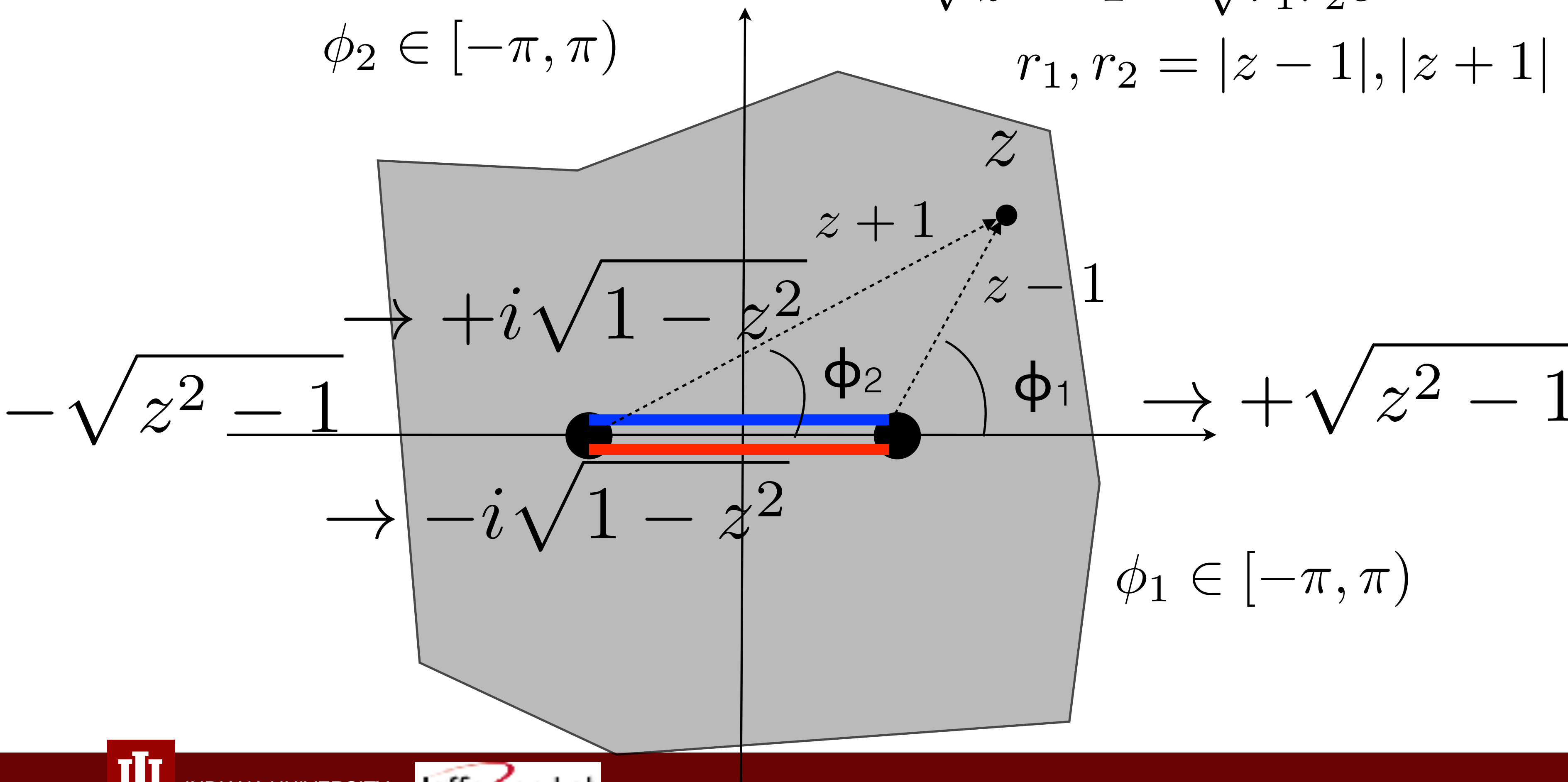
B. $z \to \sqrt{z^2 - 1}$

$= \sqrt{z-1}\sqrt{z+1}$

and use principal branches

$$\sqrt{z^2-1} = \sqrt{r_1 r_2}\, e^{i\frac{\phi_1+\phi_2}{2}}$$

$$r_1, r_2 = |z-1|, |z+1|$$



# Exercise

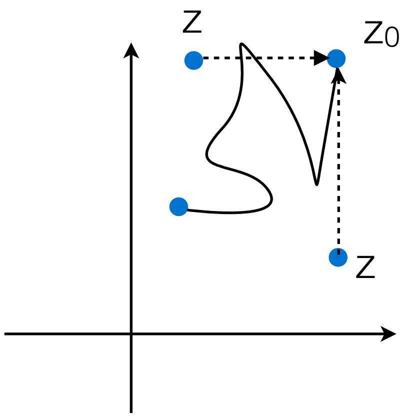


$$\int_{-1}^{1} dx \frac{1}{\sqrt{1-x^2}} = \pi$$

# Calculus: differentiation

f(z) is differentiable (holomorphic) if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \equiv f'(z_0)$ exists

write z = x + iy and f(z) as f(z) = u(x,y) + i v(x,y). Since the procedure of taking the limit in definition of $f'(z_0)$ is independent of the path taken in $z \to z_0$, you can take two independent paths e.g. path 1: $x = x_0 + \varepsilon$, $y = y_0$ and path 2: $x = x_0$, $y = y + \varepsilon$: Cauchy relations:

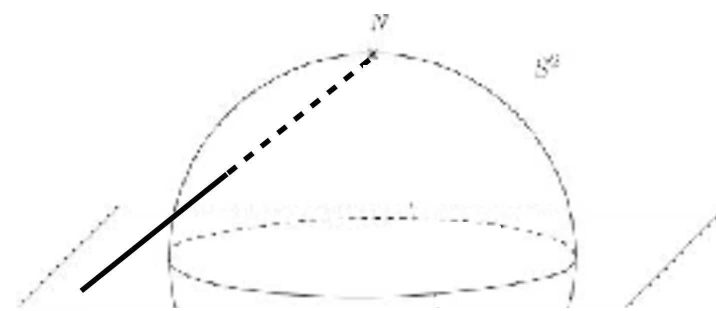



$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This implies Δu = Δv = 0 where Δ is 2-dim Laplacian u,v : harmonic functions

Infinity: on the real axis there are two (axis is oriented) but on the complex plane (calculus) there is no preferred direction: There is only ONE infinity (somewhat counter intuitive) w = 1/z

$$\left.\frac{df}{dz}\right|_{z=\infty} = -\frac{1}{w^2} \left.\frac{df}{dw}\right|_{w=0}$$



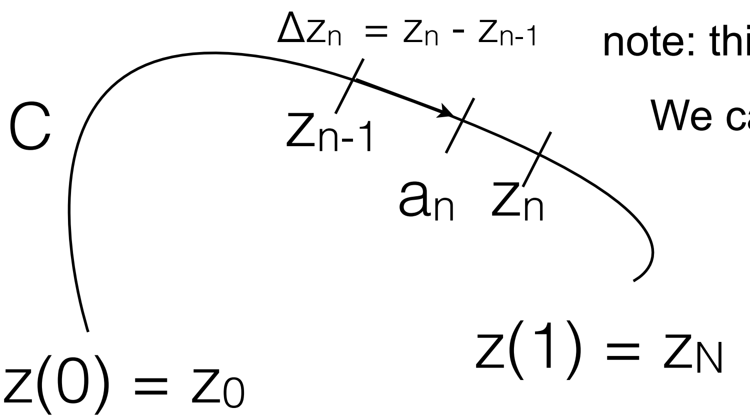
North pole corresponds to ALL points at infinity in the z-plane

# Calculus: integration



Line integrals: given a curve C in the complex plane parametrized by a real number $0 \leq t \leq 1$, $t \to z(t) = x(t) + iy(t)$ the integral of f over C is defined by

$$\int_C f(z)dz = \int_{t=0}^{1} f(z(t))\frac{dz}{dt}dt = \lim_{|\Delta z_n|\to 0, N\to\infty} \sum_{n=1}^{N} f(a_n)\Delta z_n$$




note: this is an ordered path

We can estimate the integral: if $|f(z)| \leq M > 0$ along C then

$$|\int_C f(z)dz| \leq Ms$$

where s it the length of the path

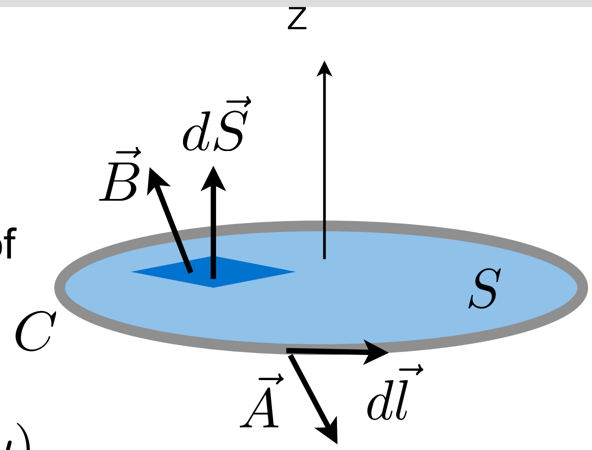
Cauchy-Goursat theorem: If f(z) is holomorphic in some region G and C is a closed contour (consisting of continuous or discontinuous cycles, double cycles, etc.) then

$$\oint f(z)dz = 0$$

(converse is also true)

# Proof: according to Stoke's theorem

$$\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{l}$$




(e.g. Magnetic flux $\vec{B} \equiv \vec{\nabla} \times \vec{A}$ over open surfaces = circulation of vector potential over its boundary)

$$\int_S \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dxdy = \oint (A_x dx + A_y dy)$$

(Cauchy relation for u,v)

use: $A_y = u(x,y)$, $A_x = v(x,y)$ then $\frac{\partial A_y}{\partial x} = \frac{\partial A_x}{\partial y}$ and l.h.s=0

$$\oint (vdx + udy) = 0$$

use: $A_y = v(x,y)$, $A_x = -u(x,y)$ then $\frac{\partial A_y}{\partial x} = \frac{\partial A_x}{\partial y}$ and l.h.s=0

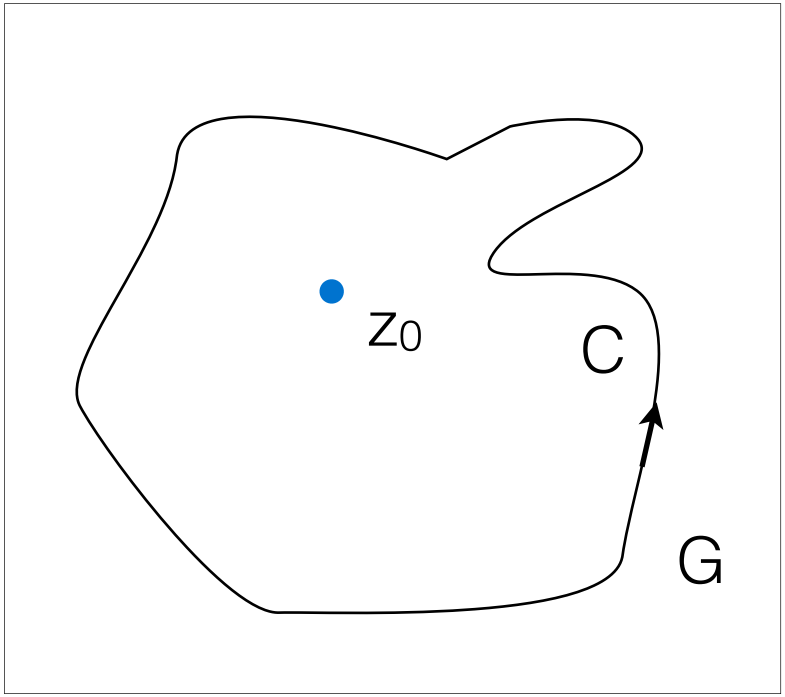
$$\oint (-udx + vdy) = 0$$

$$\oint f(z)dz = \oint [u + iv][dx + idy] = \oint [udx - vdy] + i \oint [vdx + udy] = 0$$

# Cauchy formula



The Cauchy integral formula: if f(z) holomorphic in G, $z_0 \in G$, and C a closed curve (cycle), which goes around $z_0$ once in positive (counterclockwise) direction, then

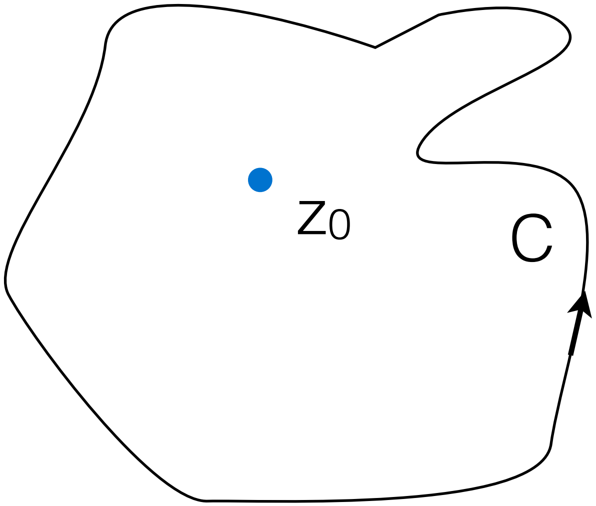



$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - z_0}$$

The Cauchy formula solves a boundary-value problem. The values of the function on C determine its value in the interior. There is no analogy in the theory of real functions. It is related though to the uniqueness of the Dirichlet boundary-value problem for harmonic functions (in 2dim)

# Proof

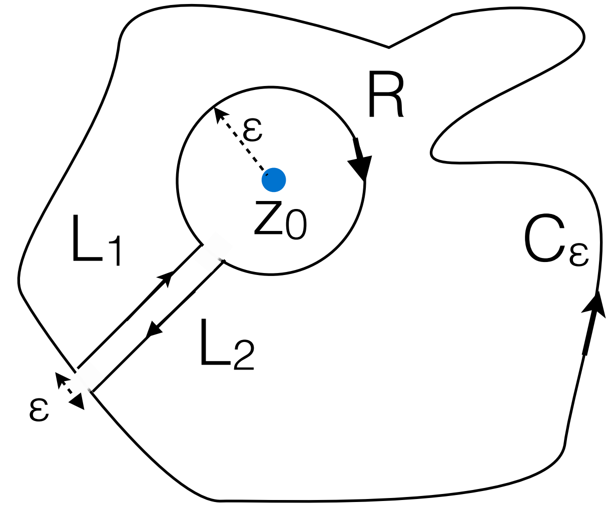





$\lim_{\varepsilon\to 0} C_\varepsilon = C$

$\lim_{\varepsilon\to 0} L_1 = -L_2$

$$\oint_{C'} \frac{f(z)dz}{z - z_0} = 0$$




$C' = C_\varepsilon + L_1 + L_2 + R$

$$0 = \oint_{C'} = \lim_{\epsilon\to 0}\left[\int_{L_1} + \int_{L_2} + \int_R + \int_{C_\epsilon}\right] = \lim_{\epsilon\to 0}\int_R + \int_{C_\epsilon}$$

$$\int_R \frac{f(z)dz}{z - z_0} = f(z_0)\int_R \frac{dz}{z - z_0} + \int_R \frac{f(z) - f(z_0)}{z - z_0}dz$$

$\varepsilon\to 0$: $\quad -2\pi i \qquad O(\epsilon) \to 0$

$$z - z_0 = \epsilon e^{i\phi}$$

$$-2\pi i f(z_0) + \int_C = 0$$

# Exercise



Integrals:

$$\int_\gamma dz \qquad \int_\gamma z^n dz$$

$$\int_\gamma \frac{dz}{z} \qquad \int_{\gamma'} \frac{dz}{z}$$

$$\int_\gamma \frac{dz}{z^2}$$

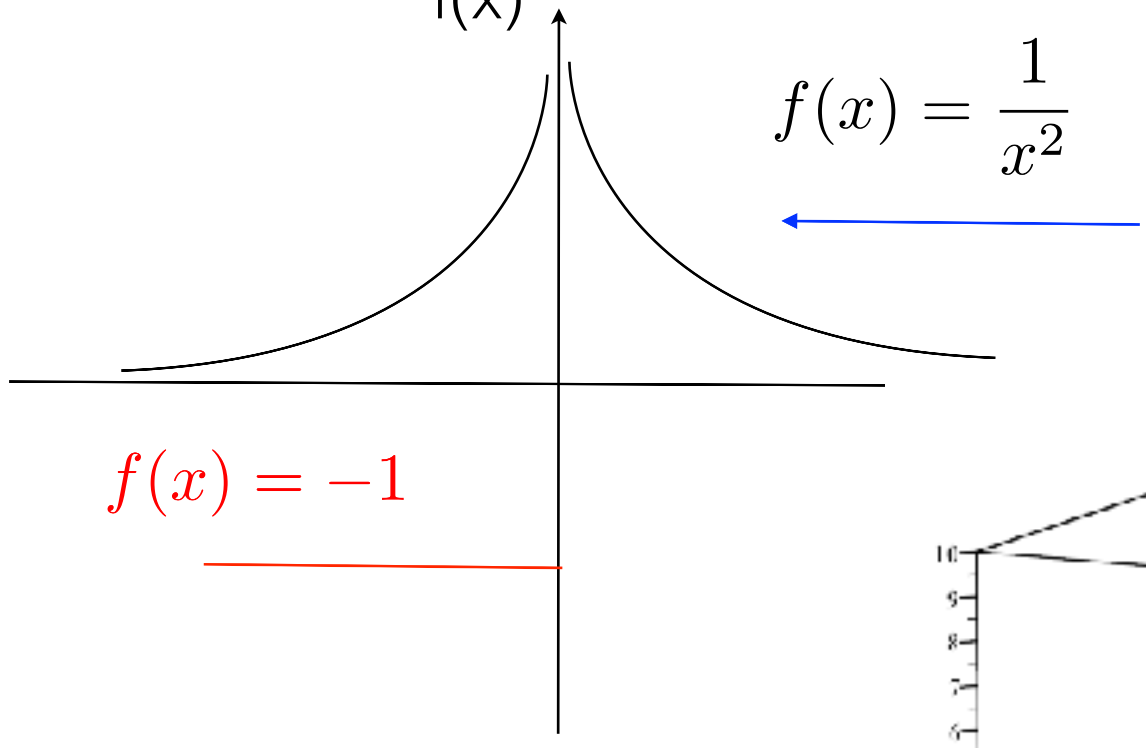
γ = unit circle

γ’ = unit square

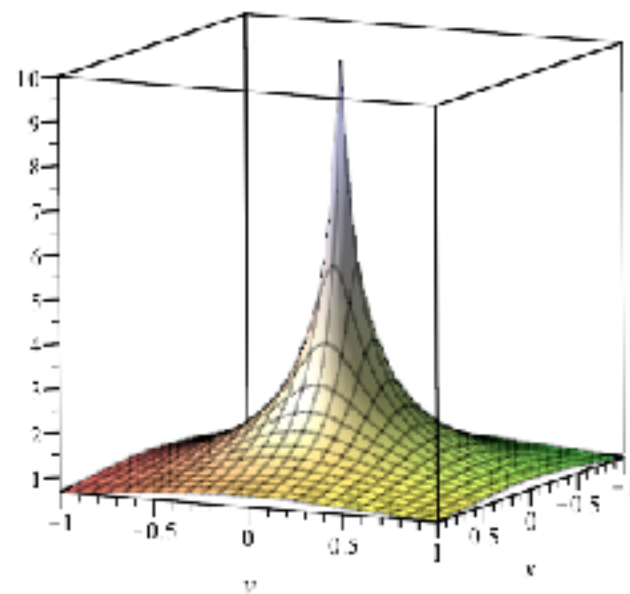
# Analytical continuation

For real functions it does not work

$$f(x) = \frac{1}{x^2}$$




$$f(x) = -1$$



but for complex functions you can go continuously around the z=0 singularity and *analytically continue* from one region to another

# Unique !

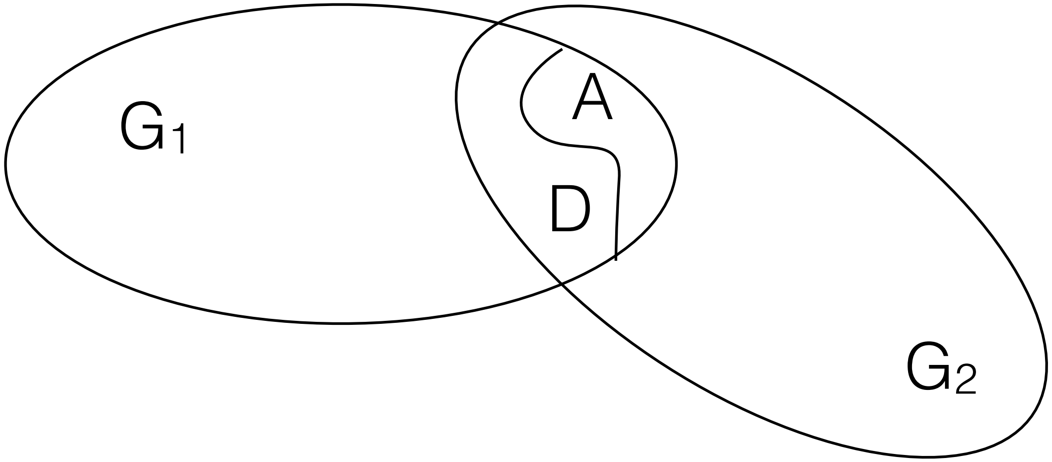


## Analytical continuation

Let $f_1(z)$ be holomorphic in $G_1$ and $f_2(z)$ in $G_2$, $G_1$ and $G_2$ intersect on an arch A (or domain D), and $f_1 = f_2$ on A (or D) then $f_1$ and $f_2$ are analytical continuation of each other and

$$f(z) = \begin{cases} f_1(z), z \in G_1 \\ f_2(z), z \in G_2 \end{cases}$$

is holomorphic in the union of $G_1$ and $G_2$



# Examples



$1 + z + z^2 + \cdots$ is holomorphic in |z|<1

$\int_0^\infty e^{-(1-z)t} dt$ is holomorphic in Re z < 1

$-(1 + 1/z + 1/z^2 + \cdots)$ is holomorphic in |z|>1

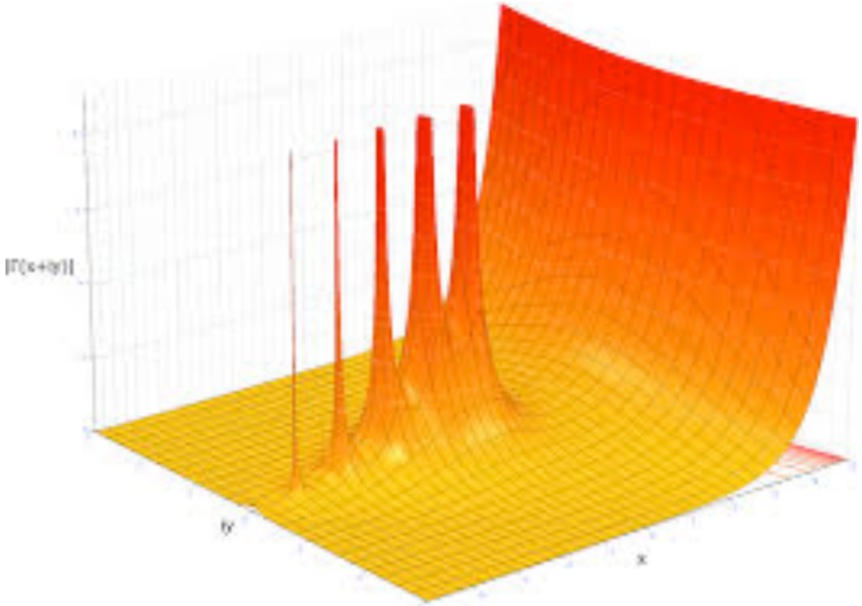
all these functions represent f(z) = 1/(1-z) in different domains, which is holomorphic everywhere except at the point z=1

# Gamma function



Γ(z+1) = zΓ(z) : generalization of factorial
n! = n (n-1)! so Γ(n) = (n-1)!

$$\Gamma(z) = \int_0^\infty \frac{dt}{t} t^z e^{-t} \qquad \Gamma(0) \sim \log 0 \qquad \Gamma(-1) \sim \frac{1}{0} \qquad \Gamma(-n) \sim \frac{1}{0^n}$$



$$\Gamma(z) \sim \frac{(-1)^n}{n!} \frac{1}{z+n}$$

for z~-n

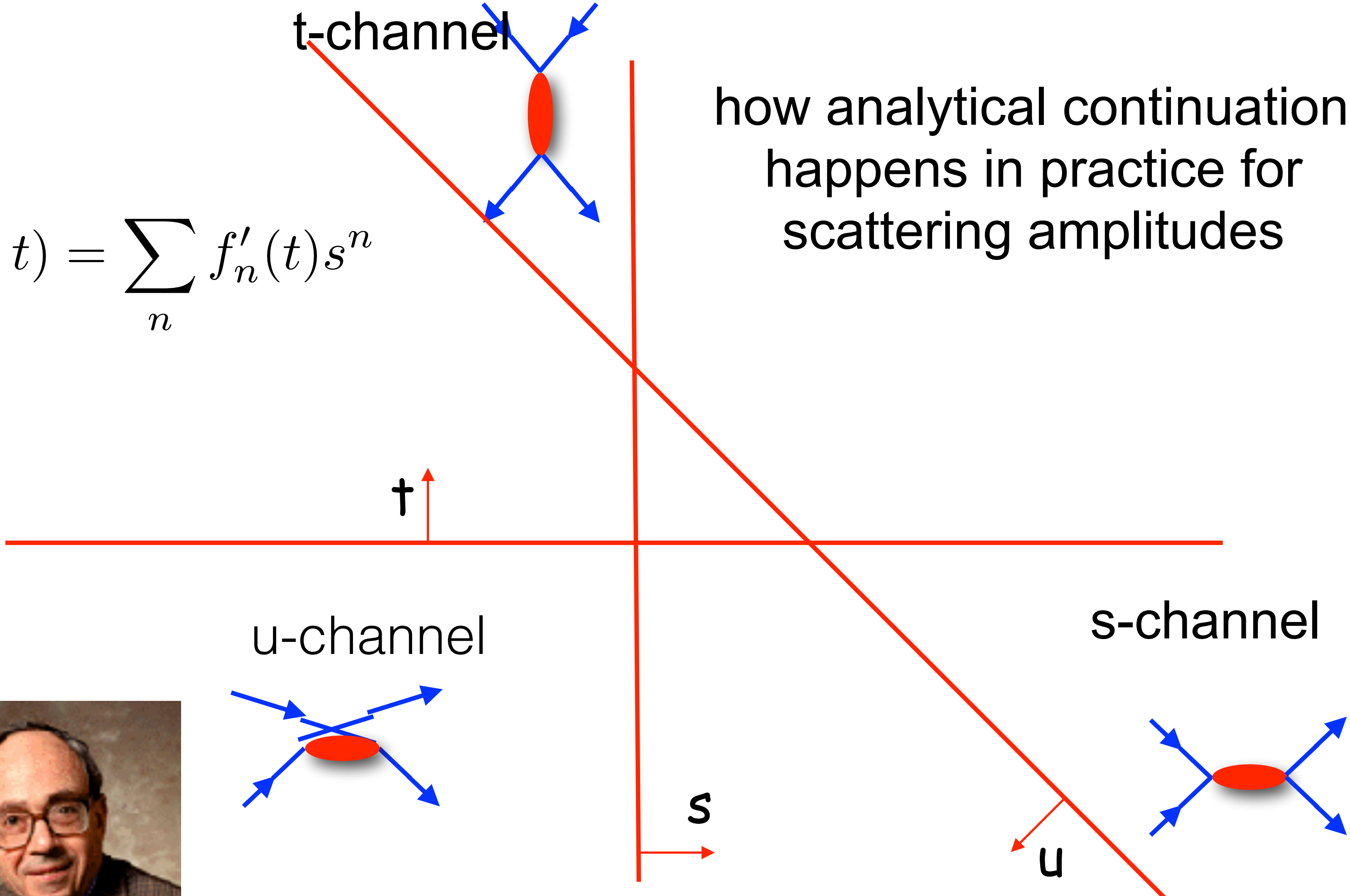
$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}$$

# A(s,t) : how to continue from between s and t



how analytical continuation happens in practice for scattering amplitudes

$$f(s,t) = \sum_n f'_n(t) s^n$$




$$f(s,t) = \sum_n f_n(s) t^n$$

# Continuation of integrals



what are the possibilities for g(w) to be singular?

$$g(w) = \int_C f(z,w)dz$$

Let D be a neighborhood of the arc C and G be a domain in the w-plane, f(z,w) be regular in both variables, except for a finite number of isolated singularities or branch points.

g(w) can be singular at $w_0 \in G$ only if

1. $f(z,w_0)$ in z-plane has a singularity coinciding with the end points of the arc C (end-point singularity)
2. two singularities of f, $z_1(w)$ and $z_2(w)$, approach the arc C from opposite sides and pinch the arc precisely at $w=w_0$. (pinch singularity)

3. a singularity z(w) tents to infinity as $w \to w_0$ deforming the contour with itself to infinity; one has to change variables to bring the point ∞ to the finite plane to see what happens.

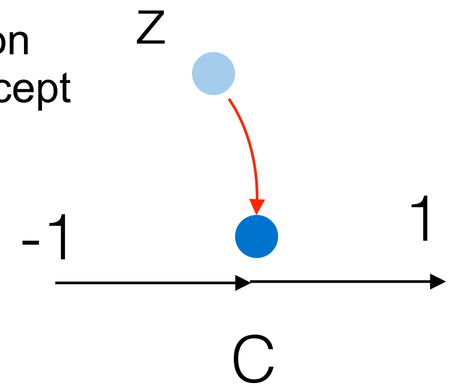
# Example

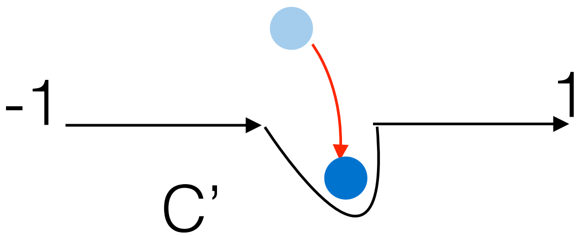


## Apparent singularities need not be there !

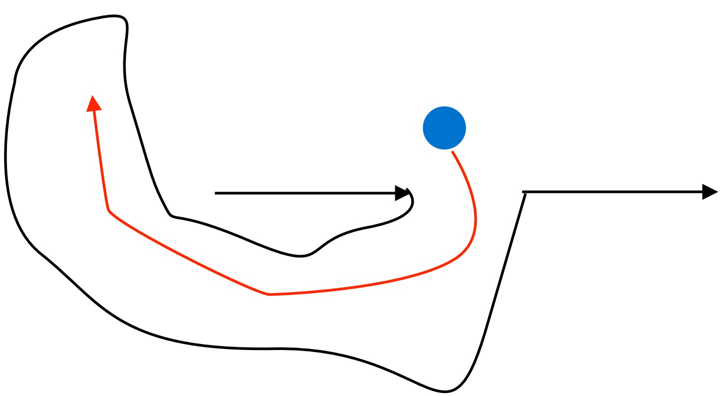
$$f(z) = \int_{-1}^{1} \frac{dx}{x - z} \qquad C = [-1,1]$$

looks like a regular function of z in the entire plane except for the interval $z \in [-1,1]$




when z approaches x deforming C allows to define a function f(z) which changes continuously



... however when z returns to the original point we end up with a different function value. f(z) is multivalued and -1 is a branch point.