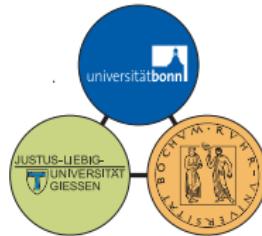


# Ambiguities in the reconstruction of partial waves from data

Yannick Wunderlich,  
work done in collaboration with Alfred Švarc

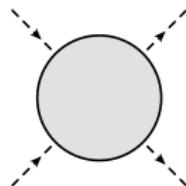
HISKP, University of Bonn

SFB School Boppard 2017



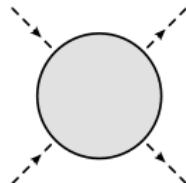
# Introduction

Consider spin-less  $2 \rightarrow 2$ -amplitude:  $A(W, \theta) =$



# Introduction

Consider spin-less  $2 \rightarrow 2$ -amplitude:  $A(W, \theta) =$

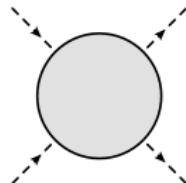


\*) One single *observable* in scattering experiment: diff. cross section

$$\sigma_0(W, \theta) = |A(W, \theta)|^2.$$

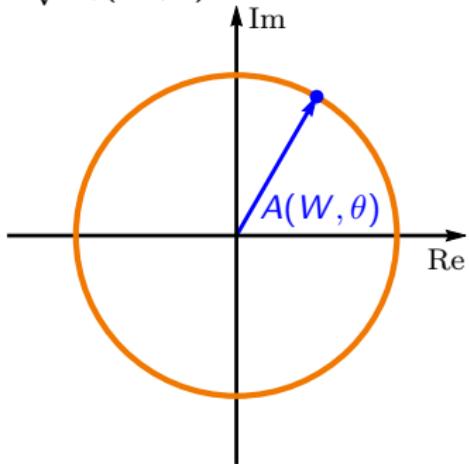
# Introduction

Consider spin-less  $2 \rightarrow 2$ -amplitude:  $A(W, \theta) =$



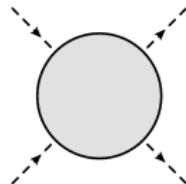
- \*) One single *observable* in scattering experiment: diff. cross section  
$$\sigma_0(W, \theta) = |A(W, \theta)|^2.$$

⇒ Complete experiment analysis:  $|A(W, \theta)| = \sqrt{\sigma_0(W, \theta)}$ , i.e.  
the amplitude lies on a circle.



# Introduction

Consider spin-less  $2 \rightarrow 2$ -amplitude:  $A(W, \theta) =$



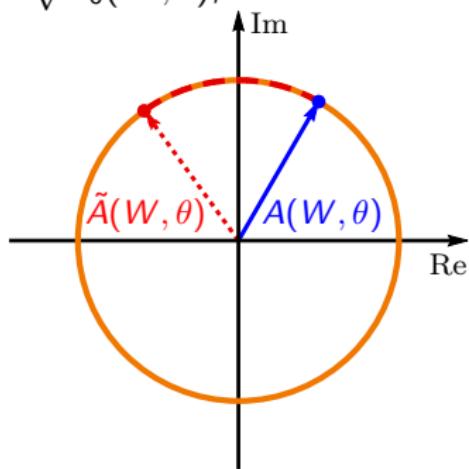
\*) One single *observable* in scattering experiment: diff. cross section

$$\sigma_0(W, \theta) = |A(W, \theta)|^2.$$

⇒ Complete experiment analysis:  $|A(W, \theta)| = \sqrt{\sigma_0(W, \theta)}$ , i.e.  
the amplitude lies on a circle.

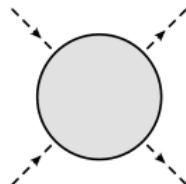
↪ Result unchanged by multiplication  
with  $W$ - and  $\theta$ -dependent phase:

$$A(W, \theta) \rightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta)$$



# Introduction

Consider spin-less  $2 \rightarrow 2$ -amplitude:  $A(W, \theta) =$



\*) One single *observable* in scattering experiment: diff. cross section

$$\sigma_0(W, \theta) = |A(W, \theta)|^2.$$

⇒ Complete experiment analysis:  $|A(W, \theta)| = \sqrt{\sigma_0(W, \theta)}$ , i.e.  
the amplitude lies on a circle.

↪ Result unchanged by multiplication  
with  $W$ - and  $\theta$ -dependent phase:

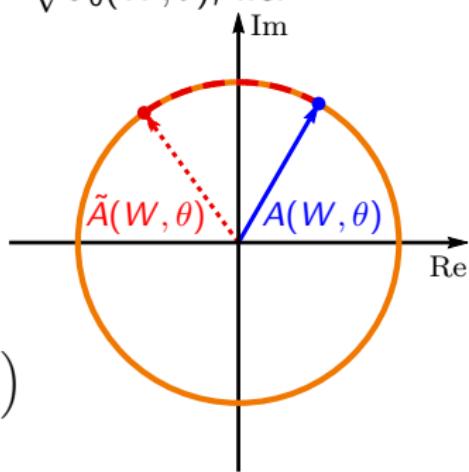
$$A(W, \theta) \rightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta)$$

⇒ Implications for partial wave decomp.

$$A(W, \theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) A_\ell(W) P_\ell(\cos \theta),$$

$$\left( \Leftrightarrow A_\ell(W) = \frac{1}{2} \int_{-1}^1 d \cos \theta A(W, \theta) P_\ell(\cos \theta) \right)$$

and in particular for truncated PWA?



# Continuum- vs. discrete ambiguities

## Continuum ambiguities

\*) Definition:

$$\tilde{A}(W, \theta) = e^{i\Phi(W, \theta)} A(W, \theta)$$

[Bowcock & Burkhardt],  
[L. P. Kok], [D. Atkinson], ...

\*) Invariance:

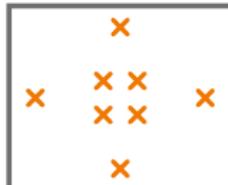
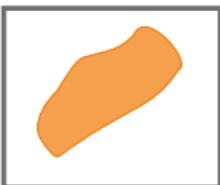
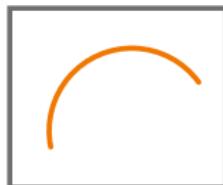
$$\begin{aligned}\sigma_0 &= |A|^2 = A^* A \\ \rightarrow \tilde{A}^* \tilde{A} &= e^{-i\Phi} A^* e^{i\Phi} A \\ &= e^{i(\Phi-\Phi)} A^* A = A^* A = \sigma_0 \checkmark\end{aligned}$$

## Discrete ambiguities

For  $A(W, \theta) = \hat{A}(W, \theta) (\cos \theta - \alpha)$ ,  
conjugate the zero/root:  $\alpha \rightarrow \alpha^*$ .  
[A. Gersten], [E. Barrelet], [L. P. Kok],  
[A. S. Omelaenko], ...

$$\begin{aligned}\sigma_0 &= |\hat{A}|^2 (\cos \theta - \alpha^*) (\cos \theta - \alpha) \\ \rightarrow |\hat{A}|^2 &(\cos \theta - [\alpha^*]^*) (\cos \theta - \alpha^*) \\ &= |\hat{A}|^2 (\cos \theta - \alpha^*) (\cos \theta - \alpha) \\ &= \sigma_0 \checkmark\end{aligned}$$

\*) Illustration:



Grey box: space of partial wave amplitudes  $\{A_0, \dots, A_\infty\}$ , or  $\{A_0, \dots, A_L\}$ .

Orange: parameter-regions of ambiguity, i.e. with same  $\sigma_0$ .

# Continuum- vs. discrete ambiguities

## Continuum ambiguities

\*) Definition:

$$\tilde{A}(W, \theta) = e^{i\Phi(W, \theta)} A(W, \theta)$$

[Bowcock & Burkhardt],  
[L. P. Kok], [D. Atkinson], ...

\*) Invariance:

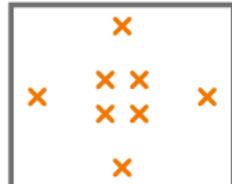
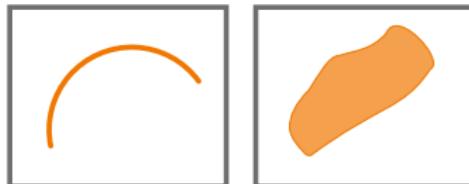
$$\begin{aligned}\sigma_0 &= |A|^2 = A^* A \\ \rightarrow \tilde{A}^* \tilde{A} &= e^{-i\Phi} A^* e^{i\Phi} A \\ &= e^{i(\Phi-\Phi)} A^* A = A^* A = \sigma_0 \checkmark\end{aligned}$$

## Discrete ambiguities

For  $A(W, \theta) = \hat{A}(W, \theta) (\cos \theta - \alpha)$ ,  
conjugate the zero/root:  $\alpha \rightarrow \alpha^*$ .  
[A. Gersten], [E. Barrelet], [L. P. Kok],  
[A. S. Omelaenko], ...

$$\begin{aligned}\sigma_0 &= |\hat{A}|^2 (\cos \theta - \alpha^*) (\cos \theta - \alpha) \\ \rightarrow |\hat{A}|^2 &(\cos \theta - [\alpha^*]^*) (\cos \theta - \alpha^*) \\ &= |\hat{A}|^2 (\cos \theta - \alpha^*) (\cos \theta - \alpha) \\ &= \sigma_0 \checkmark\end{aligned}$$

\*) Illustration:



Now: consider only mathematical ambiguities, disregarding physical constraints  
(e.g. unitarity!). Are discrete and continuum ambiguities different/related?

## Effects of the full continuum ambiguity

- \*) Transform  $A(W, \theta) \longrightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta)$  & write a Legendre-series for the rotation-function

$$e^{i\Phi(W, \theta)} = \sum_{k=0}^{\infty} L_k(W) P_k(\cos \theta).$$

How are the partial waves  $\tilde{A}_\ell$  of  $\tilde{A}(W, \theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) \tilde{A}_\ell(W) P_\ell(\cos \theta)$  expressed in terms of  $A_\ell$  from  $A(W, \theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) A_\ell(W) P_\ell(\cos \theta)$ ?

# Effects of the full continuum ambiguity

- \*) Transform  $A(W, \theta) \longrightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta)$  & write a Legendre-series for the rotation-function

$$e^{i\Phi(W, \theta)} = \sum_{k=0}^{\infty} L_k(W) P_k(\cos \theta).$$

How are the partial waves  $\tilde{A}_\ell$  of  $\tilde{A}(W, \theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) \tilde{A}_\ell(W) P_\ell(\cos \theta)$  expressed in terms of  $A_\ell$  from  $A(W, \theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) A_\ell(W) P_\ell(\cos \theta)$ ?

↪ Mixing-formula:

$$\tilde{A}_\ell(W) = \sum_{k=0}^{\infty} L_k(W) \sum_{m=|k-\ell|}^{k+\ell} \langle k, 0; \ell, 0 | m, 0 \rangle^2 A_m(W),$$

$\langle j_1, m_1; j_2, m_2 | j, m \rangle$ : Glebsch-Gordan coefficients.

# Effects of the full continuum ambiguity

\*)  $A(W, \theta) \rightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta); e^{i\Phi(W, \theta)} = \sum_k L_k(W) P_k(\cos \theta).$

↪ Mixing-formula:  $\tilde{A}_\ell(W) = \sum_{k=0}^{\infty} L_k(W) \sum_{m=|k-\ell|}^{k+\ell} \langle k, 0; \ell, 0 | m, 0 \rangle^2 A_m(W),$

$\langle j_1, m_1; j_2, m_2 | j, m \rangle$ : Glebsch-Gordan coefficients.

Explicitly:  $\tilde{A}_0(W) = L_0(W) A_0(W) + L_1(W) A_1(W) + L_2(W) A_2(W) + \dots,$

$$\tilde{A}_1(W) = L_0(W) A_1(W) + L_1(W) \left[ \frac{1}{3} A_0(W) + \frac{2}{3} A_2(W) \right]$$

$$+ L_2(W) \left[ \frac{2}{5} A_1(W) + \frac{3}{5} A_3(W) \right] + \dots,$$

$$\tilde{A}_2(W) = L_0(W) A_2(W) + L_1(W) \left[ \frac{2}{5} A_1(W) + \frac{3}{5} A_3(W) \right]$$

$$+ L_2(W) \left[ \frac{1}{5} A_0(W) + \frac{2}{7} A_2(W) + \frac{18}{35} A_4(W) \right] + \dots$$

⋮

# Effects of the full continuum ambiguity

\*)  $A(W, \theta) \rightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta); e^{i\Phi(W, \theta)} = \sum_k L_k(W) P_k(\cos \theta).$

Explicitly:  $\tilde{A}_0 = L_0 \mathbf{A}_0 + L_1 A_1 + L_2 A_2 + \dots,$

$$\tilde{A}_1 = L_0 \mathbf{A}_1 + L_1 \left[ \frac{1}{3} A_0 + \frac{2}{3} A_2 \right] + L_2 \left[ \frac{2}{5} A_1 + \frac{3}{5} A_3 \right] + \dots,$$

$$\tilde{A}_2 = L_0 \mathbf{A}_2 + L_1 \left[ \frac{2}{5} A_1 + \frac{3}{5} A_3 \right] + L_2 \left[ \frac{1}{5} A_0 + \frac{2}{7} A_2 + \frac{18}{35} A_4 \right] + \dots$$

⋮

\*) For angle-independent phase  $\Phi(W, \theta) = \Phi(W):$

$e^{i\Phi(W, \theta)} = e^{i\Phi(W)} \equiv L_0(W)$  and  $\tilde{A}_\ell(W) = L_0(W) A_\ell(W) = e^{i\Phi(W)} A_\ell(W).$   
→  $A_\ell(W)$  do not mix any more & are rotated by the same phase!

\*) Non-linearity introduced by the exp-function in the rotation  $e^{i\Phi(W, \theta)}$  generates complicated mixings, even when the phase  $\Phi(W, \theta)$  itself is simple, e.g.  $\Phi(W, \theta) = a(W) + b(W) \cos \theta.$

# Effects of the full continuum ambiguity

Illustration using a toy model:

[arXiv:1706.03211v1]

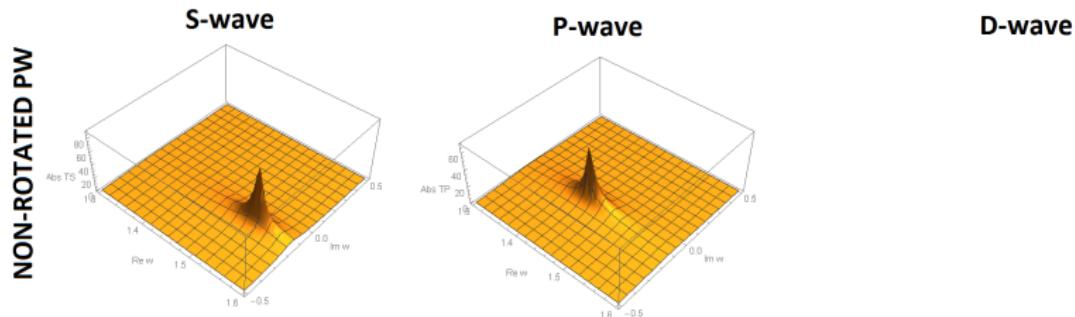
$$A(W, \theta) = T_S(W) + T_P(W) \cos(\theta),$$

$$T_{S,P}(W) = \frac{a_{S,P}}{M_{S,P} - i\Gamma_{S,P}/2 - W},$$

where

$$a_S = 0.5 + 0.4i; M_S = 1.535; \Gamma_S = 0.15,$$

$$a_P = 0.4 + 0.3i; M_P = 1.44; \Gamma_P = 0.1.$$



↪ Multiply this amplitude by a *simple* phase, e.g.  $\exp[2. + 0.5 \cos \theta]$ .

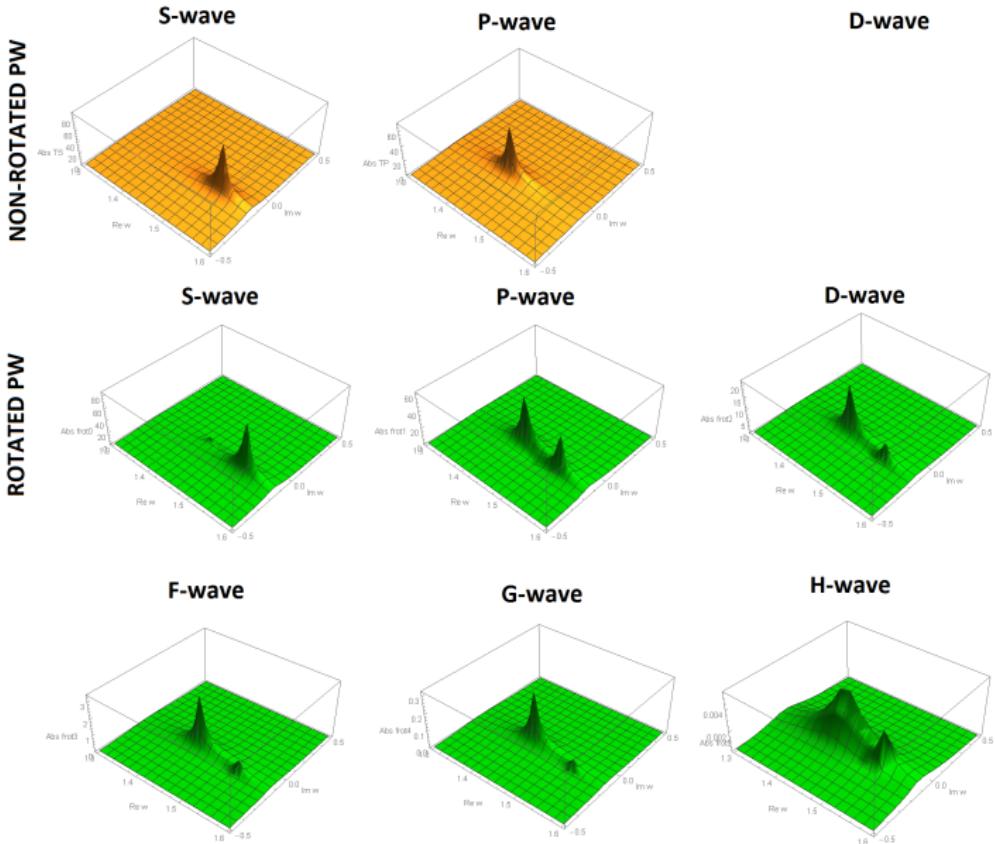
# Effects of the full continuum ambiguity

$$A(W, \theta)$$

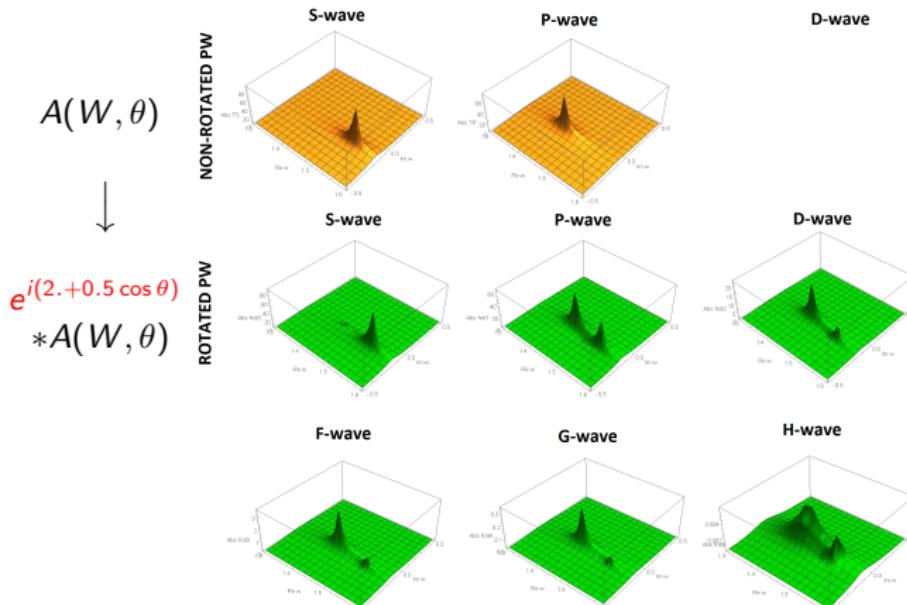


$$e^{i(2.+0.5 \cos \theta)}$$

$$*A(W, \theta)$$



# Effects of the full continuum ambiguity



$$\begin{aligned}
 \tilde{A}_0 &= \mathbf{L}_0 \mathbf{A}_0 + \mathbf{L}_1 A_1 + \mathbf{L}_2 \cancel{A}_2 + \dots, \\
 \tilde{A}_1 &= \mathbf{L}_0 \mathbf{A}_1 + \mathbf{L}_1 \left[ \frac{1}{3} A_0 + \frac{2}{3} \cancel{A}_2 \right] + \mathbf{L}_2 \left[ \frac{2}{5} A_1 + \frac{3}{5} \cancel{A}_3 \right] + \dots, \\
 \tilde{A}_2 &= \mathbf{L}_0 \cancel{A}_2 + \mathbf{L}_1 \left[ \frac{2}{5} A_1 + \frac{3}{5} \cancel{A}_3 \right] + \mathbf{L}_2 \left[ \frac{1}{5} A_0 + \frac{2}{7} \cancel{A}_4 + \frac{18}{35} \cancel{A}_5 \right] + \dots.
 \end{aligned}$$

## Discrete ambiguities: example I

\*) Model truncated at the  $P$ -wave:

$$A(W, \theta) = \sum_{\ell=0}^1 (2\ell + 1) A_\ell(W) P_\ell(\cos \theta) = A_0(W) + 3A_1(W) \cos \theta.$$

## Discrete ambiguities: example I

\*) Model truncated at the  $P$ -wave:

$$A(W, \theta) = \sum_{\ell=0}^1 (2\ell + 1) A_\ell(W) P_\ell(\cos \theta) = A_0(W) + 3A_1(W) \cos \theta.$$

\*) Cross section:

$$\sigma_0 = |A_0 + 3A_1 \cos \theta|^2 = |A_0|^2 + 6\text{Re}[A_0^* A_1] \cos \theta + 9|A_1|^2 \cos^2 \theta.$$

## Discrete ambiguities: example I

\*) Model truncated at the  $P$ -wave:

$$A(W, \theta) = \sum_{\ell=0}^1 (2\ell + 1) A_\ell(W) P_\ell(\cos \theta) = A_0(W) + 3A_1(W) \cos \theta.$$

\*) Cross section:

$$\sigma_0 = |A_0 + 3A_1 \cos \theta|^2 = |A_0|^2 + 6\operatorname{Re}[A_0^* A_1] \cos \theta + 9|A_1|^2 \cos^2 \theta.$$

| Fix phase-convention:  $A_1 \equiv \operatorname{Re}[A_1] > 0$

$$\Rightarrow \sigma_0 = |A_0|^2 + 6A_1 \operatorname{Re}[A_0^*] \cos \theta + 9A_1^2 \cos^2 \theta \equiv c_0 + c_1 \cos \theta + c_2 \cos^2 \theta.$$

## Discrete ambiguities: example I

\*) Model truncated at the  $P$ -wave:

$$A(W, \theta) = \sum_{\ell=0}^1 (2\ell + 1) A_\ell(W) P_\ell(\cos \theta) = A_0(W) + 3A_1(W) \cos \theta.$$

\*) Cross section, for phase convention  $A_1 \equiv \operatorname{Re}[A_1] > 0$ :

$$\sigma_0 = |A_0|^2 + 6A_1 \operatorname{Re}[A_0^*] \cos \theta + 9A_1^2 \cos^2 \theta \equiv c_0 + c_1 \cos \theta + c_2 \cos^2 \theta.$$

# Discrete ambiguities: example I

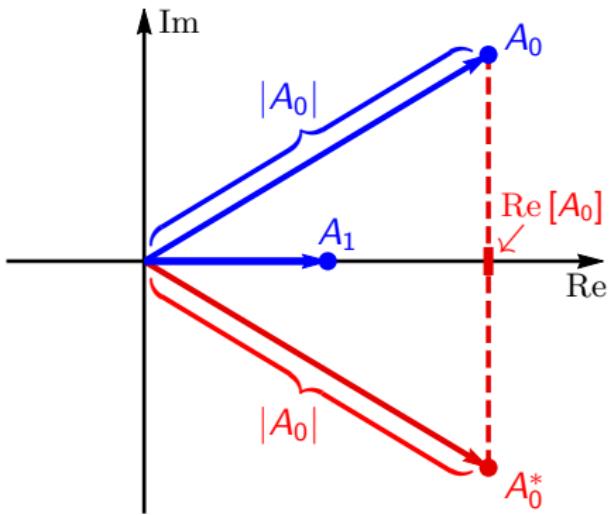
\*) Model truncated at the  $P$ -wave:

$$A(W, \theta) = \sum_{\ell=0}^1 (2\ell + 1) A_\ell(W) P_\ell(\cos \theta) = A_0(W) + 3A_1(W) \cos \theta.$$

\*) Cross section, for phase convention  $A_1 \equiv \operatorname{Re}[A_1] > 0$ :

$$\sigma_0 = |A_0|^2 + 6A_1 \operatorname{Re}[A_0^*] \cos \theta + 9A_1^2 \cos^2 \theta \equiv c_0 + c_1 \cos \theta + c_2 \cos^2 \theta.$$

⇒ Constraints on partial waves:



$$\{c_0, c_2\} \Rightarrow \{|A_0|, A_1\},$$

$$\{c_1\} \Rightarrow \{\operatorname{Re}[A_0^*] = \operatorname{Re}[A_0]\},$$

From  $|A_0|^2 = \operatorname{Re}[A_0]^2 + \operatorname{Im}[A_0]^2$ ,  
we get:

$$\operatorname{Im}[A_0] = \pm \sqrt{|A_0|^2 - \operatorname{Re}[A_0]^2}.$$

## Discrete ambiguities: example II

\*) Another point of view: linear-factor decomposition

$$A = A_0 + 3A_1 \cos \theta = 3A_1 \left( \cos \theta - \frac{[-A_0]}{3A_1} \right) \equiv \underline{\lambda} (\cos \theta - \alpha_1).$$

Phase-convention fixed via the normalization:  $A_1 \equiv \text{Re}[A_1] =: \underline{\lambda}/3$ .

## Discrete ambiguities: example II

- \*) Another point of view: linear-factor decomposition

$$A = A_0 + 3A_1 \cos \theta = 3A_1 \left( \cos \theta - \frac{[-A_0]}{3A_1} \right) \equiv \underline{\lambda (\cos \theta - \alpha_1)}.$$

Phase-convention fixed via the normalization:  $A_1 \equiv \text{Re}[A_1] =: \underline{\lambda/3}$ .

- \*) Discrete ambiguity derived from Gersten-root  $\alpha_1$ : [A. Gersten (1969)]

$$\underline{\alpha_1 \longrightarrow \alpha_1^*}.$$

## Discrete ambiguities: example II

- \*) Another point of view: linear-factor decomposition

$$A = A_0 + 3A_1 \cos \theta = 3A_1 \left( \cos \theta - \frac{[-A_0]}{3A_1} \right) \equiv \underline{\lambda (\cos \theta - \alpha_1)}.$$

Phase-convention fixed via the normalization:  $A_1 \equiv \text{Re}[A_1] =: \underline{\lambda/3}$ .

- \*) Discrete ambiguity derived from Gersten-root  $\alpha_1$ : [A. Gersten (1969)]

$$\underline{\alpha_1 \longrightarrow \alpha_1^*}.$$

- \*) Cross section  $\sigma_0$  easily seen to be invariant:

$$\begin{aligned}\sigma_0 &= |A|^2 = |\lambda|^2 (\cos \theta - \alpha_1^*) (\cos \theta - \alpha_1) \\ &\longrightarrow |\lambda|^2 (\cos \theta - [\alpha_1^*]^*) (\cos \theta - \alpha_1^*) \\ &= |\lambda|^2 (\cos \theta - \alpha_1^*) (\cos \theta - \alpha_1) = \sigma_0 \quad \checkmark\end{aligned}$$

## Discrete ambiguities: example II

- \*) Another point of view: linear-factor decomposition

$$A = A_0 + 3A_1 \cos \theta = 3A_1 \left( \cos \theta - \frac{[-A_0]}{3A_1} \right) \equiv \underline{\lambda (\cos \theta - \alpha_1)}.$$

Phase-convention fixed via the normalization:  $A_1 \equiv \text{Re}[A_1] =: \underline{\lambda/3}$ .

- \*) Discrete ambiguity derived from Gersten-root  $\alpha_1$ : [A. Gersten (1969)]

$$\underline{\alpha_1 \longrightarrow \alpha_1^*}.$$

- \*) Cross section  $\sigma_0$  easily seen to be invariant:

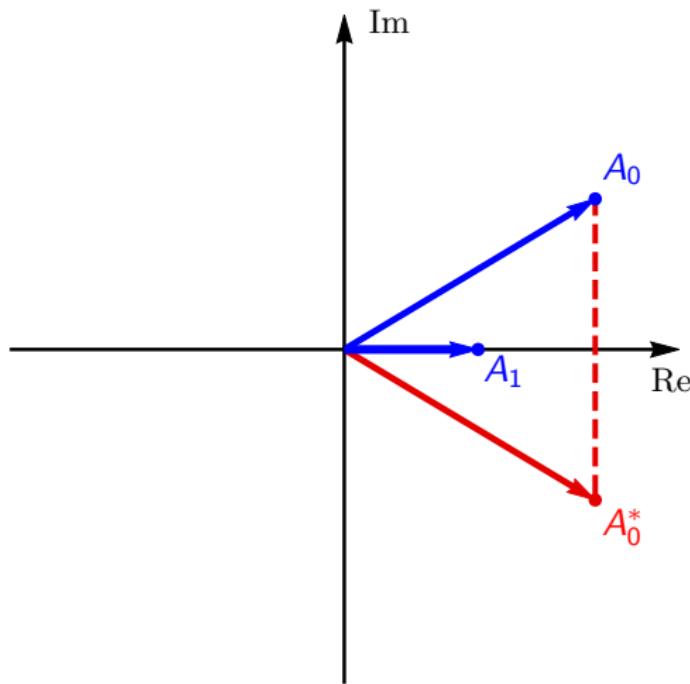
$$\begin{aligned}\sigma_0 &= |A|^2 = |\lambda|^2 (\cos \theta - \alpha_1^*) (\cos \theta - \alpha_1) \\ &\longrightarrow |\lambda|^2 (\cos \theta - [\alpha_1^*]^*) (\cos \theta - \alpha_1^*) \\ &= |\lambda|^2 (\cos \theta - \alpha_1^*) (\cos \theta - \alpha_1) = \sigma_0 \quad \checkmark\end{aligned}$$

- \*) Discrete ambiguity acting on the partial waves:

$$\alpha_1 = -\frac{A_0}{3A_1} \longrightarrow \alpha_1^* = -\frac{A_0^*}{3A_1^*} \equiv -\frac{A_0^*}{3A_1}.$$

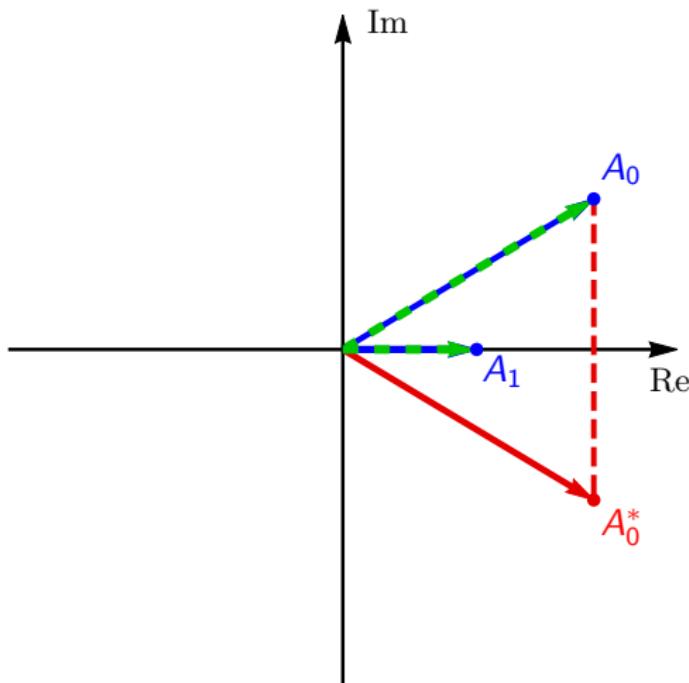
## Discrete ambiguities: example III

→ Recover the same ambiguity  $A_0 \rightarrow A_0^*$  and geometric picture as before:



## Discrete ambiguities: example III

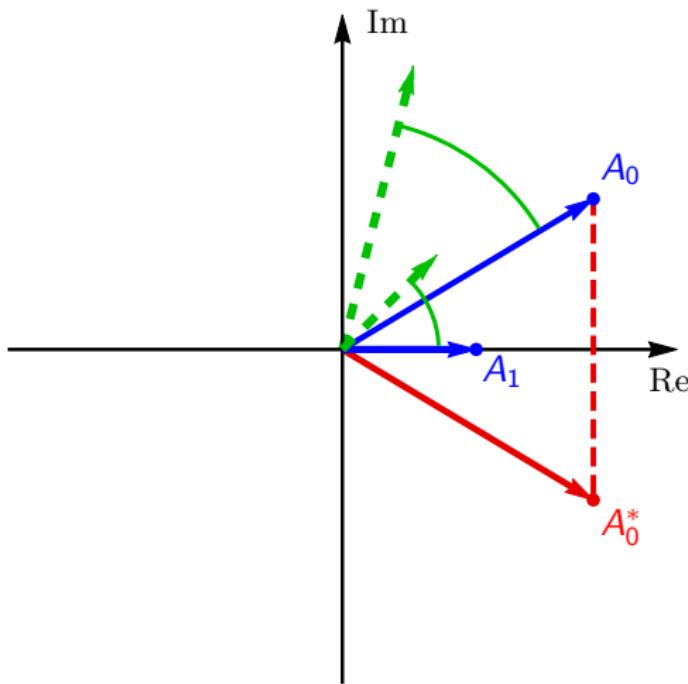
→ Recover the same ambiguity  $A_0 \rightarrow A_0^*$  and geometric picture as before:



Try to rotate  $\{A_0, A_1\}$  into  $\{A_0^*, A_1\}$  by applying the same phase-angle to both partial waves, i.e. by performing an energy-dependent phase-rotation  $\Phi(W)$ .

## Discrete ambiguities: example III

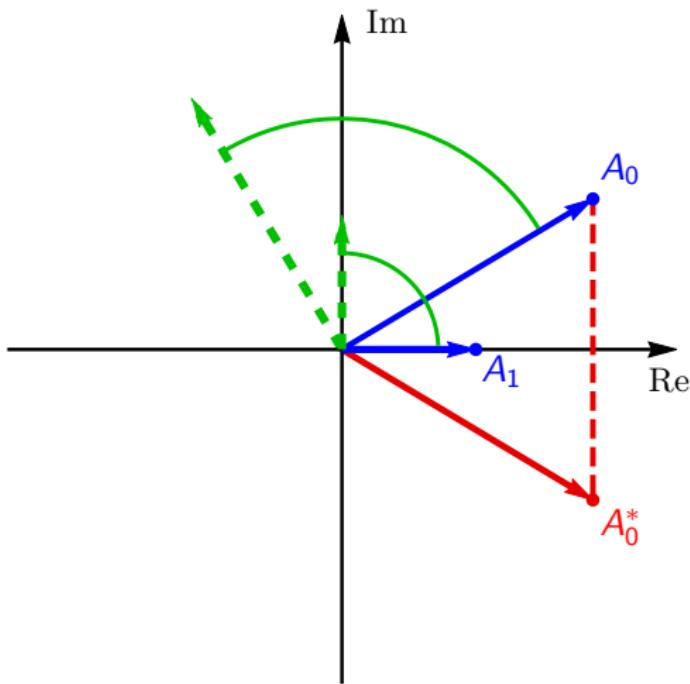
→ Recover the same ambiguity  $A_0 \rightarrow A_0^*$  and geometric picture as before:



Try to rotate  $\{A_0, A_1\}$  into  $\{A_0^*, A_1\}$  by applying the same phase-angle to both partial waves, i.e. by performing an energy-dependent phase-rotation  $\Phi(W)$ .

## Discrete ambiguities: example III

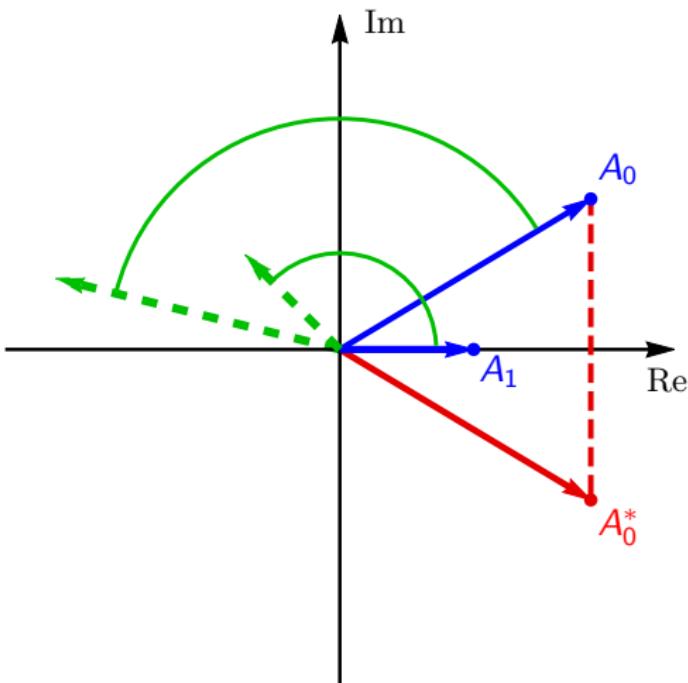
→ Recover the same ambiguity  $A_0 \rightarrow A_0^*$  and geometric picture as before:



Try to rotate  $\{A_0, A_1\}$  into  $\{A_0^*, A_1\}$  by applying the same phase-angle to both partial waves, i.e. by performing an energy-dependent phase-rotation  $\Phi(W)$ .

## Discrete ambiguities: example III

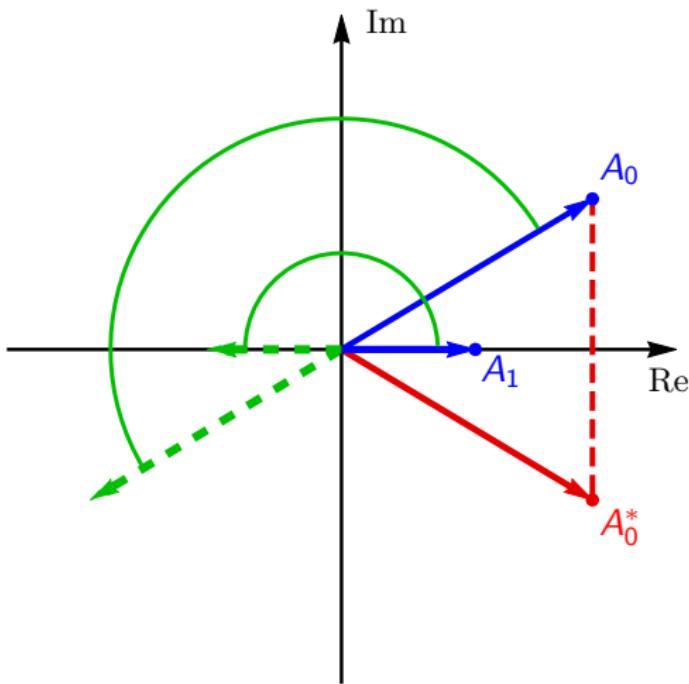
→ Recover the same ambiguity  $A_0 \rightarrow A_0^*$  and geometric picture as before:



Try to rotate  $\{A_0, A_1\}$  into  $\{A_0^*, A_1\}$  by applying the same phase-angle to both partial waves, i.e. by performing an energy-dependent phase-rotation  $\Phi(W)$ .

## Discrete ambiguities: example III

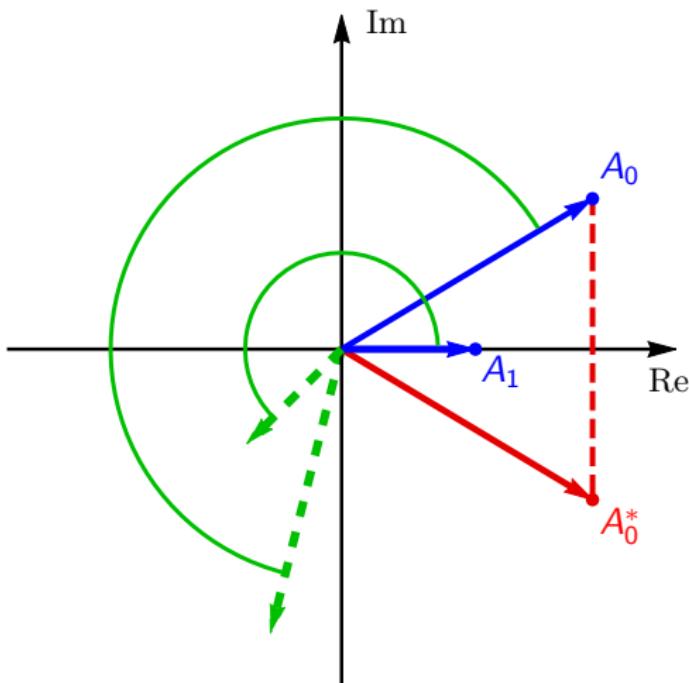
→ Recover the same ambiguity  $A_0 \rightarrow A_0^*$  and geometric picture as before:



Try to rotate  $\{A_0, A_1\}$  into  $\{A_0^*, A_1\}$  by applying the same phase-angle to both partial waves, i.e. by performing an energy-dependent phase-rotation  $\Phi(W)$ .

## Discrete ambiguities: example III

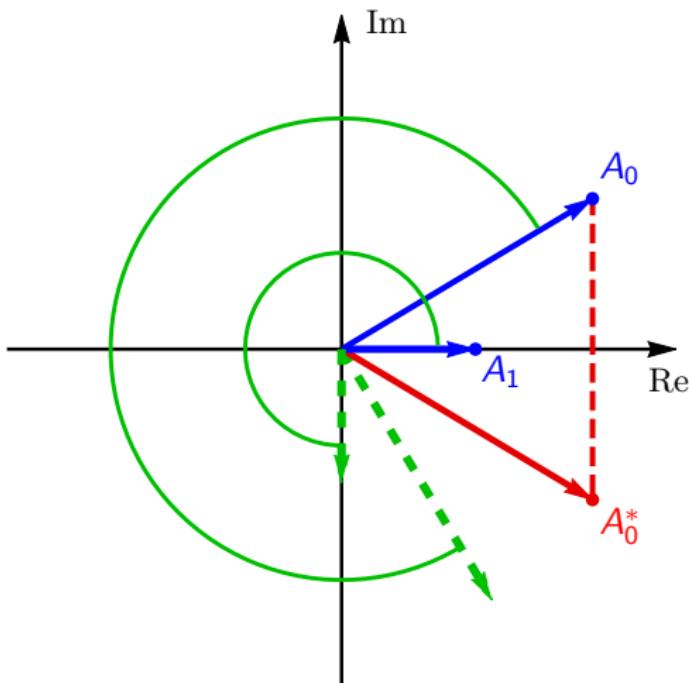
→ Recover the same ambiguity  $A_0 \rightarrow A_0^*$  and geometric picture as before:



Try to rotate  $\{A_0, A_1\}$  into  $\{A_0^*, A_1\}$  by applying the same phase-angle to both partial waves, i.e. by performing an energy-dependent phase-rotation  $\Phi(W)$ .

## Discrete ambiguities: example III

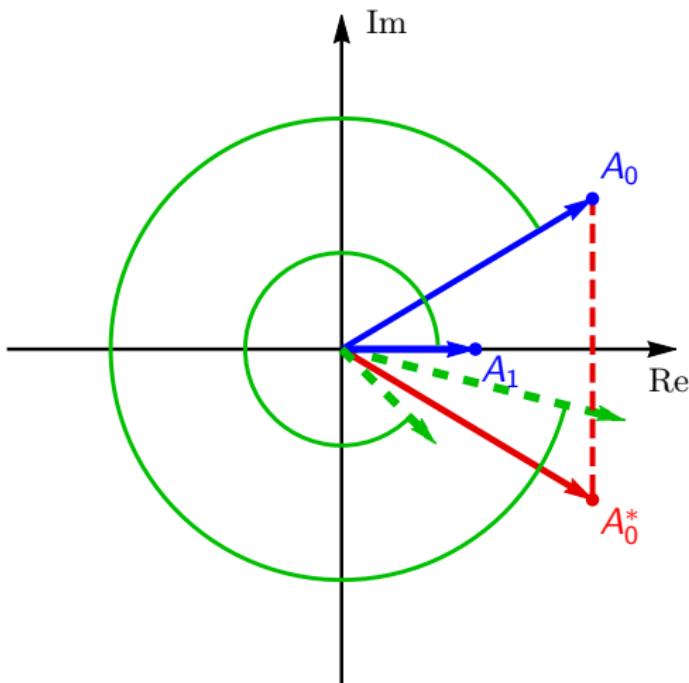
→ Recover the same ambiguity  $A_0 \rightarrow A_0^*$  and geometric picture as before:



Try to rotate  $\{A_0, A_1\}$  into  $\{A_0^*, A_1\}$  by applying the same phase-angle to both partial waves, i.e. by performing an energy-dependent phase-rotation  $\Phi(W)$ .

## Discrete ambiguities: example III

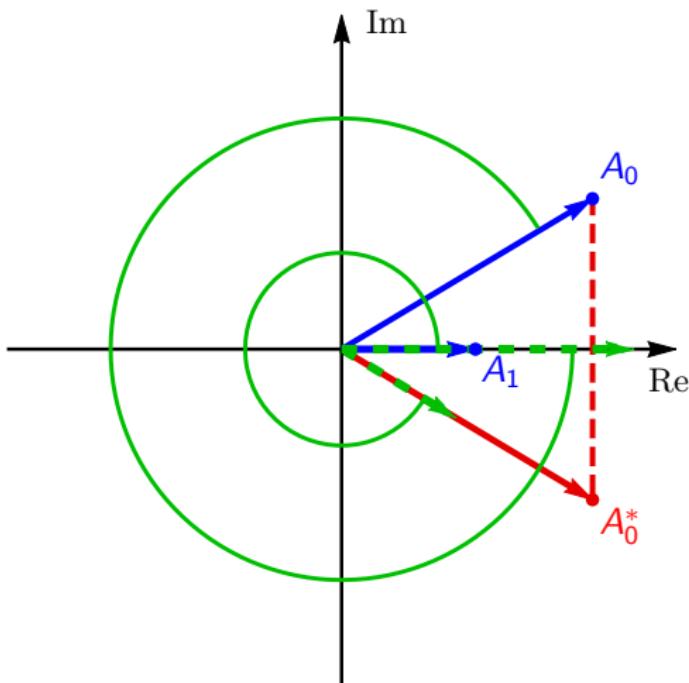
→ Recover the same ambiguity  $A_0 \rightarrow A_0^*$  and geometric picture as before:



Try to rotate  $\{A_0, A_1\}$  into  $\{A_0^*, A_1\}$  by applying the same phase-angle to both partial waves, i.e. by performing an energy-dependent phase-rotation  $\Phi(W)$ .

## Discrete ambiguities: example III

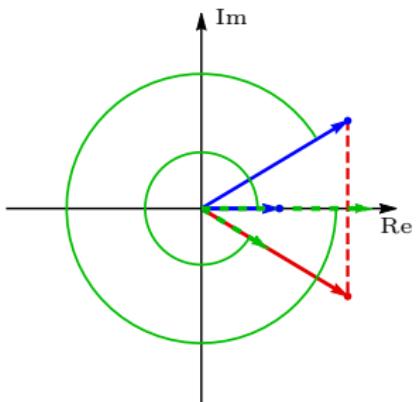
→ Recover the same ambiguity  $A_0 \rightarrow A_0^*$  and geometric picture as before:



Try to rotate  $\{A_0, A_1\}$  into  $\{A_0^*, A_1\}$  by applying the same phase-angle to both partial waves, i.e. by performing an energy-dependent phase-rotation  $\Phi(W)$ .  
↪ Impossible!

## Discrete ambiguities: example III

→ Recover the same ambiguity  $A_0 \rightarrow A_0^*$  and geometric picture as before:



Try to rotate  $\{A_0, A_1\}$  into  $\{A_0^*, A_1\}$  by applying the same phase-angle to both partial waves, i.e. by performing an energy-dependent phase-rotation  $\Phi(W)$ .

→ Impossible!

However: The modulus  $|A(W, \theta)| = \sqrt{\sigma_0(W, \theta)}$  is left invariant by the discrete ambiguity  $A_0 \rightarrow A_0^*$ .

- Transformation can (effectively) only be a rotation.
- It has to be an angle-dependent rotation!

How to generalize these results to higher  $L = \ell_{\max}$ ?

## Discrete ambiguities: general formalism I

- \*) A general truncated (i.e. polynomial-) amplitude for arbitrary  $L = \ell_{\max}$ ,  
 $A = \sum_{\ell=0}^L (2\ell + 1) A_\ell P_\ell(\cos \theta)$ , has the linear-factor decomposition:  
$$A = \lambda (\cos \theta - \alpha_1) (\cos \theta - \alpha_2) \dots (\cos \theta - \alpha_L), \text{ with } \lambda \propto A_L.$$

## Discrete ambiguities: general formalism I

- \* ) A general truncated (i.e. polynomial-) amplitude for arbitrary  $L = \ell_{\max}$ ,  
 $A = \sum_{\ell=0}^L (2\ell + 1) A_\ell P_\ell(\cos \theta)$ , has the linear-factor decomposition:  
$$A = \lambda (\cos \theta - \alpha_1) (\cos \theta - \alpha_2) \dots (\cos \theta - \alpha_L), \text{ with } \lambda \propto A_L.$$
- \* ) There exist  $2^L$  solutions that leave the differential cross section  
$$\sigma_0 = |A|^2 = |\lambda|^2 \prod_{i=1}^L (\cos \theta - \alpha_i^*) (\cos \theta - \alpha_i)$$
 invariant.

# Discrete ambiguities: general formalism I

- \* ) A general truncated (i.e. polynomial-) amplitude for arbitrary  $L = \ell_{\max}$ ,  
 $A = \sum_{\ell=0}^L (2\ell + 1) A_\ell P_\ell(\cos \theta)$ , has the linear-factor decomposition:  
$$A = \lambda (\cos \theta - \alpha_1) (\cos \theta - \alpha_2) \dots (\cos \theta - \alpha_L), \text{ with } \lambda \propto A_L.$$
- \* ) There exist  $2^L$  solutions that leave the differential cross section  
 $\sigma_0 = |A|^2 = |\lambda|^2 \prod_{i=1}^L (\cos \theta - \alpha_i^*) (\cos \theta - \alpha_i)$  invariant.
- \* ) Formally:  $2^L$  “maps”  $\pi_n$ , for  $n = 0, \dots, (2^L - 1)$ , defined by:

$$\pi_n(\alpha_i) := \begin{cases} \alpha_i & , \mu_i(n) = 0 \\ \alpha_i^* & , \mu_i(n) = 1 \end{cases}, \text{ using } n = \sum_{i=1}^L \mu_i(n) 2^{(i-1)}.$$

# Discrete ambiguities: general formalism I

- \* ) A general truncated (i.e. polynomial-) amplitude for arbitrary  $L = \ell_{\max}$ ,  
 $A = \sum_{\ell=0}^L (2\ell + 1) A_\ell P_\ell(\cos \theta)$ , has the linear-factor decomposition:  
$$A = \lambda (\cos \theta - \alpha_1)(\cos \theta - \alpha_2) \dots (\cos \theta - \alpha_L), \text{ with } \lambda \propto A_L.$$

- \* ) There exist  $2^L$  solutions that leave the differential cross section

$$\sigma_0 = |A|^2 = |\lambda|^2 \prod_{i=1}^L (\cos \theta - \alpha_i^*) (\cos \theta - \alpha_i) \text{ invariant.}$$

- \* ) Formally:  $2^L$  “maps”  $\pi_n$ , for  $n = 0, \dots, (2^L - 1)$ , defined by:

$$\pi_n(\alpha_i) := \begin{cases} \alpha_i & , \mu_i(n) = 0 \\ \alpha_i^* & , \mu_i(n) = 1 \end{cases}, \text{ using } n = \sum_{i=1}^L \mu_i(n) 2^{(i-1)}.$$

- \* ) Example: truncation at  $L = 2$ , i.e.  $S$ -,  $P$ - and  $D$ -waves  $\{A_0, A_1, A_2\}$ .  
Then, there are 2 roots  $\{\alpha_1, \alpha_2\}$  and  $2^2 = 4$  ambiguities:

$$\begin{aligned} \{\pi_0(\alpha_1), \pi_0(\alpha_2)\} &= \{\alpha_1, \alpha_2\} & \{\pi_1(\alpha_1), \pi_1(\alpha_2)\} &= \{\alpha_1^*, \alpha_2\}, \\ \{\pi_2(\alpha_1), \pi_2(\alpha_2)\} &= \{\alpha_1, \alpha_2^*\} & \{\pi_3(\alpha_1), \pi_3(\alpha_2)\} &= \{\alpha_1^*, \alpha_2^*\}. \end{aligned}$$

## Discrete ambiguities: general formalism II

\*) From the original truncated model  $A = \lambda \prod_{i=1}^L (\cos \theta - \alpha_i)$ , one can transform to  $2^L$  ambiguous amplitudes, i.e. for  $n = 0, \dots, (2^L - 1)$ :

$$A^{(n)} = \lambda \prod_{i=1}^L (\cos \theta - \pi_n[\alpha_i]) \equiv \sum_{\ell=0}^L (2\ell + 1) A_\ell^{(n)}(W) P_\ell(\cos \theta),$$

## Discrete ambiguities: general formalism II

- \*) From the original truncated model  $A = \lambda \prod_{i=1}^L (\cos \theta - \alpha_i)$ , one can transform to  $2^L$  ambiguous amplitudes, i.e. for  $n = 0, \dots, (2^L - 1)$ :

$$A^{(n)} = \lambda \prod_{i=1}^L (\cos \theta - \pi_n[\alpha_i]) \equiv \sum_{\ell=0}^L (2\ell + 1) A_\ell^{(n)}(W) P_\ell(\cos \theta),$$

- \*) Important point from the Gersten-paper:

*"For fixed  $L$  these (i.e. the  $\pi_n$ -maps) are the only ambiguities in determining the phase shifts from  $(d\sigma/d\Omega)$  (i.e.  $\sigma_0$ )."*

[cf.: A. Gersten, Nucl. Phys. B 12, p. 538 (1969).]

However, no proof of this claim is given by the author.

## Discrete ambiguities: general formalism II

- \*) From the original truncated model  $A = \lambda \prod_{i=1}^L (\cos \theta - \alpha_i)$ , one can transform to  $2^L$  ambiguous amplitudes, i.e. for  $n = 0, \dots, (2^L - 1)$ :

$$A^{(n)} = \lambda \prod_{i=1}^L (\cos \theta - \pi_n[\alpha_i]) \equiv \sum_{\ell=0}^L (2\ell + 1) A_\ell^{(n)}(W) P_\ell(\cos \theta),$$

- \*) Important point from the Gersten-paper:

*"For fixed  $L$  these (i.e. the  $\pi_n$ -maps) are the only ambiguities in determining the phase shifts from  $(d\sigma/d\Omega)$  (i.e.  $\sigma_0$ )."*

[cf.: A. Gersten, Nucl. Phys. B 12, p. 538 (1969).]

However, no proof of this claim is given by the author.

- Question: Is this really the case? Could one not contrive more complicated discrete ambiguities transmitted by some maps  $\zeta_n$  via

$$\alpha_i \rightarrow \zeta_n(\alpha_i),$$

and which go beyond complex conjugations?

## Discrete ambiguities: general formalism II

- \* From the original truncated model  $A = \lambda \prod_{i=1}^L (\cos \theta - \alpha_i)$ , one can transform to  $2^L$  ambiguous amplitudes, i.e. for  $n = 0, \dots, (2^L - 1)$ :

$$A^{(n)} = \lambda \prod_{i=1}^L (\cos \theta - \pi_n[\alpha_i]) \equiv \sum_{\ell=0}^L (2\ell + 1) A_\ell^{(n)}(W) P_\ell(\cos \theta),$$

- \* Important point from the Gersten-paper:

*"For fixed  $L$  these (i.e. the  $\pi_n$ -maps) are the only ambiguities in determining the phase shifts from  $(d\sigma/d\Omega)$  (i.e.  $\sigma_0$ )."*

[cf.: A. Gersten, Nucl. Phys. B 12, p. 538 (1969).]

However, no proof of this claim is given by the author.

- Question: Is this really the case? Could one not contrive more complicated discrete ambiguities transmitted by some maps  $\zeta_n$  via

$$\alpha_i \rightarrow \zeta_n(\alpha_i),$$

and which go beyond complex conjugations?

- ⇒ We believe Gersten's statement! The reason why should become more clear by comparing discrete to continuum ambiguities.

# Discrete ambiguities & angle-dependent phase rotations

\*) Naively assumed equivalence:

$\Phi(W, \theta) = \Phi(W)$ , phase  
only energy-dependent.



$A(W, \theta)$  truncated at  $L$   
 $\downarrow$   
 $e^{i\Phi(W, \theta)} A(W, \theta)$  truncated at  $L$ .

# Discrete ambiguities & angle-dependent phase rotations

\*) Naively assumed equivalence:

$\Phi(W, \theta) = \Phi(W)$ , phase  
only energy-dependent.

→  
easy ✓

$A(W, \theta)$  truncated at  $L$   
↓  
 $e^{i\Phi(W, \theta)} A(W, \theta)$  truncated at  $L$ .

# Discrete ambiguities & angle-dependent phase rotations

\*) Naively assumed equivalence:

$\Phi(W, \theta) = \Phi(W)$ , phase  
only energy-dependent.

→  
easy ✓  
←  
wrong ✗

$A(W, \theta)$  truncated at  $L$   
↓  
 $e^{i\Phi(W, \theta)} A(W, \theta)$  truncated at  $L$ .

# Discrete ambiguities & angle-dependent phase rotations

\*) Naively assumed equivalence:

$\Phi(W, \theta) = \Phi(W)$ , phase  
only energy-dependent.

→  
easy ✓  
←  
wrong ✗

$A(W, \theta)$  truncated at  $L$

↓  
 $e^{i\Phi(W, \theta)} A(W, \theta)$  truncated at  $L$ .

⇒ Instead: discrete ambiguities are angle-dependent rotations, for certain phases  $\Phi_n(W, \theta)$ ,  $n = 0, \dots, (2^L - 1)$ :

$$e^{i\Phi_n(W, \theta)} = \frac{A^{(n)}(W, \theta)}{A(W, \theta)} = \frac{(\cos \theta - \pi_n[\alpha_1]) \dots (\cos \theta - \pi_n[\alpha_L])}{(\cos \theta - \alpha_1) \dots (\cos \theta - \alpha_L)}$$

# Discrete ambiguities & angle-dependent phase rotations

\*) Naively assumed equivalence:

$\Phi(W, \theta) = \Phi(W)$ , phase  
only energy-dependent.

→  
easy ✓  
←  
wrong ✗

$A(W, \theta)$  truncated at  $L$

↓  
 $e^{i\Phi(W, \theta)} A(W, \theta)$  truncated at  $L$ .

⇒ Instead: discrete ambiguities are angle-dependent rotations, for certain phases  $\Phi_n(W, \theta)$ ,  $n = 0, \dots, (2^L - 1)$ :

$$e^{i\Phi_n(W, \theta)} = \frac{A^{(n)}(W, \theta)}{A(W, \theta)} = \frac{(\cos \theta - \pi_n[\alpha_1]) \dots (\cos \theta - \pi_n[\alpha_L])}{(\cos \theta - \alpha_1) \dots (\cos \theta - \alpha_L)}$$

\*) Amazing: Even though  $e^{i\Phi(W, \theta)}$  is highly non-linear, one can rotate truncated models again into truncated ones!

⇒ Moreover: The  $\Phi_n(W, \theta)$  are the only angle-dependent phases which leave the truncation order  $L$  un-touched!

# Discrete ambiguities & angle-dependent phase rotations

\*) Naively assumed equivalence:

$\Phi(W, \theta) = \Phi(W)$ , phase  
only energy-dependent.

→ easy ✓  
← wrong ✗

$A(W, \theta)$  truncated at  $L$

↓  
 $e^{i\Phi(W, \theta)} A(W, \theta)$  truncated at  $L$ .

⇒ Instead: discrete ambiguities are angle-dependent rotations, for certain phases  $\Phi_n(W, \theta)$ ,  $n = 0, \dots, (2^L - 1)$ :

$$e^{i\Phi_n(W, \theta)} = \frac{A^{(n)}(W, \theta)}{A(W, \theta)} = \frac{(\cos \theta - \pi_n[\alpha_1]) \dots (\cos \theta - \pi_n[\alpha_L])}{(\cos \theta - \alpha_1) \dots (\cos \theta - \alpha_L)}$$

\*) Amazing: Even though  $e^{i\Phi(W, \theta)}$  is highly non-linear, one can rotate truncated models again into truncated ones!

⇒ Moreover: The  $\Phi_n(W, \theta)$  are the only angle-dependent phases which leave the truncation order  $L$  un-touched!

↪ Can this be substantiated *without* using the **Gersten-formalism**?

## Functional methods and Conclusions

For fixed  $W$  and angular variable  $x = \cos \theta$ , start with an amplitude  $A(W, \theta) = A(\theta) \equiv A(x)$  truncated at some  $L$ .

→ Search (numerically) for functions  $F(x)$  satisfying the 2 requirements:

- (i) Unimodularity:  $|F(x)|^2 = 1, \forall x \in [-1, 1]$ ,
- (ii) The rotated model  $\tilde{A}(x) = e^{i\Phi(x)} A(x) \equiv F(x)A(x)$  is truncated at  $L$ :

$$\tilde{A}_{L+k} = \frac{1}{2} \int_{-1}^{+1} dx F(x) A(x) P_{L+k}(x) \equiv 0, \forall k = 1, \dots, \infty.$$

## Functional methods and Conclusions

For fixed  $W$  and angular variable  $x = \cos \theta$ , start with an amplitude  $A(W, \theta) = A(\theta) \equiv A(x)$  truncated at some  $L$ .

→ Search (numerically) for functions  $F(x)$  satisfying the 2 requirements:

- (i) Unimodularity:  $|F(x)|^2 = 1, \forall x \in [-1, 1]$ ,
- (ii) The rotated model  $\tilde{A}(x) = e^{i\Phi(x)} A(x) \equiv F(x)A(x)$  is truncated at  $L$ :

$$\tilde{A}_{L+k} = \frac{1}{2} \int_{-1}^{+1} dx F(x) A(x) P_{L+k}(x) \equiv 0, \forall k = 1, \dots, \infty.$$

In practice: Implement condition (i) on a grid  $\{x_n\} \in [-1, 1]$  & truncate condition (ii) at some (large) value  $K = k_{\max}$ .

⇒ The functions  $F_n(x) = e^{i\Phi_n(x)}$ , known from the **Gersten**-formalism, are the only solutions returned by this numerical procedure!

# Functional methods and Conclusions

For fixed  $W$  and angular variable  $x = \cos \theta$ , start with an amplitude  $A(W, \theta) = A(\theta) \equiv A(x)$  truncated at some  $L$ .

→ Search (numerically) for functions  $F(x)$  satisfying the 2 requirements:

- (i) Unimodularity:  $|F(x)|^2 = 1, \forall x \in [-1, 1]$ ,
- (ii) The rotated model  $\tilde{A}(x) = e^{i\Phi(x)} A(x) \equiv F(x)A(x)$  is truncated at  $L$ :

$$\tilde{A}_{L+k} = \frac{1}{2} \int_{-1}^{+1} dx F(x) A(x) P_{L+k}(x) \equiv 0, \forall k = 1, \dots, \infty.$$

⇒ The functions  $F_n(x) = e^{i\Phi_n(x)}$ , known from the **Gersten**-formalism, are the only solutions returned by this numerical procedure!

**Conclusion:** The discrete ambiguities generated by  $F_n(x) = e^{i\Phi_n(x)}$  are the remnants of the full continuum ambiguity  $A(x) \rightarrow e^{i\Phi(x)} A(x)$ , once the latter is restricted to truncated models!

## Some references

- [A. Gersten (1969)]: A. Gersten, Nucl. Phys. B **12**, p. 537 (1969).
- [Bowcock & Burkhardt (1975)]: J. E. Bowcock and H. Burkhardt, Rep. Prog. Phys. **38**, 1099 (1975).
- [L. P. Kok (1976)]: L.P. Kok., *Ambiguities in Phase Shift Analysis*, In \*Delhi 1976, Conference On Few Body Dynamics\*, Amsterdam 1976, 43-46.
- [D. Atkinson]: D. Atkinson, *Phase-shift Ambiguities - A Review*, available at servers of U. Groningen.
- [E. Barrelet (1972)]: E. Barrelet, Nuovo Cimento **8A**, 331 (1972).
- [Dean & Lee (1972)]: N. W. Dean and P. Lee, Phys. Rev. D **5**, 2741 (1972).
- [A. S. Omelaenko (1981)]: A. S. Omelaenko, Sov. J. Nucl. Phys. **34**, 406 (1981).

Thank You!

## Additional Slides

# Derivation of the mixing-formula

\*) General continuum ambiguity transformation:

$$A(W, \theta) \longrightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta).$$

# Derivation of the mixing-formula

\*) General continuum ambiguity transformation:

$$A(W, \theta) \longrightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta).$$

\*) Write  $e^{i\Phi(W, \theta)}$  as a Legendre-series, defined by complex coefficients:

$$e^{i\Phi(W, \theta)} = \sum_{\ell'=0}^{\infty} L_{\ell'}(W) P_{\ell'}(\cos \theta).$$

# Derivation of the mixing-formula

\*) General continuum ambiguity transformation:

$$A(W, \theta) \longrightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta).$$

\*) Write  $e^{i\Phi(W, \theta)}$  as a Legendre-series, defined by complex coefficients:

$$e^{i\Phi(W, \theta)} = \sum_{\ell'=0}^{\infty} L_{\ell'}(W) P_{\ell'}(\cos \theta).$$

Investigate partial waves  $\tilde{A}_\ell$ , projected out of the rotated amplitude  $\tilde{A}$ :

$$\tilde{A}_\ell(W) = \frac{1}{2} \int_{-1}^1 d(\cos \theta) \tilde{A}(W, \theta) P_\ell(\cos \theta)$$

# Derivation of the mixing-formula

\*) General continuum ambiguity transformation:

$$A(W, \theta) \longrightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta).$$

\*) Write  $e^{i\Phi(W, \theta)}$  as a Legendre-series, defined by complex coefficients:

$$e^{i\Phi(W, \theta)} = \sum_{\ell'=0}^{\infty} L_{\ell'}(W) P_{\ell'}(\cos \theta).$$

Investigate partial waves  $\tilde{A}_\ell$ , projected out of the rotated amplitude  $\tilde{A}$ :

$$\begin{aligned}\tilde{A}_\ell(W) &= \frac{1}{2} \int_{-1}^1 d(\cos \theta) \tilde{A}(W, \theta) P_\ell(\cos \theta) \\ &= \frac{1}{2} \int_{-1}^1 d(\cos \theta) e^{i\phi(W, \theta)} A(W, \theta) P_\ell(\cos \theta)\end{aligned}$$

# Derivation of the mixing-formula

\*) General continuum ambiguity transformation:

$$A(W, \theta) \longrightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta).$$

\*) Write  $e^{i\Phi(W, \theta)}$  as a Legendre-series, defined by complex coefficients:

$$e^{i\Phi(W, \theta)} = \sum_{\ell'=0}^{\infty} L_{\ell'}(W) P_{\ell'}(\cos \theta).$$

Investigate partial waves  $\tilde{A}_\ell$ , projected out of the rotated amplitude  $\tilde{A}$ :

$$\begin{aligned}\tilde{A}_\ell(W) &= \frac{1}{2} \int_{-1}^1 d(\cos \theta) \tilde{A}(W, \theta) P_\ell(\cos \theta) \\ &= \frac{1}{2} \int_{-1}^1 d(\cos \theta) e^{i\phi(W, \theta)} A(W, \theta) P_\ell(\cos \theta) \\ &= \frac{1}{2} \int_{-1}^1 d(\cos \theta) \sum_{\ell'=0}^{\infty} L_{\ell'}(W) P_{\ell'}(\cos \theta) A(W, \theta) P_\ell(\cos \theta)\end{aligned}$$

# Derivation of the mixing-formula

\*) General continuum ambiguity transformation:

$$A(W, \theta) \longrightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta).$$

\*) Write  $e^{i\Phi(W, \theta)}$  as a Legendre-series, defined by complex coefficients:

$$e^{i\Phi(W, \theta)} = \sum_{\ell'=0}^{\infty} L_{\ell'}(W) P_{\ell'}(\cos \theta).$$

Investigate partial waves  $\tilde{A}_\ell$ , projected out of the rotated amplitude  $\tilde{A}$ :

$$\begin{aligned}\tilde{A}_\ell(W) &= \frac{1}{2} \int_{-1}^1 d(\cos \theta) \tilde{A}(W, \theta) P_\ell(\cos \theta) \\ &= \frac{1}{2} \int_{-1}^1 d(\cos \theta) e^{i\phi(W, \theta)} A(W, \theta) P_\ell(\cos \theta) \\ &= \frac{1}{2} \int_{-1}^1 d(\cos \theta) \sum_{\ell'=0}^{\infty} L_{\ell'}(W) P_{\ell'}(\cos \theta) A(W, \theta) P_\ell(\cos \theta) \\ &= \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \frac{1}{2} \int_{-1}^1 d(\cos \theta) A(W, \theta) P_{\ell'}(\cos \theta) P_\ell(\cos \theta).\end{aligned}$$

# Derivation of the mixing-formula

- \* General continuum ambiguity transformation:

$$A(W, \theta) \longrightarrow \tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta).$$

- \* Write  $e^{i\Phi(W, \theta)}$  as a Legendre-series, defined by complex coefficients:

$$e^{i\Phi(W, \theta)} = \sum_{\ell'=0}^{\infty} L_{\ell'}(W) P_{\ell'}(\cos \theta).$$

Investigate partial waves  $\tilde{A}_\ell$ , projected out of the rotated amplitude  $\tilde{A}$ :

$$\begin{aligned}\tilde{A}_\ell(W) &= \frac{1}{2} \int_{-1}^1 d(\cos \theta) \tilde{A}(W, \theta) P_\ell(\cos \theta) \\ &= \frac{1}{2} \int_{-1}^1 d(\cos \theta) e^{i\phi(W, \theta)} A(W, \theta) P_\ell(\cos \theta) \\ &= \frac{1}{2} \int_{-1}^1 d(\cos \theta) \sum_{\ell'=0}^{\infty} L_{\ell'}(W) P_{\ell'}(\cos \theta) A(W, \theta) P_\ell(\cos \theta) \\ &= \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \frac{1}{2} \int_{-1}^1 d(\cos \theta) A(W, \theta) \underbrace{P_{\ell'}(\cos \theta) P_\ell(\cos \theta)}_{?}.\end{aligned}$$

## Derivation of the mixing-formula

\*) Tr.:  $\tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta)$ ,  $e^{i\Phi(W, \theta)} = \sum_{\ell'} L_{\ell'}(W) P_{\ell'}(\cos \theta)$ .

## Derivation of the mixing-formula

\*) Tr.:  $\tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta)$ ,  $e^{i\Phi(W, \theta)} = \sum_{\ell'} L_{\ell'}(W) P_{\ell'}(\cos \theta)$ .

\*) What to do about product  $P_{\ell'}(x)P_{\ell}(x)$ ?  $\longrightarrow$  Re-expand it!:

$$P_{\ell'}(x)P_{\ell}(x) = \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \langle \ell', 0; \ell, 0 | m, 0 \rangle^2 P_m(x). \quad [\text{J. C. Adams (1878)}]$$

# Derivation of the mixing-formula

\*) Tr.:  $\tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta)$ ,  $e^{i\Phi(W, \theta)} = \sum_{\ell'} L_{\ell'}(W) P_{\ell'}(\cos \theta)$ .

\*) What to do about product  $P_{\ell'}(x)P_{\ell}(x)$ ?  $\rightarrow$  Re-expand it!:

$$P_{\ell'}(x)P_{\ell}(x) = \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \underbrace{\langle \ell', 0; \ell, 0 | m, 0 \rangle}_\text{Clebsch–Gordan coeff.}^2 P_m(x). \quad [\text{J. C. Adams (1878)}]$$

# Derivation of the mixing-formula

\*) Tr.:  $\tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta)$ ,  $e^{i\Phi(W, \theta)} = \sum_{\ell'} L_{\ell'}(W) P_{\ell'}(\cos \theta)$ .

\*) What to do about product  $P_{\ell'}(x)P_{\ell}(x)$ ?  $\rightarrow$  Re-expand it!:

$$P_{\ell'}(x)P_{\ell}(x) = \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \underbrace{\langle \ell', 0; \ell, 0 | m, 0 \rangle^2}_{\text{Clebsch-Gordan coeff.}} P_m(x). \quad [\text{J. C. Adams (1878)}]$$

Thus, we obtain for the partial waves of the *rotated* amplitude  $\tilde{A}$ :

$$\begin{aligned} \tilde{A}_{\ell}(W) &= \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \frac{1}{2} \int_{-1}^1 d(\cos \theta) A(W, \theta) \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \langle \ell', 0; \ell, 0 | m, 0 \rangle^2 P_m(\cos \theta) \\ &= \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \langle \ell', 0; \ell, 0 | m, 0 \rangle^2 \frac{1}{2} \int_{-1}^1 d(\cos \theta) A(W, \theta) P_m(\cos \theta) \end{aligned}$$

# Derivation of the mixing-formula

\*) Tr.:  $\tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta)$ ,  $e^{i\Phi(W, \theta)} = \sum_{\ell'} L_{\ell'}(W) P_{\ell'}(\cos \theta)$ .

\*) What to do about product  $P_{\ell'}(x)P_{\ell}(x)$ ?  $\rightarrow$  Re-expand it!:

$$P_{\ell'}(x)P_{\ell}(x) = \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \underbrace{\langle \ell', 0; \ell, 0 | m, 0 \rangle^2}_{\text{Clebsch-Gordan coeff.}} P_m(x). \quad [\text{J. C. Adams (1878)}]$$

Thus, we obtain for the partial waves of the *rotated* amplitude  $\tilde{A}$ :

$$\begin{aligned} \tilde{A}_{\ell}(W) &= \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \frac{1}{2} \int_{-1}^1 d(\cos \theta) A(W, \theta) \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \langle \ell', 0; \ell, 0 | m, 0 \rangle^2 P_m(\cos \theta) \\ &= \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \langle \ell', 0; \ell, 0 | m, 0 \rangle^2 \underbrace{\frac{1}{2} \int_{-1}^1 d(\cos \theta) A(W, \theta) P_m(\cos \theta)}_{A_m(W)} \end{aligned}$$

# Derivation of the mixing-formula

\*) Tr.:  $\tilde{A}(W, \theta) := e^{i\Phi(W, \theta)} A(W, \theta)$ ,  $e^{i\Phi(W, \theta)} = \sum_{\ell'} L_{\ell'}(W) P_{\ell'}(\cos \theta)$ .

\*) What to do about product  $P_{\ell'}(x)P_{\ell}(x)$ ?  $\rightarrow$  Re-expand it!:

$$P_{\ell'}(x)P_{\ell}(x) = \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \underbrace{\langle \ell', 0; \ell, 0 | m, 0 \rangle^2}_{\text{Clebsch-Gordan coeff.}} P_m(x). \quad [\text{J. C. Adams (1878)}]$$

Thus, we obtain for the partial waves of the *rotated* amplitude  $\tilde{A}$ :

$$\begin{aligned} \tilde{A}_{\ell}(W) &= \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \frac{1}{2} \int_{-1}^1 d(\cos \theta) A(W, \theta) \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \langle \ell', 0; \ell, 0 | m, 0 \rangle^2 P_m(\cos \theta) \\ &= \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \langle \ell', 0; \ell, 0 | m, 0 \rangle^2 \underbrace{\frac{1}{2} \int_{-1}^1 d(\cos \theta) A(W, \theta) P_m(\cos \theta)}_{A_m(W)} \\ &= \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \langle \ell', 0; \ell, 0 | m, 0 \rangle^2 A_m(W). \end{aligned}$$

## Functional problem: definition

2 requirements for the rotation  $F(x)$  which generates  $\tilde{A}(x) = F(x)A(x)$ :

(i)  $|F(x)|^2 = 1, \forall x \in [-1, 1]$  & (ii)  $\frac{1}{2} \int_{-1}^{+1} dx F(x)A(x)P_{L+k}(x) \equiv 0, \forall k \geq 1$ .

## Functional problem: definition

2 requirements for the rotation  $F(x)$  which generates  $\tilde{A}(x) = F(x)A(x)$ :

(i)  $|F(x)|^2 = 1, \forall x \in [-1, 1]$  & (ii)  $\frac{1}{2} \int_{-1}^{+1} dx F(x)A(x)P_{L+k}(x) \equiv 0, \forall k \geq 1$ .

Define minimization-functional in a suitable way:

$$\begin{aligned} W[F(x)] &:= \sum_x \left( \operatorname{Re}[F(x)]^2 + \operatorname{Im}[F(x)]^2 - 1 \right)^2 + \operatorname{Im} \left[ \frac{1}{2} \int_{-1}^{+1} dx F(x)A(x) \right]^2 \\ &+ \sum_{k \geq 1} \left\{ \operatorname{Re} \left[ \frac{1}{2} \int_{-1}^{+1} dx F(x)A(x)P_{L+k}(x) \right]^2 + \operatorname{Im} \left[ \frac{1}{2} \int_{-1}^{+1} dx F(x)A(x)P_{L+k}(x) \right]^2 \right\}, \end{aligned}$$

and find phase-rotation functions that minimize of this functional:

$$\begin{aligned} W[F(x)] &\rightarrow \min. \equiv 0, \\ \text{for } F(x) &\rightarrow F_n(x), n = 0, \dots, (2^L - 1). \end{aligned}$$

## Functional problem: solution Ansatz

\*) Discretize the interval  $[-1, 1]$  into  $N_I$  equidistant points  $\{x_n\}$  via:

$$\Delta x \equiv \frac{1 - (-1)}{N_I} = \frac{2}{N_I}; x_n := -1 + \left( \frac{1+2(n-1)}{2} \right) \Delta x, \forall n = 1, \dots, N_I.$$

## Functional problem: solution Ansatz

\*) Discretize the interval  $[-1, 1]$  into  $N_I$  equidistant points  $\{x_n\}$  via:

$$\Delta x \equiv \frac{1 - (-1)}{N_I} = \frac{2}{N_I}; x_n := -1 + \left( \frac{1+2(n-1)}{2} \right) \Delta x, \forall n = 1, \dots, N_I.$$

\*) Parametrize  $F(x)$  using a finite Legendre-expansion (with  $\mathcal{L}_{\text{cut}} \gg 1$ ):

$$F(\{y_{\ell'}, w_{\ell'}\})(x) := \sum_{\ell'=0}^{\mathcal{L}_{\text{cut}}} (y_{\ell'} + iw_{\ell'}) P_{\ell'}(x).$$

$y_{\ell'}$  = Re  $[L_{\ell'}]$  and  $w_{\ell'}$  = Im  $[L_{\ell'}]$  are parameters for which to solve.

# Functional problem: solution Ansatz

\*) Discretize the interval  $[-1, 1]$  into  $N_I$  equidistant points  $\{x_n\}$  via:

$$\Delta x \equiv \frac{1 - (-1)}{N_I} = \frac{2}{N_I}; x_n := -1 + \left( \frac{1+2(n-1)}{2} \right) \Delta x, \forall n = 1, \dots, N_I.$$

\*) Parametrize  $F(x)$  using a finite Legendre-expansion (with  $\mathcal{L}_{\text{cut}} \gg 1$ ):

$$F(\{y_{\ell'}, w_{\ell'}\})(x) := \sum_{\ell'=0}^{\mathcal{L}_{\text{cut}}} (y_{\ell'} + iw_{\ell'}) P_{\ell'}(x).$$

$y_{\ell'} = \text{Re}[L_{\ell'}]$  and  $w_{\ell'} = \text{Im}[L_{\ell'}]$  are parameters for which to solve.

\*) Use mixing-formula, for input-model  $A(x) = \sum_{\ell \leq L} (2\ell + 1) A_{\ell} P_{\ell}(x)$  truncated at  $L$ :

$$\tilde{A}_{L+k}(\{y_{\ell'}, w_{\ell'}\}) = \sum_{\ell'=k}^{\min(2L+k, \mathcal{L}_{\text{cut}})} (y_{\ell'} + iw_{\ell'}) \sum_{m=|L+k-\ell'|}^L \langle \ell', 0; \ell, 0 | m, 0 \rangle^2 A_m,$$

$$\forall k = 1, \dots, K \Rightarrow \underline{K \equiv \mathcal{L}_{\text{cut}}}.$$

## Functional problem: solution Ansatz

- \*) Discr.:  $\Delta x \equiv \frac{2}{N_I}; x_n := -1 + \left( \frac{1+2(n-1)}{2} \right) \Delta x, \forall n = 1, \dots, N_I.$
- \*) Legendre-expansion:  $F(\{y_{\ell'}, w_{\ell'}\})(x) := \sum_{\ell' \leq \mathcal{L}_{\text{cut}}} (y_{\ell'} + iw_{\ell'}) P_{\ell'}(x)$
- \*)  $\tilde{A}_{L+k}(\{y_{\ell'}, w_{\ell'}\}) = \sum_{\ell'=k}^{\min(2L+k, \mathcal{L}_{\text{cut}})} (y_{\ell'} + iw_{\ell'}) \sum_{m=|L+k-\ell'|}^L \langle \ell', 0; \ell, 0 | m, 0 \rangle^2 A_m,$   
 $\forall k = 1, \dots, K \Rightarrow K \equiv \mathcal{L}_{\text{cut}}.$

## Functional problem: solution Ansatz

- \*) Discr.:  $\Delta x \equiv \frac{2}{N_I}; x_n := -1 + \left( \frac{1+2(n-1)}{2} \right) \Delta x, \forall n = 1, \dots, N_I.$
- \*) Legendre-expansion:  $F(\{y_{\ell'}, w_{\ell'}\})(x) := \sum_{\ell' \leq \mathcal{L}_{\text{cut}}} (y_{\ell'} + iw_{\ell'}) P_{\ell'}(x)$
- \*)  $\tilde{A}_{L+k}(\{y_{\ell'}, w_{\ell'}\}) = \sum_{\ell'=k}^{\min(2L+k, \mathcal{L}_{\text{cut}})} (y_{\ell'} + iw_{\ell'}) \sum_{m=|L+k-\ell'|}^L \langle \ell', 0; \ell, 0 | m, 0 \rangle^2 A_m,$   
 $\forall k = 1, \dots, K \Rightarrow K \equiv \mathcal{L}_{\text{cut}}.$

Minimize the quantity

$$\begin{aligned} W_{\mathcal{L}}(\{y_{\ell'}, w_{\ell'}\}) := & \sum_{\{x_n\}} \left( \operatorname{Re}[F(\{y_{\ell'}, w_{\ell'}\})(x_n)]^2 + \operatorname{Im}[F(\{y_{\ell'}, w_{\ell'}\})(x_n)]^2 - 1 \right)^2 \\ & + \operatorname{Im} \left[ \tilde{A}_0(\{y_{\ell'}, w_{\ell'}\}) \right]^2 + \\ & \sum_{k=1}^K \left( \operatorname{Re} \left[ \tilde{A}_{L+k}(\{y_{\ell'}, w_{\ell'}\}) \right]^2 + \operatorname{Im} \left[ \tilde{A}_{L+k}(\{y_{\ell'}, w_{\ell'}\}) \right]^2 \right), \end{aligned}$$

starting from randomly chosen initial parameters

$$\left( y_{\ell'}^{(0)} \right)_j \in [-1, 1], \quad \left( w_{\ell'}^{(0)} \right)_j \in [-1, 1], \quad j = 1, \dots, N_{\text{MonteCarlo}}.$$

## Functional problem for a toy model

\*) Run codes using the following typical (suitable) parameter values:

- Number of grid-points  $|\{x_n\}|: N_I = 400$ .
- Truncation-order in the Legendre-expansion of  $F(x)$ :

$$\mathcal{L}_{\text{cut}} = 20, \dots, 100$$

$\equiv K$  = number of higher p.w.'s on whom requirement (ii) is imposed.

- Number of initial conditions:  $N_{\text{MonteCarlo}} = 50, \dots, 100, \dots$

## Functional problem for a toy model

\*) Run codes using the following typical (suitable) parameter values:

- Number of grid-points  $|\{x_n\}|: N_I = 400$ .
- Truncation-order in the Legendre-expansion of  $F(x)$ :

$$\mathcal{L}_{\text{cut}} = 20, \dots, 100$$

$\equiv K =$  number of higher p.w.'s on whom requirement (ii) is imposed.

- Number of initial conditions:  $N_{\text{MonteCarlo}} = 50, \dots, 100, \dots$

\*) Try a toy model truncated at the  $D$ -waves, i.e.  $L = 2$  (arbitr. units!):

$$\begin{aligned} A(x) &= \sum_{\ell \leq 2} (2\ell + 1) A_\ell P_\ell(x) = A_0 + 3A_1 P_1(x) + 5A_2 P_2(x) \\ &= 5. + 3(0.4 + 0.3i)P_1(x) + 5(0.02 + 0.01i)P_2(x). \end{aligned}$$

## Functional problem for a toy model

\*) Run codes using the following typical (suitable) parameter values:

- Number of grid-points  $|\{x_n\}|: N_I = 400$ .
- Truncation-order in the Legendre-expansion of  $F(x)$ :

$$\mathcal{L}_{\text{cut}} = 20, \dots, 100$$

$\equiv K =$  number of higher p.w.'s on whom requirement (ii) is imposed.

- Number of initial conditions:  $N_{\text{MonteCarlo}} = 50, \dots, 100, \dots$

\*) Try a toy model truncated at the  $D$ -waves, i.e.  $L = 2$  (arbitr. units!):

$$\begin{aligned} A(x) &= \sum_{\ell \leq 2} (2\ell + 1) A_\ell P_\ell(x) = A_0 + 3A_1 P_1(x) + 5A_2 P_2(x) \\ &= 5. + 3(0.4 + 0.3i)P_1(x) + 5(0.02 + 0.01i)P_2(x). \end{aligned}$$

\*) Toy model has **Gersten**-decomposition  $A(x) = \lambda(x - \alpha_1)(x - \alpha_2)$ , with complex normalization  $\lambda = 0.15 + 0.075$  and two roots

$$\alpha_1 = -7.05858 - 4.63163i, \quad \alpha_2 = -1.74142 + 3.03163i.$$

## Toy model: Gersten-rotations

\*) 2 roots  $\{\alpha_1, \alpha_2\}$  result in  $2^2 = 4$  discrete ambiguities

$$\pi_0 = \{\alpha_1, \alpha_2\}, \pi_1 = \{\alpha_1^*, \alpha_2\}, \pi_2 = \{\alpha_1, \alpha_2^*\}, \pi_3 = \{\alpha_1^*, \alpha_2^*\}.$$

## Toy model: Gersten-rotations

\*) 2 roots  $\{\alpha_1, \alpha_2\}$  result in  $2^2 = 4$  discrete ambiguities

$$\pi_0 = \{\alpha_1, \alpha_2\}, \pi_1 = \{\alpha_1^*, \alpha_2\}, \pi_2 = \{\alpha_1, \alpha_2^*\}, \pi_3 = \{\alpha_1^*, \alpha_2^*\}.$$

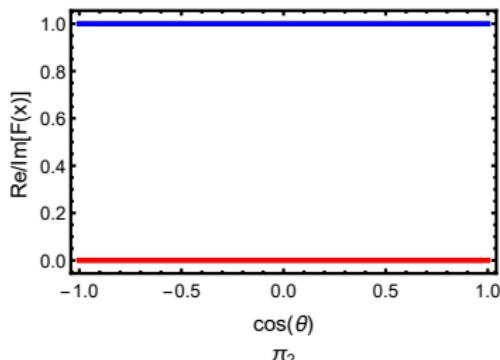
\*) 4 rotations  $e^{i\varphi_0(x)} = 1, e^{i\varphi_1(x)}, e^{i\varphi_2(x)}, e^{i\varphi_3(x)}$ , generating the discrete ambiguities, are defined by

$$F_n(x) = e^{i\varphi_n(x)} = \frac{(x - \pi_n[\alpha_1])(x - \pi_n[\alpha_2])}{(x - \alpha_1)(x - \alpha_2)}$$

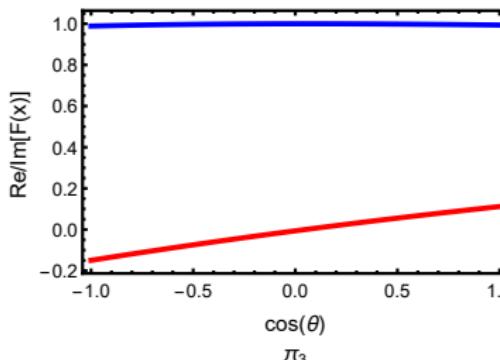
# Toy model: Gersten-rotations

Phases  $e^{i\varphi_0(x)} = 1, e^{i\varphi_1(x)}, e^{i\varphi_2(x)}, e^{i\varphi_3(x)}$ , plotted versus  $x = \cos \theta$ :

$\pi_0$

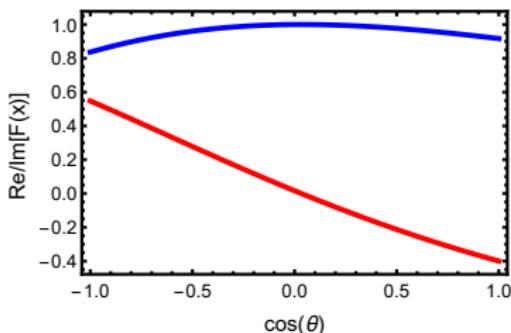


$\pi_1$

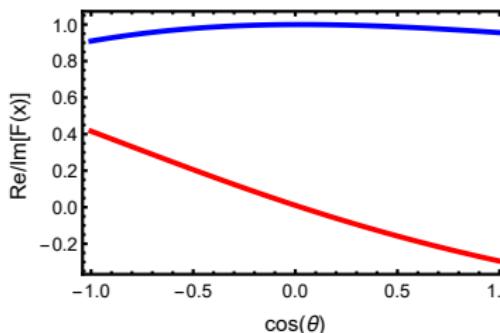


Real part

$\pi_2$



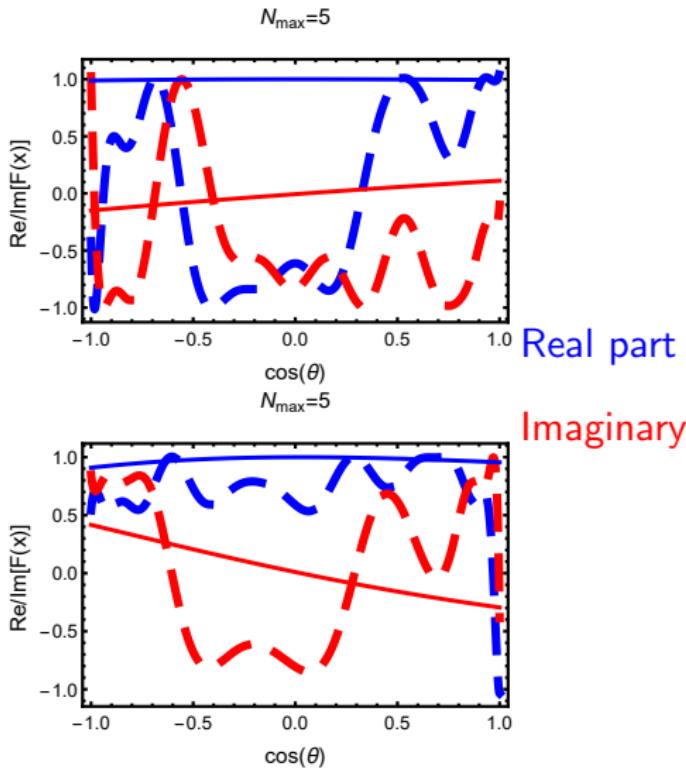
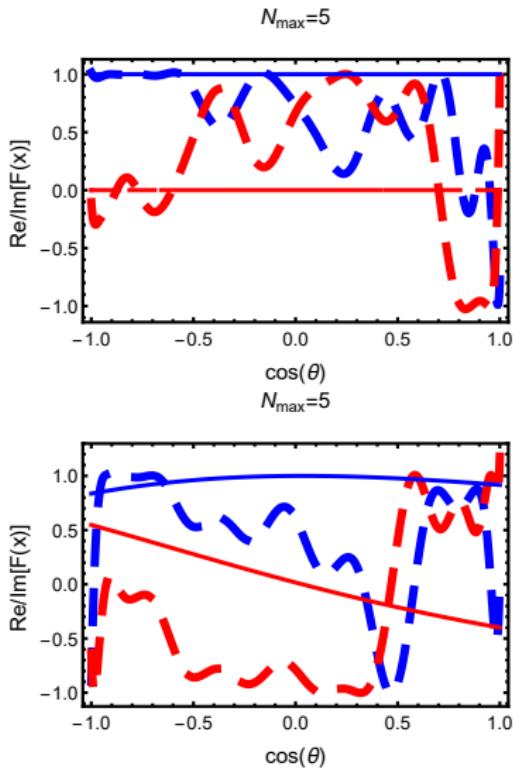
$\pi_3$



Imaginary part

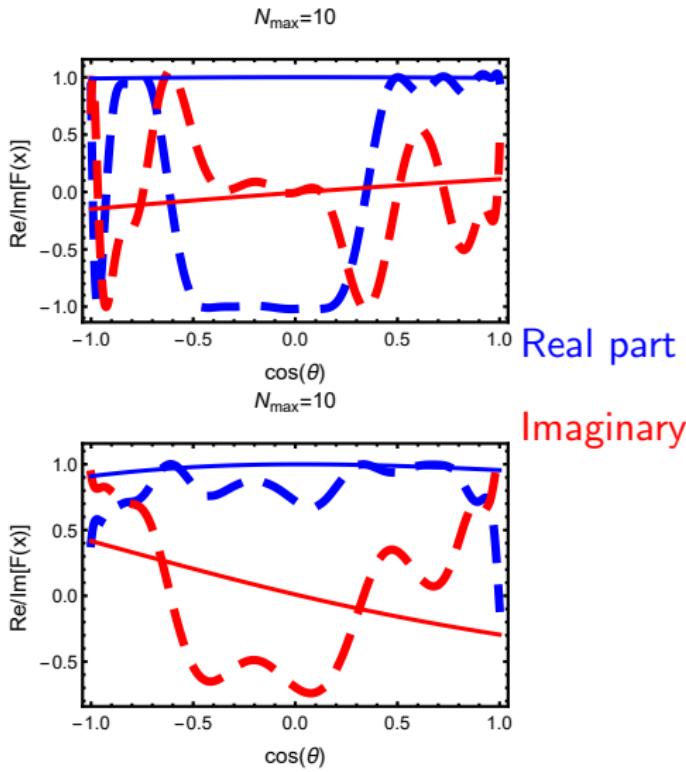
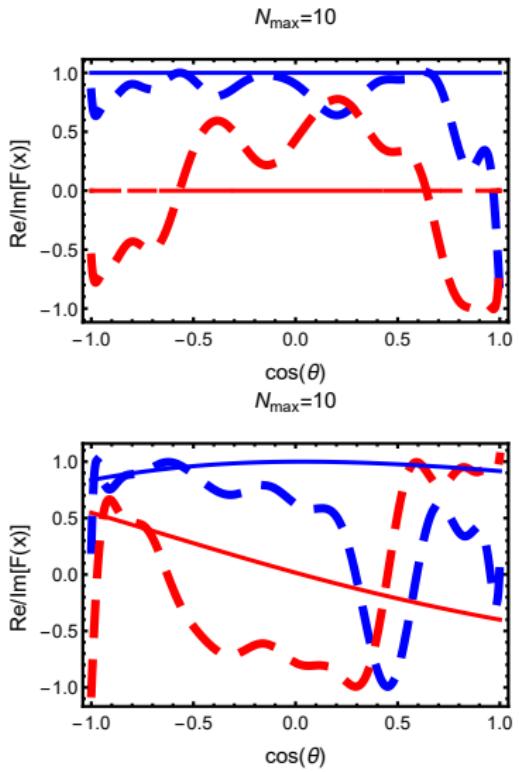
# Functional problem: results

$W$ -min.'s starting from 4 different random functions;  $N_{\max}$ : iterations



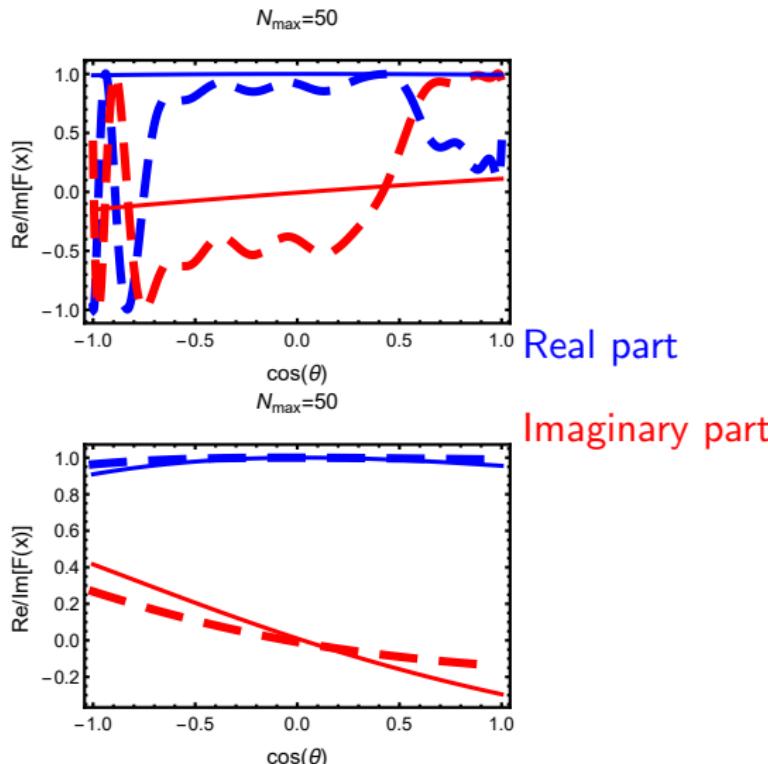
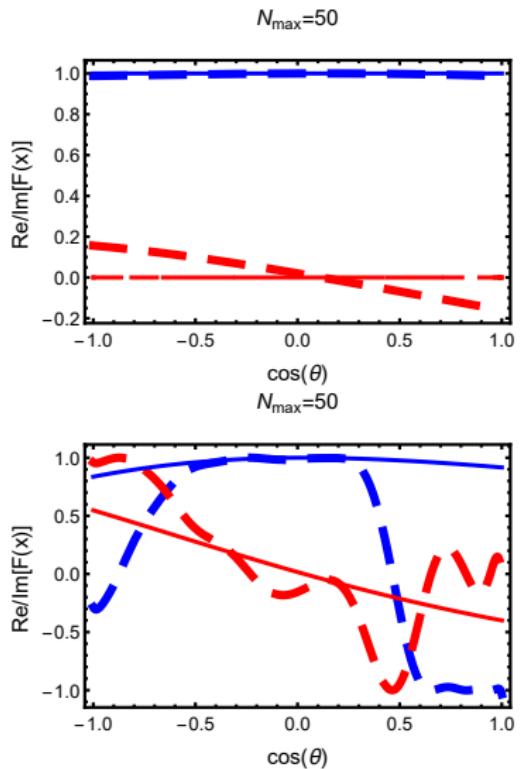
# Functional problem: results

$W$ -min.'s starting from 4 different random functions;  $N_{\max}$ : iterations



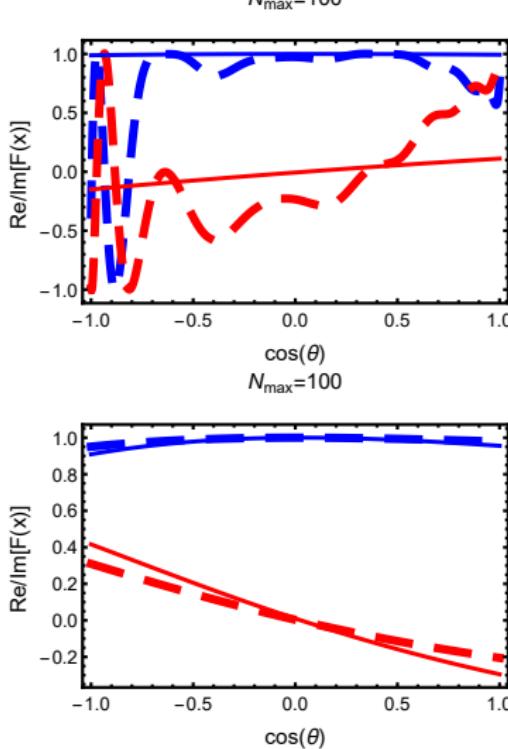
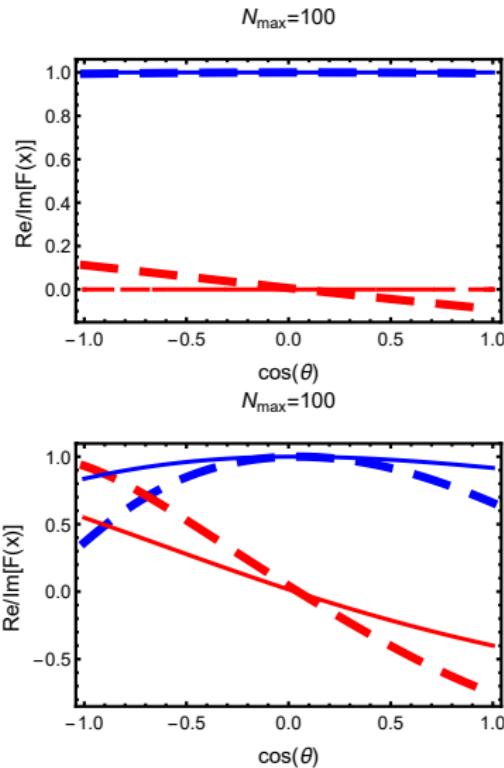
# Functional problem: results

$W$ -min.'s starting from 4 different random functions;  $N_{\max}$ : iterations

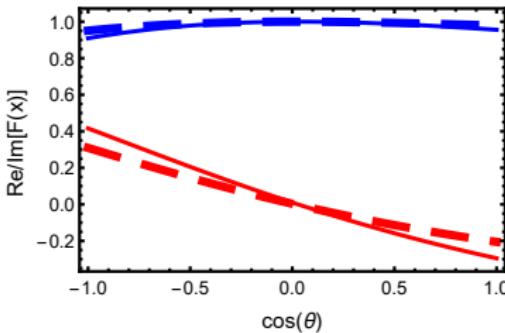


# Functional problem: results

$W$ -min.'s starting from 4 different random functions;  $N_{\max}$ : iterations

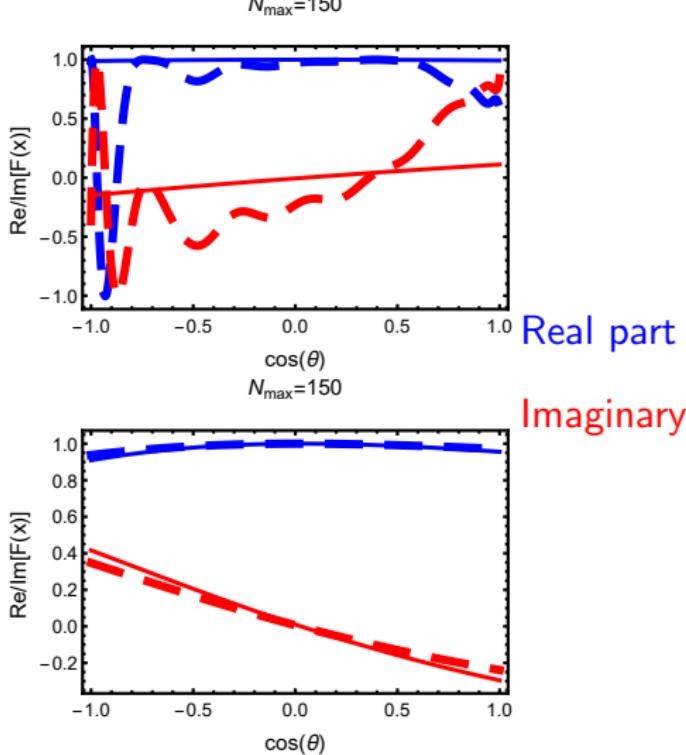
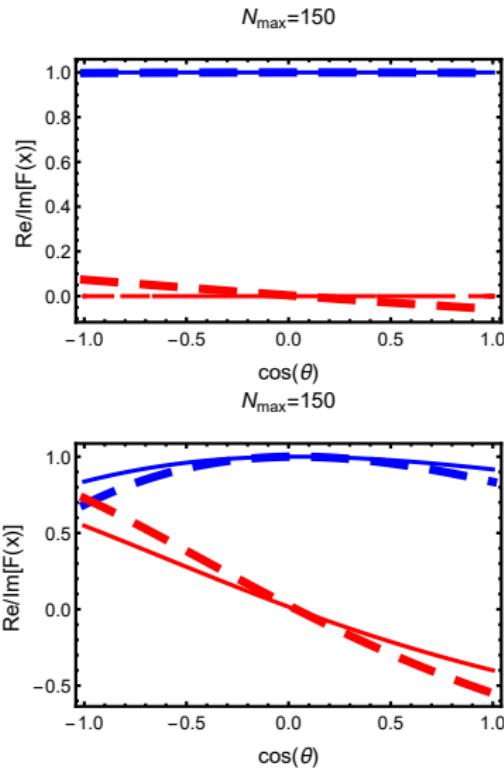


**Imaginary part**



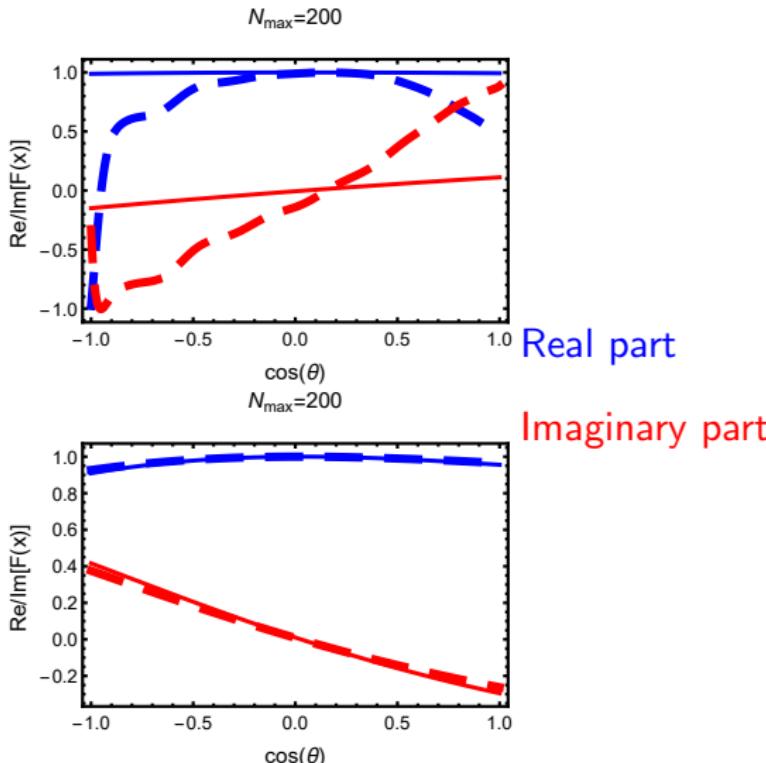
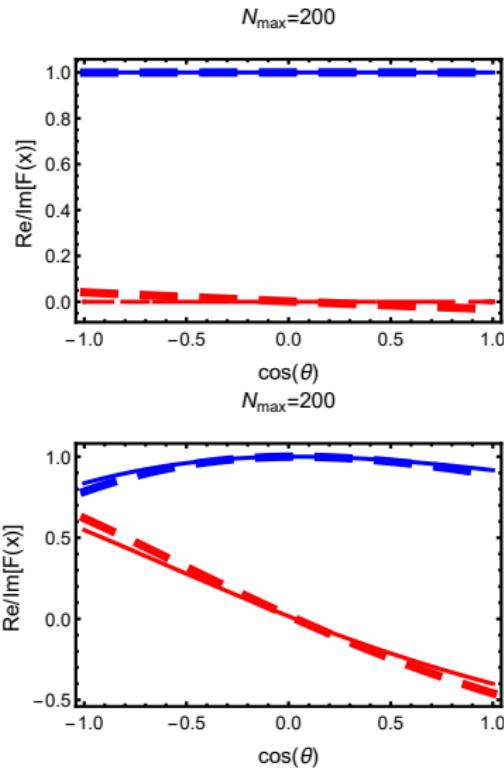
# Functional problem: results

$W$ -min.'s starting from 4 different random functions;  $N_{\max}$ : iterations



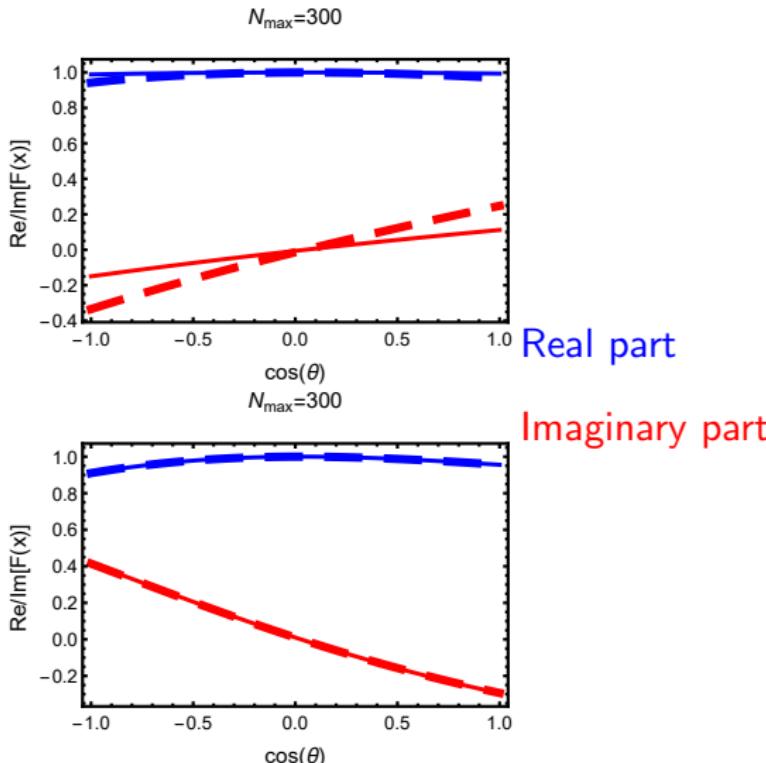
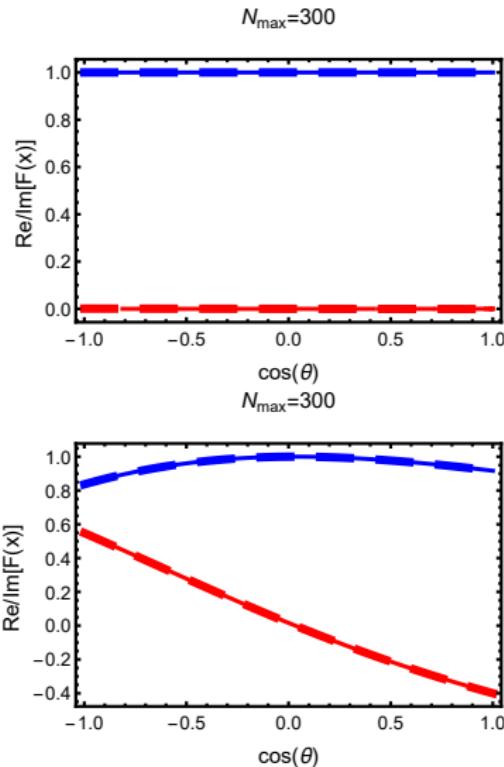
# Functional problem: results

$W$ -min.'s starting from 4 different random functions;  $N_{\max}$ : iterations



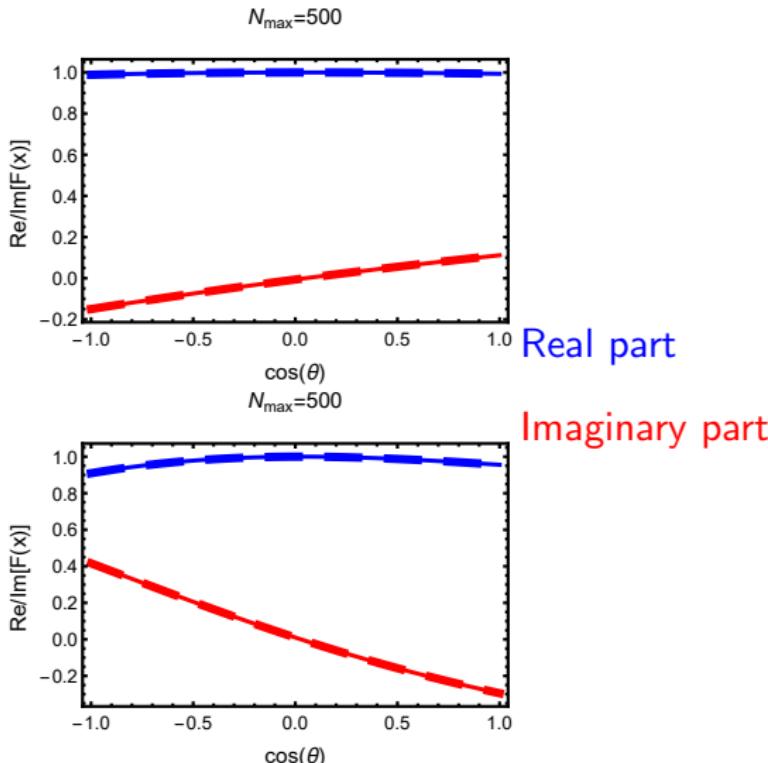
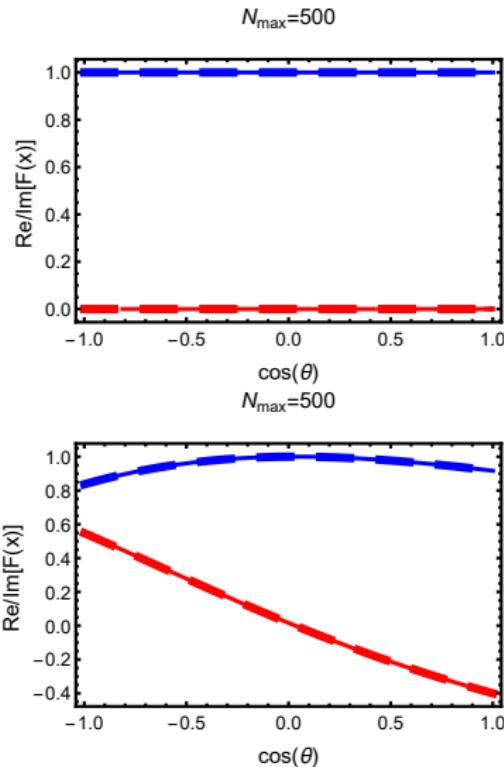
# Functional problem: results

$W$ -min.'s starting from 4 different random functions;  $N_{\max}$ : iterations



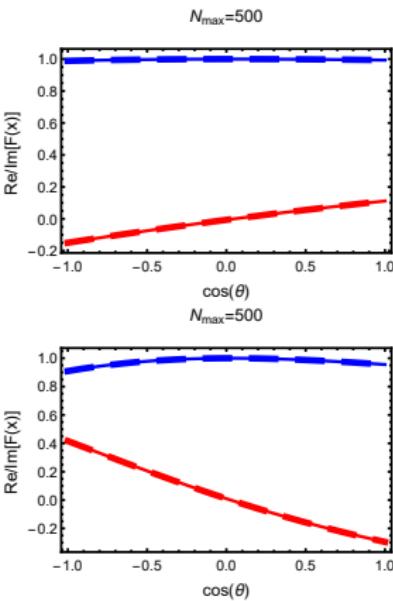
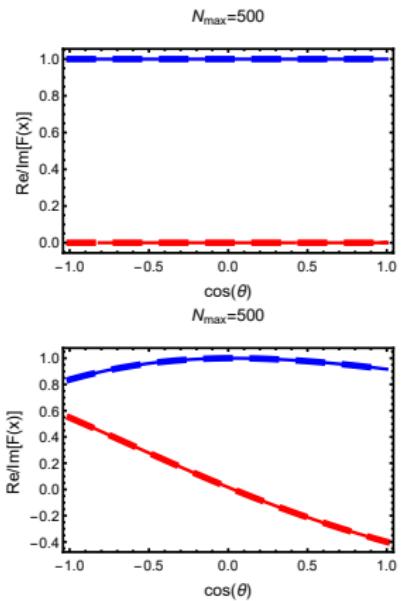
# Functional problem: results

$W$ -min.'s starting from 4 different random functions;  $N_{\max}$ : iterations



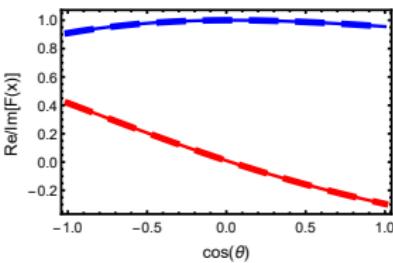
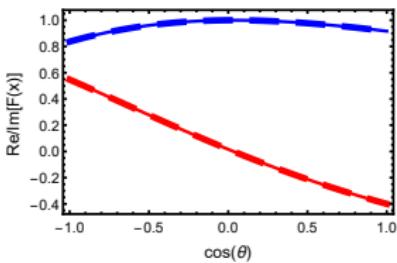
# Functional problem: results

$\mathbf{W}$ -min.'s starting from 4 different random functions;  $N_{\max}$ : iterations



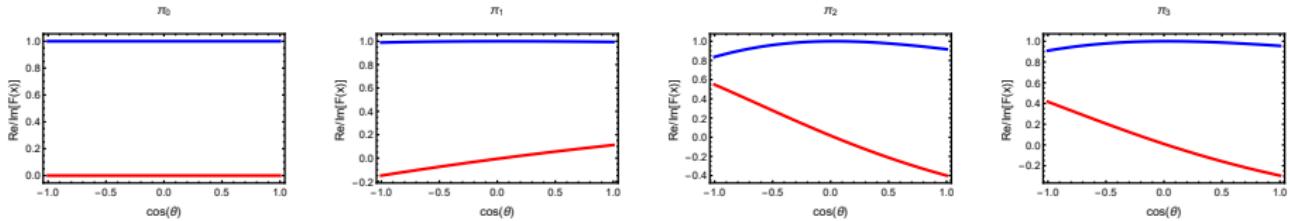
Real part

Imaginary part



→ The Gersten-phases are the only minima of  $\mathbf{W}$  found, which are consistent with zero up to a very good approximation (i.e. with values within the range  $10^{-29}, \dots, 10^{-30}$  in A.U.)!!

# Toy-model: Legendre-series' of Gersten-phases



$L_k$	$e^{i\varphi_0(x)}$	$e^{i\varphi_1(x)}$	$e^{i\varphi_2(x)}$	$e^{i\varphi_3(x)}$
$L_0$	1	$0.997 - 0.01049i$	$0.95864 + 0.03697i$	$0.97741 + 0.02781i$
$L_1$	0	$0.00182 + 0.13038i$	$0.02769 - 0.48277i$	$0.01563 - 0.35988i$
$L_2$	0	$-0.00581 - 0.00852i$	$-0.08227 + 0.03939i$	$-0.04507 + 0.03429i$
$L_3$	0	$0.00068 + 0.00028i$	$0.0126 + 0.009i$	$0.00769 + 0.00427i$
$L_4$	0	$-0.00005 + 9.6 * 10^{-6}i$	$0.00029 - 0.00249i$	$0.00005 - 0.00148i$
$L_5$	0	$2.3 * 10^{-6} - 2.3 * 10^{-6}i$	$-0.00037 + 0.00015i$	$-0.00021 + 0.00011i$
$L_6$	0	$-4.4 * 10^{-8} + 2.1 * 10^{-7}i$	$0.00005 + 0.00004i$	$0.00003 + 0.00002i$
$L_7$	0	$-4.98 * 10^{-9} - 1.3 * 10^{-8}i$	$1.6 * 10^{-6} - 9.1 * 10^{-6}i$	$4.0 * 10^{-7} - 5.5 * 10^{-6}i$
$L_8$	0	$7.0 * 10^{-10} + 4.9 * 10^{-10}i$	$-1.3 * 10^{-6} + 4.5 * 10^{-7}i$	$-7.6 * 10^{-7} + 3.5 * 10^{-7}i$

# Omelaenko's warning about angle-dependent phase

The large amount of experimental information which is needed for the complete experiment does not allow one, however, to obtain values of partial amplitudes  $F_i$  from model assumptions. In fact, in a complete

free

experiment the amplitudes  $F_i$  are determined with accuracy to the transformation

$$F_i(E_\gamma, \theta) = \exp(i\varphi(E_\gamma, \theta)) F_i(E_\gamma, \theta),$$



where  $\varphi(E_\gamma, \theta)$  is an independent real function. By choosing  $\varphi(E_\gamma, \theta)$  one can vary the angular distributions of the amplitudes  $F_i$ , although the observables remain unchanged. Going over then to multipole expansions, one obtains as a result various sets of partial amplitudes differing both in the number of excited waves and in their magnitudes.

Killer  
Argument

In a multipole analysis with  $l \leq L$  the uncertainty in the phase manifests itself as an ambiguity in the choice of  $L$ . In the amplitude corresponding to the solution with some  $L$  one can also introduce a phase depending arbitrarily on angle, and the number of terms in the multipole expansions then changes. Having this in mind, obviously it is expedient to use the smallest value  $L$  for which one achieves a description of the experimental data.



Warning written on  
[Omelaenko (1981), page 6]