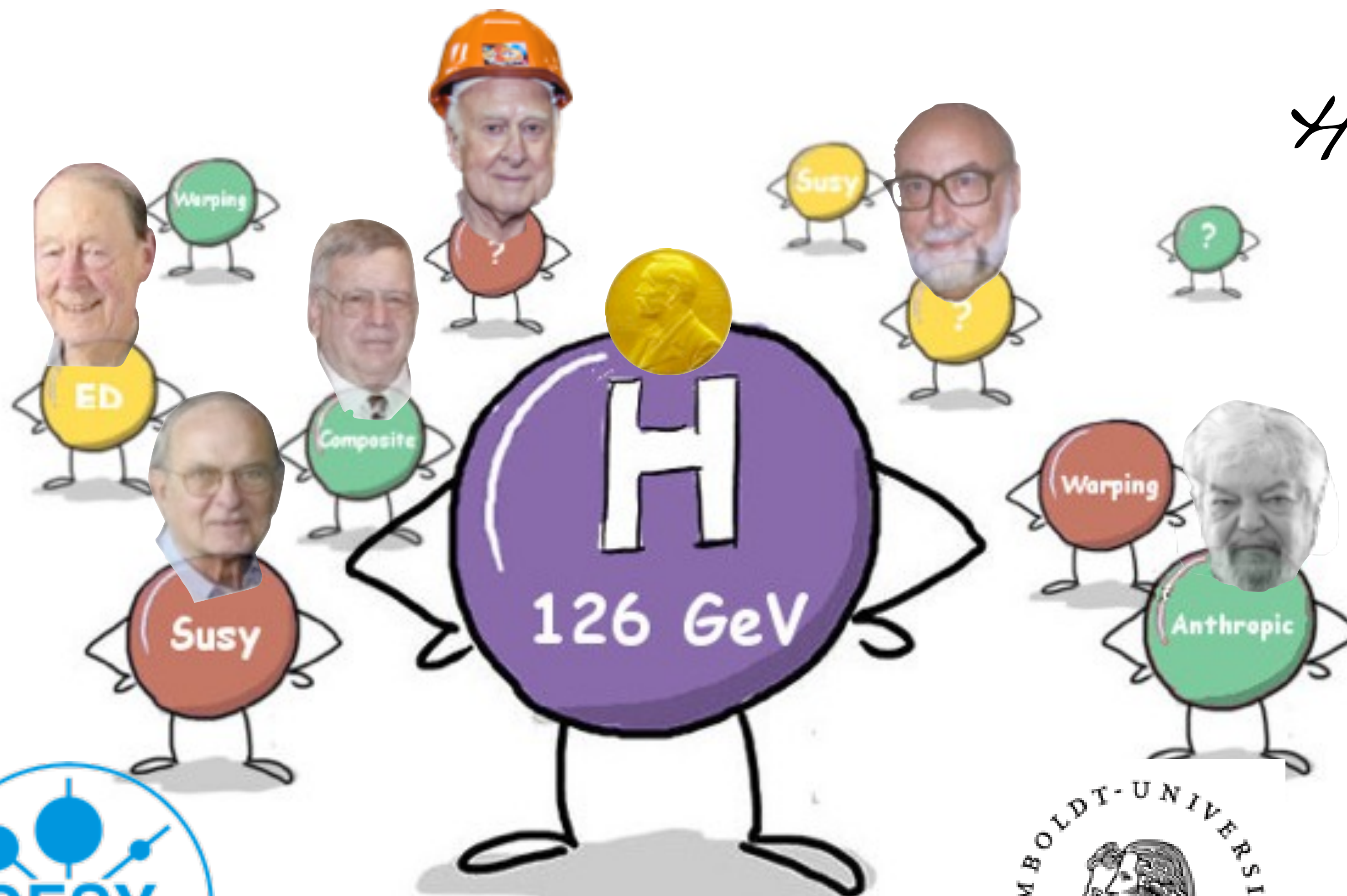
SM, EWSB & Higgs

MITP Summer School 2017

Joint Challenges for Cosmology and Colliders



Homework & Exercises

Christophe Grojean
DESY (Hamburg)
Humboldt University (Berlin)
(christophe.grojean@desy.de)

Ex 0: Order of magnitude estimates

- **Special relativity:**

  1. How long does it take for a photon to travel from your feet to your brain?
     What is the maximal frequency at which the brain can function?
     When is AI going to take over humans? (all this because of finite light speed and human size!)

  2. Estimate the energy of the cosmic rays given that the lifetime of a muon is about 1μs

- **Quantum field theory:**

  1. By dimensional analysis, derive the expression of the Casimir pressure between two neutral plates assumed to be ideal conductors and uncharged. For which distance is this pressure of the order of 1 atm?

- **Particle physics:**

  1. A 1cm^3 cube of ice melts in about 40 minutes under the sun. Compute the volume of oil to burn a 1cm-thick ice cap surrounding the sun at a distance of 150 million kms (the effects of the atmosphere will be neglected). What do you conclude concerning the origin of the energy radiated by the Sum?

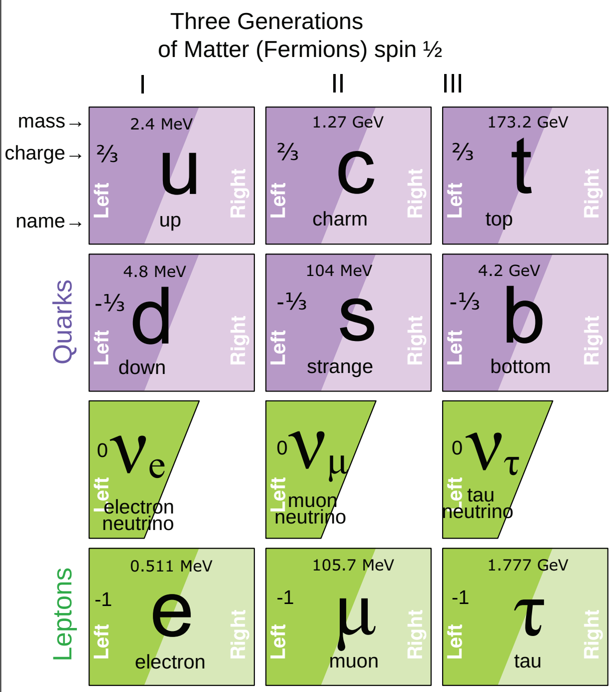
Ex 1: Mass and ħ dimensions

in natural units (c=ħ=1), the couplings are dimensionless numbers
But it is often useful to remember that couplings are dimensionful quantities

- Compute the dimensions of spin-0, -1/2, -1 fields in 4D in units of ħ and length
- Compute the dimensions of a gauge coupling, a Yukawa coupling, a Higgs quartic coupling
- From dimensional arguments, derive the functional form of the $V_L V_L$ scattering amplitude
- In the MSSM, the Higgs quartic coupling is related to the gauge couplings. Based on dimensional arguments, find the functional form of this relation

we will see in the lectures that h-dimensions are particularly useful to derive
the power-counting of various higher-dimensional operators in Effective Field Theories

Ex 2: SM Matter - Anomaly Cancellation



- Count the number of quarks and leptons
- Check the cancelation of the gauge anomalies (Tr T^a^3)
- Check the cancelation of the gravitational anomalies (Tr T^a)
- Check that the baryon and lepton numbers are anomalous but B-L is anomaly-free

- Within the SM, the anomaly cancelation fixes the relative electric charges of the leptons and quarks. Show that with the addition of a right-handed neutrino, this ratio of electric charges is free. Still the cancelation of the anomaly imposes that the proton is electrically neutral

Recap: a symmetry is anomaly free if the trace Tr($T^a\{T^b,T^c\}$) for any generators $T^{a,b,c}$ is vanishing. In the SM, the only non-trivial anomaly coefficients are the ones associated to SU(2)^2 U(1), U(1)^3 and U(1)

Ex 3: rho parameter - Higgs self-coupling

General expression of the ρ parameter

If $SU(2)_L xU(1)_Y$ is broken not only through a doublet, but also through a collection of scalar fields in the $2s_i+1$ representation of $SU(2)_L$, carrying a hypercharge y_i and acquiring a vev v_i, show that the ρ parameter is now given by

$$\rho = \frac{\sum_i (s_i(s_i+1) - y_i^2) v_i^2}{\sum_i 2 y_i^2 v_i^2}$$

Higgs self-coupling

1/ In the SM, the Higgs potential takes the following form

$$V(H) = \lambda \left(|H|^2 - v^2/2 \right)^2$$

The two free parameters of this potential are fixed by the Fermi constant and the Higgs mass. In terms of these two physical quantities, compute the Higgs boson self-interactions after expanding around the vacuum.

2/ Add a non-renormalizable term to the potential:

$$\frac{c_6}{\Lambda^2} \left(|H|^2 - v^2/2 \right)^3$$

Compute the corrections to the Higgs self-couplings

Ex 4: Perturbative unitarity - Partial Waves

for a 2-to-2 process, the (angular) differential cross-section is related to the amplitude by

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}|\mathcal{A}|^2$$

Partial wave amplitude decomposition: the partial waves are defined by

$$\mathcal{A} = 16\pi \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta)\, a_l$$

where P_l are the & polynomials ($P_0(x) = 1,\ P_1(x) = x, P_2(x) = 3x^2/2 - 1/2 \ldots$ & $\int_{-1}^{1} dx P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}$)

Show that $a_l = \frac{1}{32\pi}\int_{-1}^{+1} d(\cos\theta) P_l(\cos\theta)\mathcal{A}$ and $\sigma = \frac{16\pi}{s}\sum_{l=0}^{\infty}(2l+1)|a_l|^2$

Optical theorem: using the optical theorem: $\sigma = \mathrm{Im}\left(\mathcal{A}_{|\theta=0}\right)/s$

show that $|\mathrm{Re}\,(a_l)\,| \leq 1/2$

Application: WW scattering: $W^+ W^- \to W^+ W^-$ $\quad \mathcal{A} = \mathcal{A}(s) + \mathcal{A}(t)$

1/ Compute the a0 wave:

SM without a Higgs

$\mathcal{A}(s) = s/v^2$ $\qquad a_0 = \frac{s}{8\pi v^2}$

SM with a Higgs

$\mathcal{A}(s) = (s - s^2/(s - m_h^2))/v^2$ $\qquad a_0 = \frac{m_h^2}{16\pi v^2}$

2/ Derive the perturbative unitarity bound:

➡ $\sqrt{s} = 2E \leq 4\sqrt{\pi} v \approx 1.7\ \mathrm{TeV}$

➡ $m_h \leq 2\sqrt{2\pi} v \approx 1.2\ \mathrm{TeV}$

Ex 5: Coleman-Weinberg potential

The general expression of the 1-loop Coleman-Weinberg potential is given by:

$$V(h) = \int \frac{d^4 k_E}{2(2\pi)^4} \mathrm{STr} \ln \left(k_E^2 + M^2(h)\right)$$

1. Compute this integral but regularizing the divergences with with a UV cutoff and obtain:

$$V(h) = -\frac{\Lambda^4}{128\pi^2}\mathrm{STr}1 + \frac{\Lambda^2}{64\pi^2}\mathrm{STr}M^2(h) + \frac{1}{64\pi^2}\mathrm{STr}M^4(h)\ln\frac{M^2(h)}{\Lambda^2}$$

2. Obtain the quadratically divergent contribution to the Higgs mass in the SM:

$$V(h) = \left(2m_W^2 + m_Z^2 + m_H^2 - 4m_t^2\right)\frac{3G_F\Lambda^2}{32\sqrt{2}\pi^2}h^2$$

Hint: You will first derive the SM mass spectrum for a background value h of the Higgs field

Ex 6: Composite Higgs potential

Ex 7: S and T from higher dim-6 operators

The oblique parameters are defined from physical observables:

$$\Delta m_W = -\frac{\alpha m_W}{4(c_0^2 - s_0^2)} S + \frac{\alpha c_0^2 m_W}{2(c_0^2 - s_0^2)} T + \frac{\alpha m_W}{8 s_0^2} U$$

$$\Delta s^2_{\text{eff}} = \frac{\alpha}{4(c_0^2 - s_0^2)} S - \frac{\alpha c_0^2 s_0^2}{c_0^2 - s_0^2} T$$

$$\Delta \Gamma_{ll} = -\frac{2(1-4s_0^2)\alpha \Gamma^0_{ll}}{(1+(1-4s_0^2)^2)(c_0^2 - s_0^2)} S + \left(1 + \frac{8(1-4s_0^2) s_0^2 c_0^2}{(1+(1-4s_0^2)^2)(c_0^2 - s_0^2)}\right) \alpha \Gamma^0_{ll}\, T$$

We are using α_{em}, G_F and m_Z as input parameters

s_{eff} is defined via the LR asymmetry in Z-decay: $A_{LR} = \frac{(-1/2 + s^2_{\text{eff}})^2 - s^4_{\text{eff}}}{(-1/2 + s^2_{\text{eff}})^2 + s^4_{\text{eff}}}$

c_0 and s_0 are SM tree-level values of the sin and cos of the weak mixing angle

Alternatively, they can be computed by looking at the gauge boson self-energies:

oblique parameters = modified propagators of $W^\pm$ and Z

$$\mathcal{L} = -\Pi_{+-}(p^2)\, W_+^\mu W_{-\,\mu} - \tfrac{1}{2}\Pi_{33}(p^2)\, W_3^\mu W_{3\,\mu} - \Pi_{3B}(p^2)\, W_3^\mu B_\mu - \tfrac{1}{2}\Pi_{BB}(p^2)\, B^\mu B_\mu$$

$$\widehat{S} = \frac{\alpha_{em}}{4 s_W^2} S = \frac{g}{g'} \Pi'_{3B}(0) \qquad \widehat{T} = \alpha_{em} T = \frac{(\Pi_{33}(0) - \Pi_{+-}(0))}{m_W^2} \qquad \widehat{U} = -\frac{\alpha_{em}}{4 s_W^2} U = \Pi'_{+-}(0) - \Pi'_{33}(0)$$

Ex 7: S and T from higher dim-6 operators

We want to compute the oblique parameters
when the following dimension-6 operator is added to the SM

$$\mathcal{L} = \frac{1}{\Lambda^2} \left| H^\dagger D_\mu H \right|^2$$

1. In the unitary gauge, show that this operator gives only a correction to Π_{33} equal to

$$\Delta \Pi_{33} = -\frac{g^2 v^4}{8\Lambda^2}$$

2. Conclude that

$$T = -\frac{1}{2\sqrt{2}\alpha G_F \Lambda^2}$$

3. Write m_W, and s_{eff} in terms of the input observables: α, G_F and m_Z. And rederive the expression of the T oblique parameter:

$$\Delta m_W = \frac{c_0^2 m_W}{4\sqrt{2}(c_0^2 - s_0^2) G_F \Lambda^2} \qquad \Delta s^2_{\text{eff}} = -\frac{-s_0^2 c_0^2}{2\sqrt{2}(s_0^2 - c_0^2) G_F \Lambda^2}$$

4. Check that T measures the deviation to $\rho=1$

$$\rho = \frac{m_W^2}{c_W^2 m_Z^2} \approx 1 + \alpha T$$

Ex 7: S and T from higher dim-6 operators

We want to compute the oblique parameters
when the following dimension-6 operator is added to the SM

$$\mathcal{L} = \frac{1}{\Lambda^2} H^\dagger W_{\mu\nu} H B_{\mu\nu}$$

1. In the unitary gauge, show that this operator gives only a correction to Π_{30} equal to $\Delta\Pi'_{3B} = \frac{v^2}{\Lambda^2}$

2. Conclude that $S = \frac{4 s_W c_W}{\sqrt{2}\alpha_{em} G_F \Lambda^2}$

3. Because of the kinetic mixing, the Z and the γ are not obtained from the usual weak rotation from W_3 and B. Find the correct expressions of Z and γ.

4. The expression of e in terms of g and g' receives some corrections compared to its SM expression. Derive these corrections.

5. Write m_W in terms of the input observables: α, G_F and m_Z.
And re-derive the expression of the S oblique parameter

Ex 8: non-linear field redefinition of Higgs field

The Higgs boson doesn't have to be an elementary particle. It could be a bound state emerging from a strongly coupled sector. Below the compositeness scale of the Higgs, f, the dynamics of such a composite Higgs boson is well captured by the SM Lagrangian supplemented by a few dimension-six operators:

$$\mathcal{L} = |D_\mu H|^2 + \frac{c_H}{2f^2}\left(\partial_\mu |H|^2\right)^2 + \mu^2 |H|^2 - \lambda |H|^4 - \frac{c_6 \lambda}{3f^2}|H|^6 - y_f \left(H \bar{f}_L f_R \left(1 + \frac{c_y}{f^2}|H|^2\right) + \text{h.c.}\right)$$

1. Compute the corrections to the Higgs self-couplings to the lowest order in $\xi = v^2/f^2$
   (check that $\hat{h} = \left(1 + \frac{c_H \xi}{2}\right) h + \frac{c_H \xi}{2}\frac{h^2}{v} + \frac{c_H \xi}{6}\frac{h^3}{v^2}$ is canonically normalized)

2. Compute the coefficients a,b,c for the effective Lagrangian:

$$\mathcal{L}_{\text{EWSB}} = \frac{v^2}{4}\text{Tr}\left(D_\mu \Sigma^\dagger D_\mu \Sigma\right)\left(1 + 2a\frac{h}{v} + b\frac{h^2}{v^2}\right) - \lambda \bar{\psi}_L \Sigma \psi_R \left(1 + c\frac{h}{v}\right)$$

3. What is the high-energy behavior of the amplitudes for WW→WW and WW→ hh?

4. Compute the corrections to the decay width of the Higgs into a pair of fermions

5. Compute the corrections to the decay width of the Higgs into a pair of bosons

Ex 9: Beta function - unification

The one-loop β function giving the running of the coupling constant of an SU(N) gauge symmetry is given by

$$\beta = \frac{dg}{d\log\mu} = -\frac{1}{16\pi^2} b_0 g^3 \qquad \text{ie} \qquad \frac{d\alpha}{d\log\mu} = -\frac{1}{2\pi} b_0 \alpha^2$$

where the coefficient b_0 is computed to

$$b_0 = \frac{11}{3} T_2(\texttt{spin-1}) - \frac{2}{3} T_2(\texttt{chiral spin-1/2}) - \frac{1}{3} T_2(\texttt{complex spin-0})$$

$T_2(R)$ is defined from the traces of the product of two generators of SU(N) in the representation R

$$\mathrm{Tr}\left(T^a(R) T^b(R)\right) = T_2(R)\delta^{ab}$$

1. What should be the sign of b_0 to get an asymptotically free theory?
2. Compute $\alpha(\mu)$ from $\alpha(\mu_0)$
3. Compute b_0 for $U(1)_{em}$ with a single massive electron.

   What is the value of the Landau pole of QED, ie the energy at which α_{em} blows up?

   At which energy do we get a 1% departure from $\alpha(0)$?
4. Compute the coefficients b_0 for the 3 gauge groups of the SM
5. Compute the coefficients b_0 for the 3 gauge groups of the MSSM
6. In N=4 supersymmetric gauge theories, a supermultiplet contains 1 spin-1 field, 4 spin-1/2 chiral fields and 6 real spin-0 fields in the adjoint representation. Compute the coefficient b_0 for that theory? What do you conclude?

Ex 10: AdS$_5$ and Randall-Sundrum

- Consider a conformally flat space-time: $ds^2 = \Omega^2(x)\eta_{MN}dx^M dx^N$
  - Show that the Einstein tensor is given by (D is the total space-time dimension)

$$G_{MN} = (2-D)\frac{\partial_M\partial_N\Omega}{\Omega} + (D-2)\frac{\partial_P\partial^P\Omega}{\Omega}\eta_{MN} + 2(D-2)\frac{\partial_M\Omega\partial_N\Omega}{\Omega^2} + \frac{(D-5)(D-2)}{2}\frac{\partial_P\Omega\partial^P\Omega}{\Omega^2}\eta_{MN}$$

  - In the specific case of a 5D space-time when the conformal factor depends only on z:
    - the metric writes: $ds^2 = \Omega^2(z)(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2)$
    - show that the only non-vanishing component of the Einstein tensor are

$$G_{\mu\nu} = -3\frac{\Omega''}{\Omega}\eta_{\mu\nu} \qquad G_{zz} = 6\left(\frac{\Omega'}{\Omega}\right)^2$$

  - Consider a 5D space-time with a (negative) vacuum energy: $\int d^5x\sqrt{g}\left(-M_5^3\mathcal{R}+\Lambda\right)$
  - Show that a solution of the Einstein equation is the anti-de Sitter metric

$$\Omega = \frac{R}{z} \qquad \text{with} \qquad R = \sqrt{12M_5^3/\Lambda}$$

  - Note that the dilation of the 4D space-time can be compensated by a translation along the 5th dimension to leave the metric invariant
  - If the space has two boundaries (at z=R and z=R'), find the values of the vacuum energy localized at the two boundaries to satisfy the boundary conditions (continuity of the Einstein equations. at the boundaries)