Hadronic contributions to $(g-2)_\mu$ from Lattice QCD: Results from the Mainz group

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PRISMA

SFB 1044

THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

The Mainz $(g-2)_\mu$ project

Collaborators:

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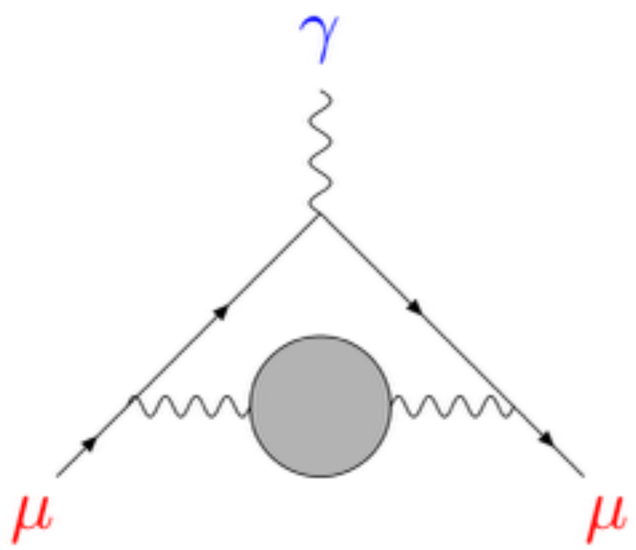
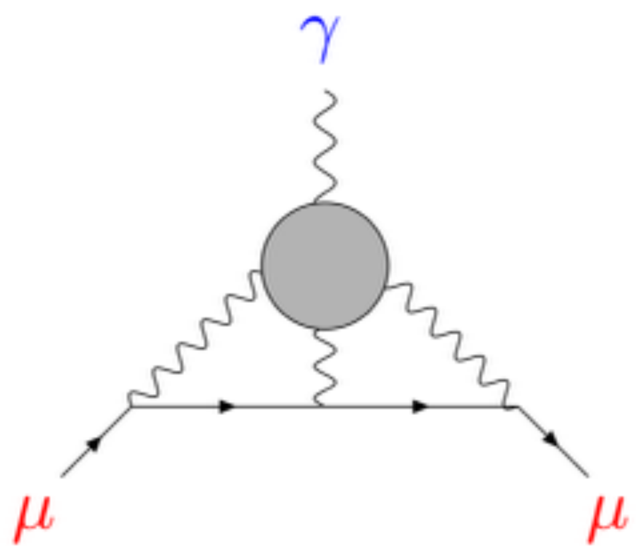
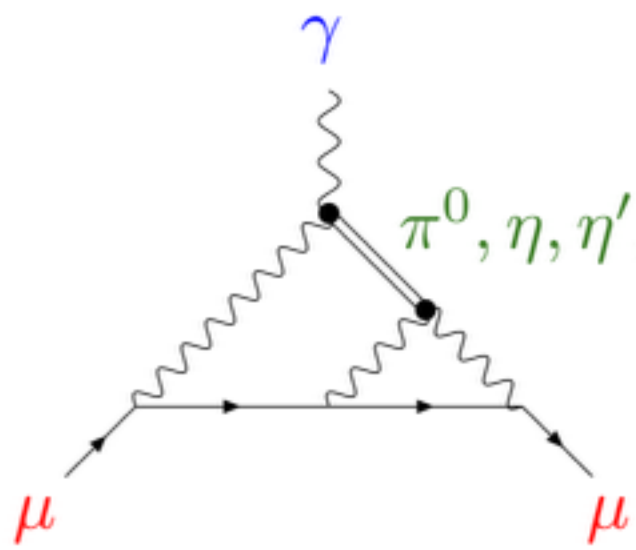
M. Della Morte, A. Francis, J. Green, V. Gülpers, B. Jäger, G. Herdoíza



- Direct determinations of LO $a_\mu^{\rm hvp}$



- Exact QED kernel
- Forward scattering amplitude



- Transition form factor for $\pi^0 \to \gamma^*\gamma^*$

Hadronic Vacuum Polarisation

Lattice QCD approach to HVP

* Convolution integral over Euclidean momenta: *[Lautrup & de Rafael; Blum]*

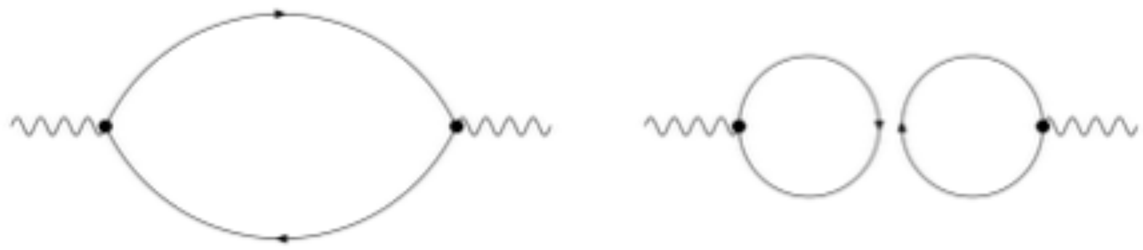
$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2\, f(Q^2)\hat{\Pi}(Q^2), \quad \hat{\Pi}(Q^2) = 4\pi^2\left(\Pi(Q^2) - \Pi(0)\right)$$

* Weight function $f(Q^2)$ strongly peaked near muon mass

* Accurate determination of $\Pi(Q^2)$ near $Q^2 \approx 0$

* Control effects of finite volume; $m_\pi L \geq 4$ not sufficient

* Include quark-disconnected diagrams:



* Include isospin breaking: $m_u \neq m_d$, QED corrections

Lattice QCD approach to HVP

* **Direct method**: determine $\Pi(Q^2)$ from VP tensor

$$\Pi_{\mu\nu}(Q) = \int d^4x \, e^{iQ\cdot x} \left\langle J_\mu(x) J_\nu(0) \right\rangle \equiv (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)\Pi(Q^2)$$

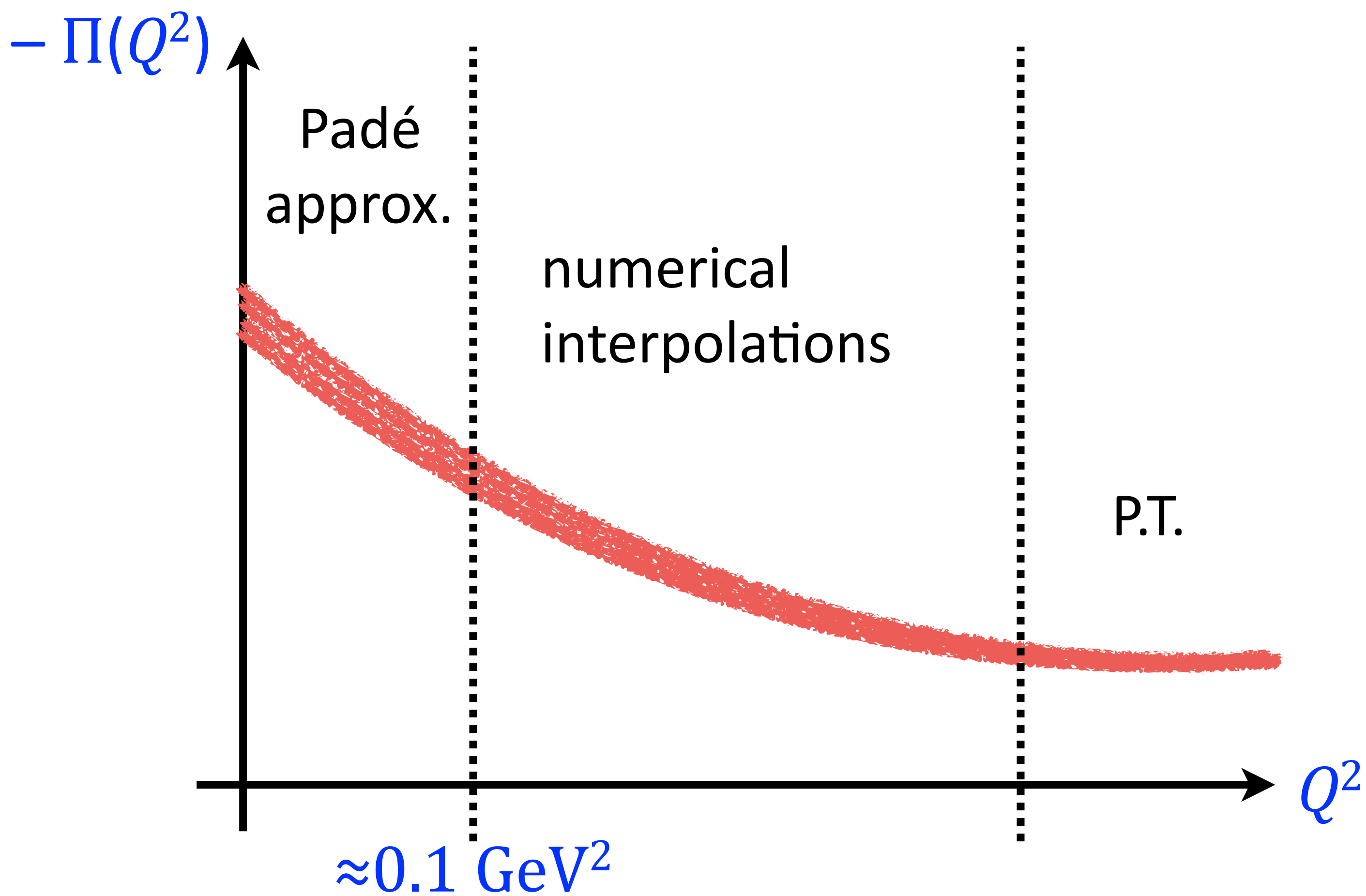
$$J_\mu = \tfrac{2}{3}\bar{u}\gamma_\mu u - \tfrac{1}{3}\bar{d}\gamma_\mu d - \tfrac{1}{3}\bar{s}\gamma_\mu s + \ldots$$

* Obtain Padé representation of $\Pi(Q^2)$ from fits for

$$Q^2 \leq Q^2_{\rm cut} \approx 0.1 - 0.5\,\mathrm{GeV}^2$$

"Hybrid method"

[Golterman, Maltman & Peris 2014]



Lattice QCD approach to HVP

* **Time-momentum representation:** *[Bernecker & Meyer 2011]*

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{f}(x_0)\, G(x_0), \quad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle$$

$$\tilde{f}(x_0) = 4\pi^2 \int_0^\infty dQ^2\, f(Q^2) \left[x_0^2 - \frac{4}{Q^2} \sin^2\left(\tfrac{1}{2} Q x_0\right) \right]$$

* Control long-distance behaviour of $G(x_0)$ — large statistical noise

$$G(x_0) = \begin{cases} G(x_0)_{\rm data}, & x_0 \leq x_{0,\rm cut} \\ G(x_0)_{\rm ext}, & x_0 > x_{0,\rm cut} \end{cases}$$

* $G(x_0)$ dominated by two-pion state for $x_0 \to \infty$

HVP: current data sets

CLS consortium — "Coordinated Lattice Simulations"

- $N_f = 2$ flavours of $O(a)$ improved Wilson fermions
- Three values of the lattice spacing: $a = 0.076, 0.066, 0.049$ fm
- Pion masses and volumes: $m_\pi^{\text{min}} = 185\,\text{MeV}, \quad m_\pi L > 4$
- Focus on methodology and systematics `arXiv:1705.1775`

- $N_f = 2+1$ flavours of $O(a)$ improved Wilson fermions
- Three values of the lattice spacing: $a = 0.085, 0.065, 0.050$ fm
- Pion masses and volumes: $m_\pi^{\text{min}} = 200\,\text{MeV}, \quad m_\pi L > 4$
- To be included: two more lattice spacings; physical pion mass

Hybrid Method

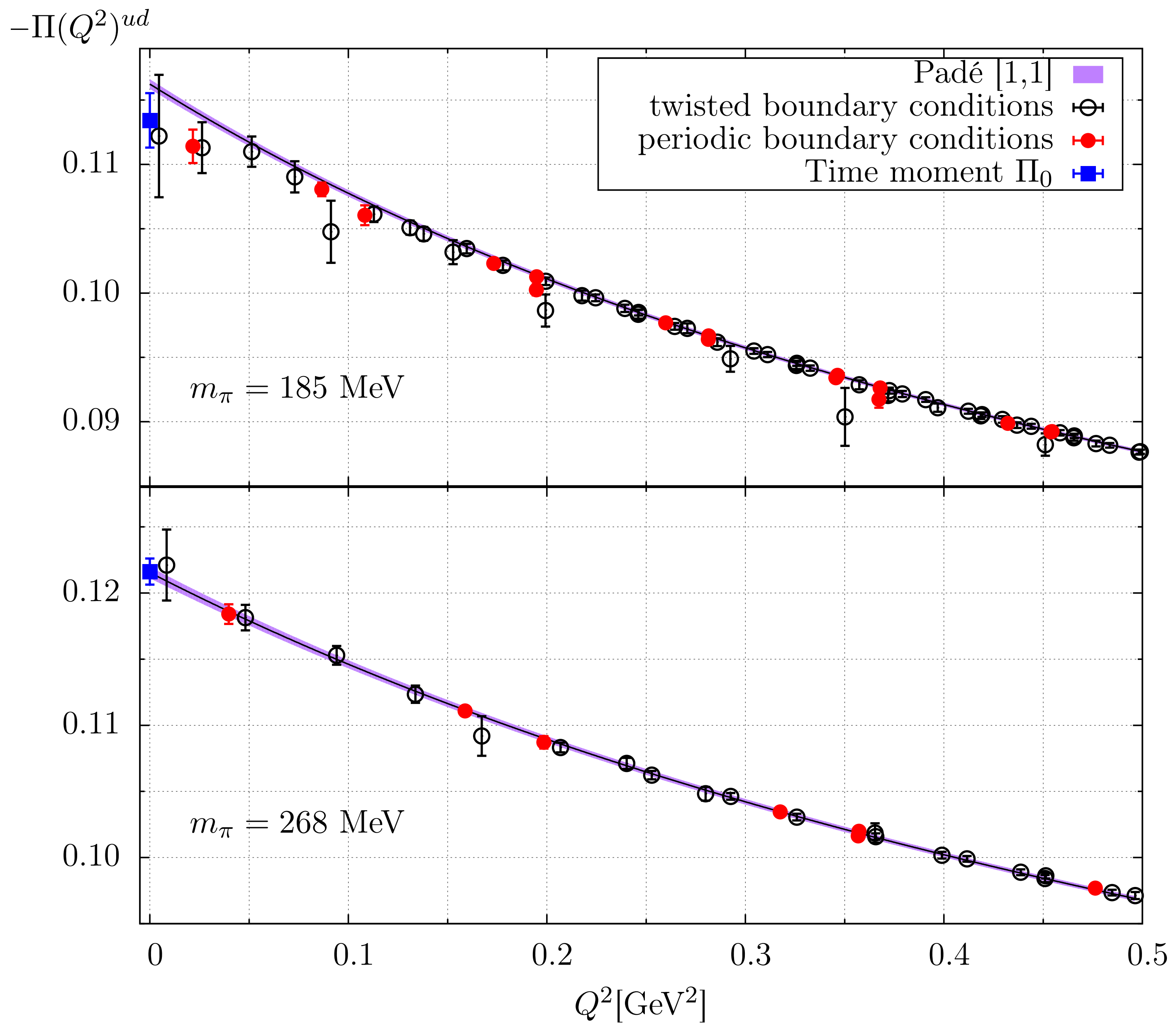
* Lattice observable:

$$a^4 \sum_f q_f^2 Z_V \sum_x \left(e^{iQ(x+a\hat{\mu}/2)} - 1\right) \left\langle V_{\mu,f}^{\text{con}}(x) V_{\nu,f}^{\text{loc}}(0) \right\rangle = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)\Pi(\hat{Q})$$

* Use twisted boundary conditions to reach smaller Q^2

  [Della Morte et al. 2011]

* Fit $\Pi(Q^2)$ to low-order Padé approximants



Time-momentum representation

* Lattice observable: $G^f(x_0) = -\frac{a^3}{3}\sum_k\sum_{\vec{x}} q_f^2\, Z_V \left\langle V_{k,f}^{\rm con}(x_0,\vec{x})V_{k,f}^{\rm loc}(0)\right\rangle$

$$(a_\mu^{\rm hvp})^f = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{f}(x_0)\, G^f(x_0)$$

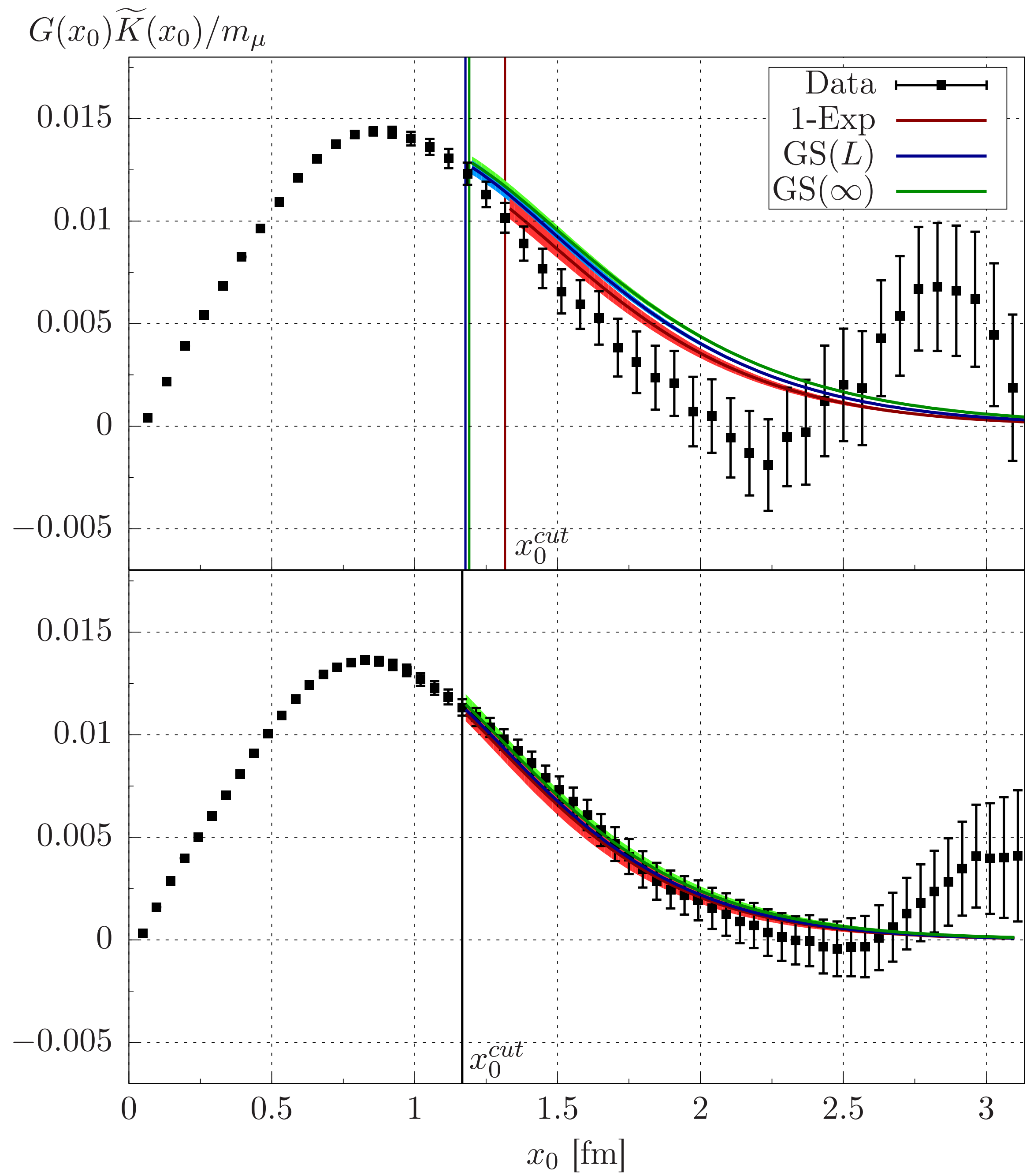
* Control tail of integrand:
  - Naive single exponential:

    $G(x_0)_{\rm ext} = A\,{\rm e}^{-m_\rho x_0}$

  - Single exponential plus 2-pion state:

    $G(x_0)_{\rm ext} = A\,{\rm e}^{-m_\rho x_0} + B\,{\rm e}^{-E_{2\pi}(\vec{p})x_0}$

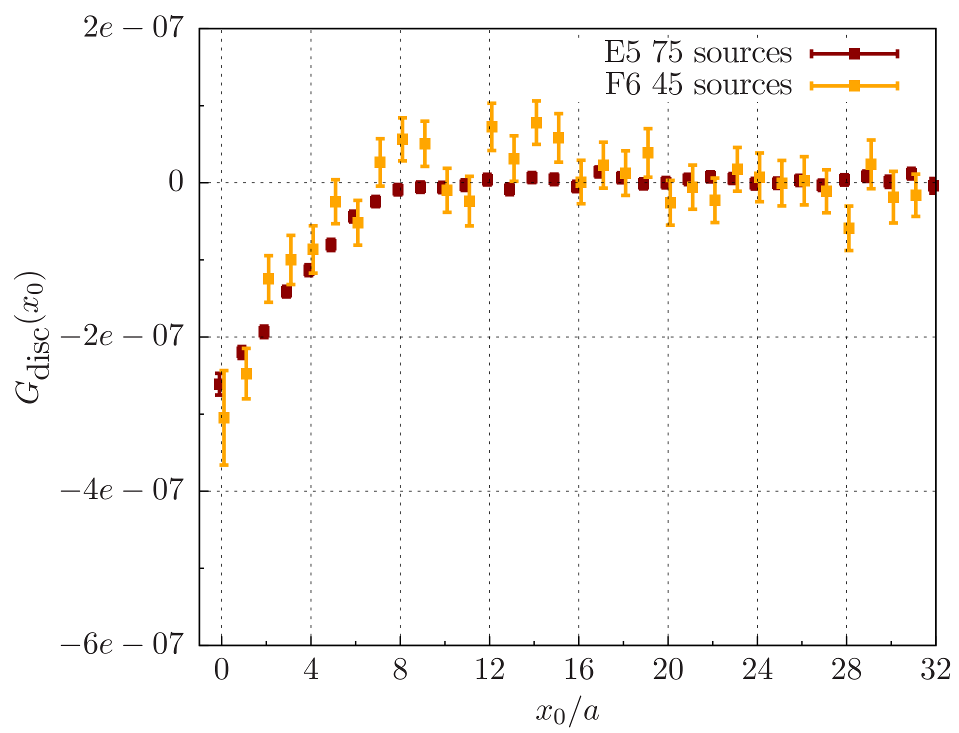
  - Gounaris-Sakurai parameterisation of timeline pion form factor

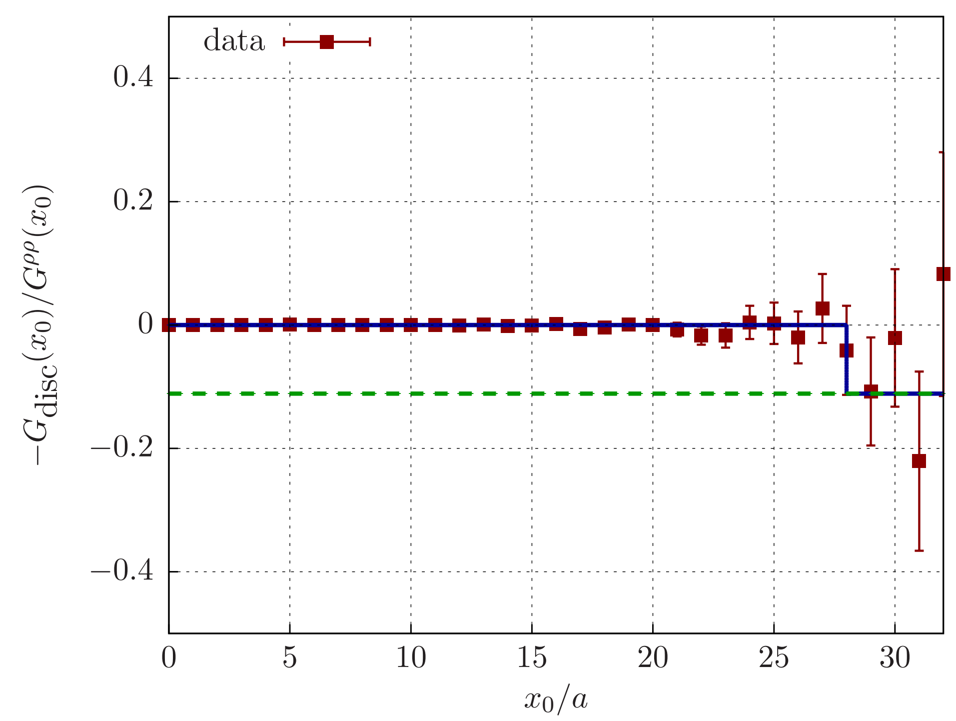


Disconnected Contributions

* Exploit stochastic noise cancellation between (ud) and s quarks

  [Gülpers et al., arXiv:1411.7592; V. Gülpers, PhD Thesis 2015]





Run	$N_{\rm cfg}$	$N_{\rm r}$	T/a	x_0^*	$\Delta a_\mu^{\rm hvp}$
E5	1000	75	64	25	0.7%
				28	0.3%
F6	300	45	96	22	1.8%
				23	1.5%

$$\Rightarrow \quad \Delta a_\mu^{\rm hvp} \equiv -\frac{(a_\mu^{\rm hvp})_{\rm disc}}{(a_\mu^{\rm hvp})_{\rm con}} \leq 2\%$$

TMR analysis of finite-volume effects

* Iso-vector correlator in infinite and finite volume:

$$G^{\rho\rho}(x_0,\infty) = \int_0^\infty d\omega\,\omega^2\rho(\omega^2)\mathrm{e}^{-\omega|x_0|}, \quad \rho(\omega^2) = \frac{1}{48\pi^2}\left(1-\frac{4m_\pi^2}{\omega^2}\right)^{3/2}|F_\pi(\omega)|^2$$

⇒ Continuum of states with $E \geq 2m_\pi$

$$G^{\rho\rho}(x_0,L) = \sum_n |A_n|^2\,\mathrm{e}^{-\omega_n x_0}, \quad \omega_n = 2\sqrt{m_\pi^2+k_n^2}$$

$$|A_n|^2 = \frac{2k_n^2}{3\pi\omega_n^2}\,\frac{|F_\pi(\omega_n)|^2}{\left\{k\phi'(k)+k\delta_1'(k)\right\}_{k=k_n}}, \qquad \delta_{11}(k)+\phi\left(\frac{kL}{2\pi}\right) = n\pi, \quad n=1,2,\ldots$$

⇒ Discrete energy levels: $E \geq 2\sqrt{m_\pi^2+(2\pi/L)^2}$

* Use information on $F_\pi(\omega)$ to determine finite-volume shift:

$$a_\mu^{\mathrm{hvp}}(\infty) - a_\mu^{\mathrm{hvp}}(L) = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dx_0\,\tilde{f}(x_0)\,\left[G^{\rho\rho}(x_0,\infty)-G^{\rho\rho}(x_0,L)\right]$$

TMR analysis of finite-volume effects

* Compute $F_\pi(\omega)$ via energy levels in isovector channel in finite volume

* Approximate $F_\pi(\omega)$ by Gounaris-Sakurai parameterisation: (m_ρ, Γ_ρ)

TMR analysis of finite-volume effects

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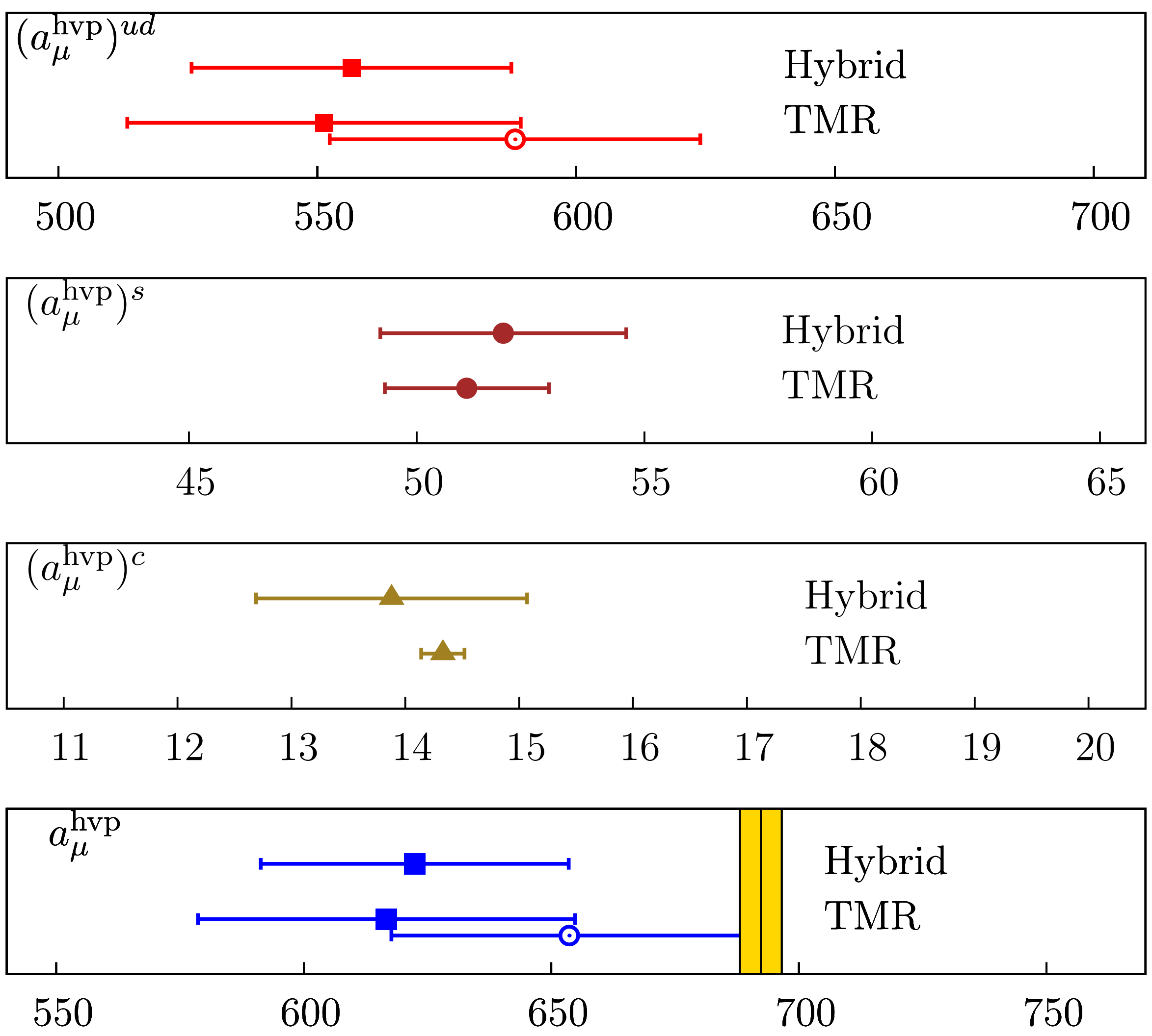
- Approximate $F_\pi(\omega)$ by Gounaris-Sakurai parameterisation: (m_ρ, Γ_ρ)



- Finite-volume corrections sizeable
- Good consistency between hybrid method and TMR

Final result for $N_f = 2$

* Estimate from TMR including finite-volume correction:

$$a_\mu^{\rm hvp} = (654 \pm 32_{\rm stat} \pm 17_{\rm syst} \pm 10_{\rm scale} \pm 7_{\rm FV}\,{}^{+\ 0}_{-10}{}_{\rm disc}) \cdot 10^{-10}$$

- Statistical error: 4.8% at the physical point
- Systematic error: 2.6% from procedural variations
- Scale setting: 1.5% from uncertainty in am_μ
- Finite volume shift: 1.0% from variations in (m_ρ, Γ_ρ)
- Disconnected parts: – 1.5% from upper bound on magnitude

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3.3% total systematic error

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* Isospin-breaking effects not (yet) included

* Analysis of CLS ensembles with $N_f = 2+1$ flavours has started

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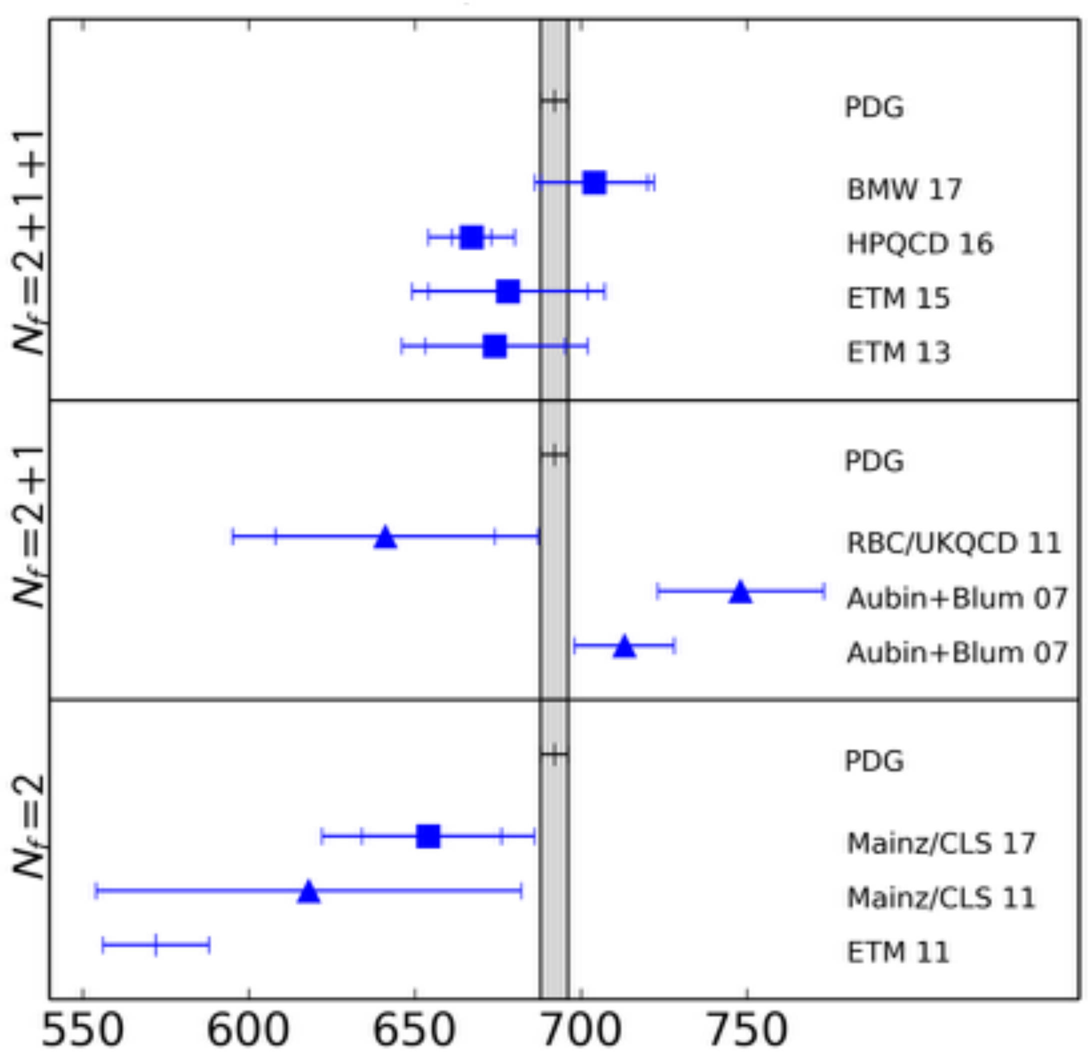


Hadronic Light-by-Light Scattering

Lattice QCD approaches to HLbL

* Matrix element of e.m. current between muon initial and final states:

$$\left\langle \mu(\boldsymbol{p}', s') \left| J_\mu(0) \right| \mu(\boldsymbol{p}, s) \right\rangle = -e\, \overline{u}(\boldsymbol{p}', s') \left(F_1(Q^2)\gamma_\mu + \frac{F_2(Q^2)}{2m}\sigma_{\mu\nu}Q_\nu \right) u(\boldsymbol{p}, s)$$

$$a_\mu^{\rm hlbl} = F_2(0)$$

RBC/UKQCD:

* QCD + QED simulations *[Hayakawa et al. 2005; Blum et al. 2015]*
* QCD + stochastic QED *[Blum et al. 2016, 2017]*

Mainz group:

* Exact QED kernel in position space *[Asmussen et al. 2015, 2016, and in prep.]*
* Transition form factors of sub-processes *[Gérardin, Meyer, Nyffeler 2016]*
* Forward scattering amplitude *[Green et al. 2015, and in prep.]*

Exact QED kernel in position space

* Determine QED part perturbatively in the continuum in infinite volume

⇒ no power-law volume effects

$$a_\mu^{\rm hlbl} = F_2(0) = \frac{me^6}{3} \int d^4y \int d^4x \, \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \, i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y)$$

* QCD four-point function: $i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z \, z_\rho \left\langle J_\mu(x) J_\nu(y) J_\sigma(z) J_\lambda(0) \right\rangle$

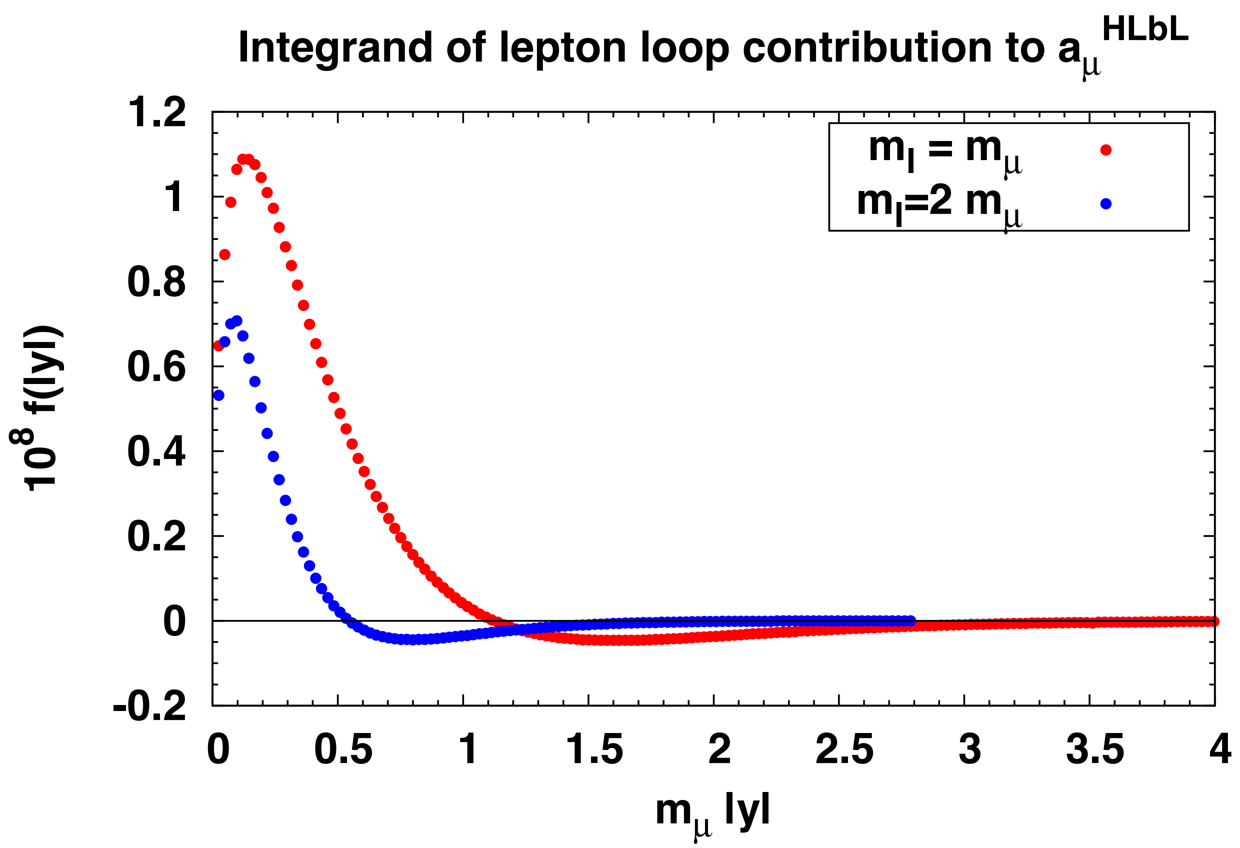
* QED kernel function: $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ *[Asmussen, Green, Meyer, Nyffeler, in prep.]*

  - Infra-red finite; can be computed semi-analytically
  - Admits a tensor decomposition in terms of six weight functions which depend on $x^2, \quad y^2, \quad x \cdot y$

  ⇒ 3D integration instead of $\int d^4x \int d^4y$

* Weight functions computed and stored on disk

Testing the exact QED kernel

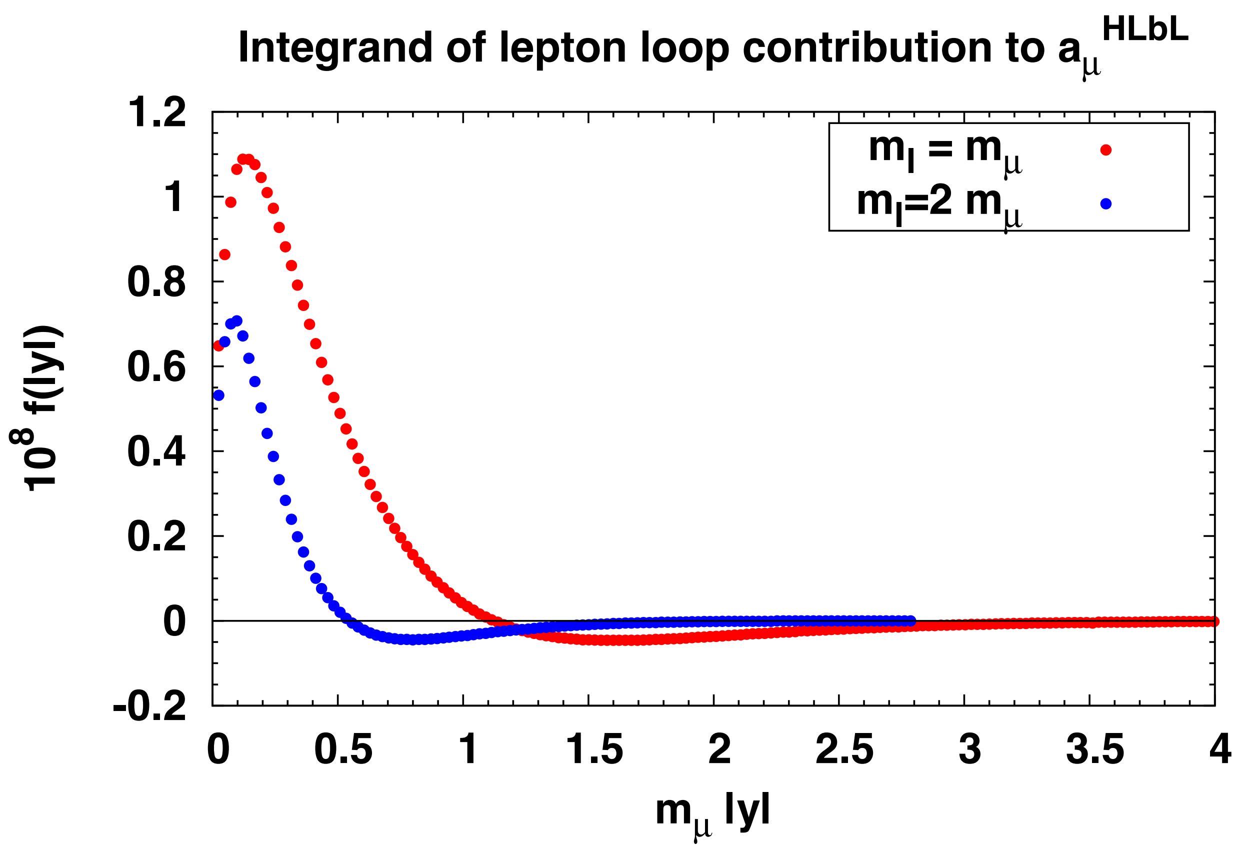


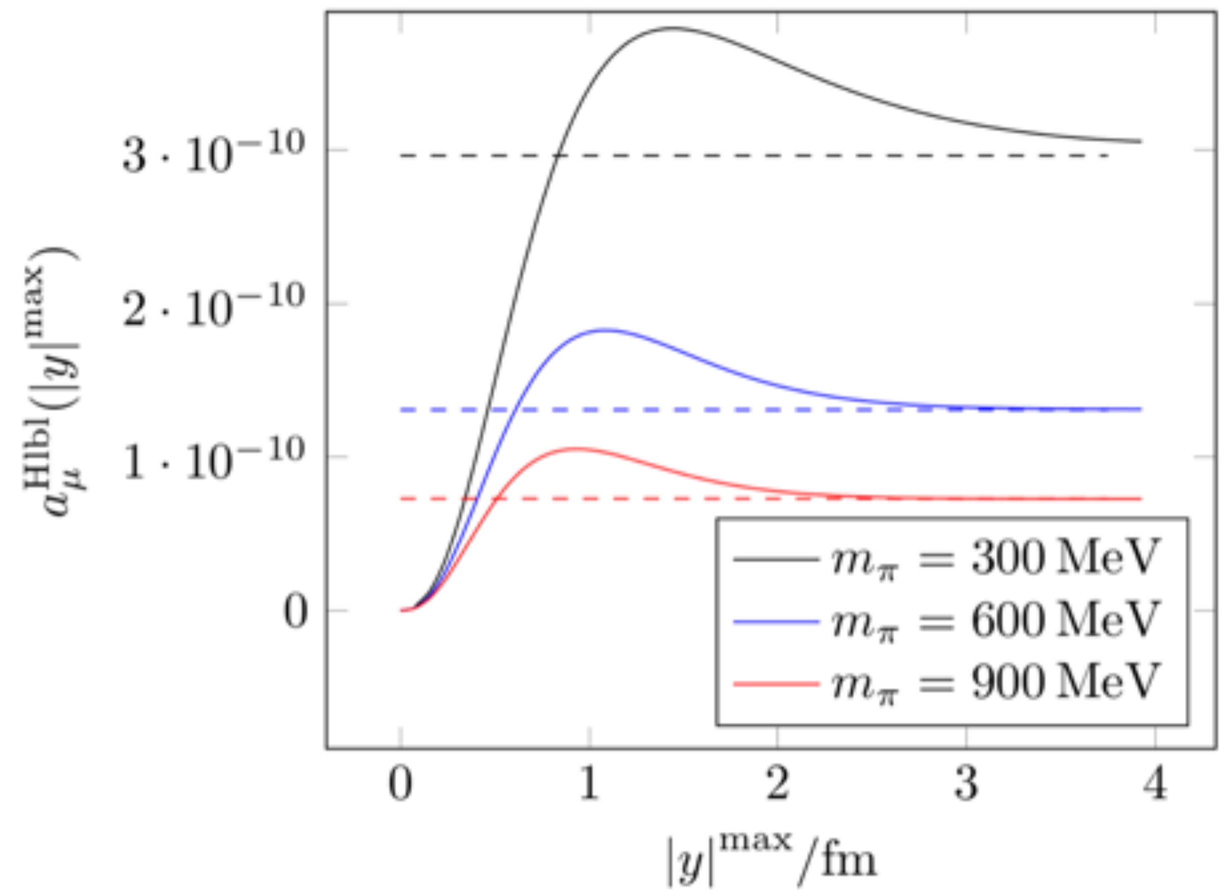
- Analytic result for the lepton-loop reproduced at percent level

$$f(|y|) \equiv \frac{me^6}{3} 2\pi^2 |y|^3 \int d^4x\, \overline{\mathcal{L}}_{\ldots}(x,y)\, i\Pi_{\ldots}(x,y)$$

[Meyer @ FNAL 2017, Asmussen @ Lattice 2017]

Testing the exact QED kernel





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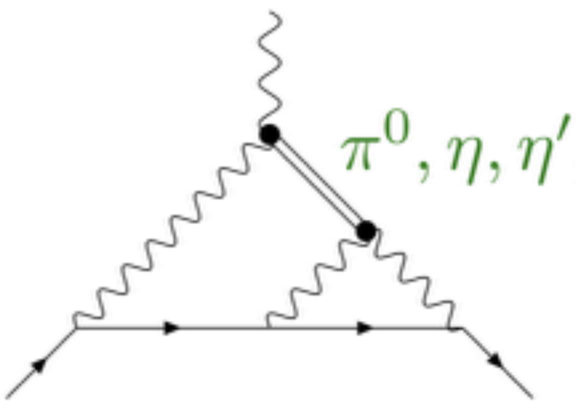
* π^0 pole contribution: assume VMD model for TFF

* Contribution (surprisingly?) long-range

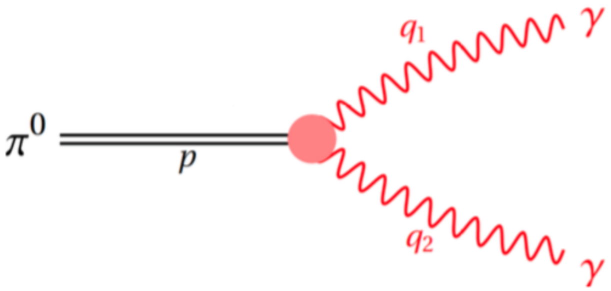
[Meyer @ FNAL 2017, Asmussen @ Lattice 2017]

Transition form factor $\pi^0 \longrightarrow \gamma^*\gamma^*$

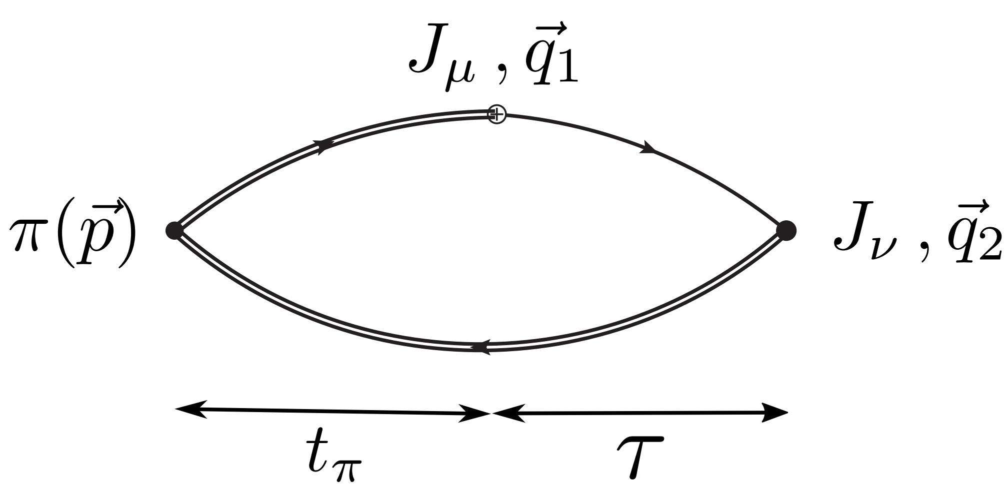
* Pseudoscalar meson pole expected to dominate LbL scattering



* Compute transition form factor between π^0 and two off-shell photons:



$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2; q_1^2, q_2^2) \equiv M_{\mu\nu}$$



$$M_{\mu\nu} \sim C_{\mu\nu}^{(3)}(\tau, t_\pi; \vec{p}, \vec{q}_1, \vec{q}_2) =$$

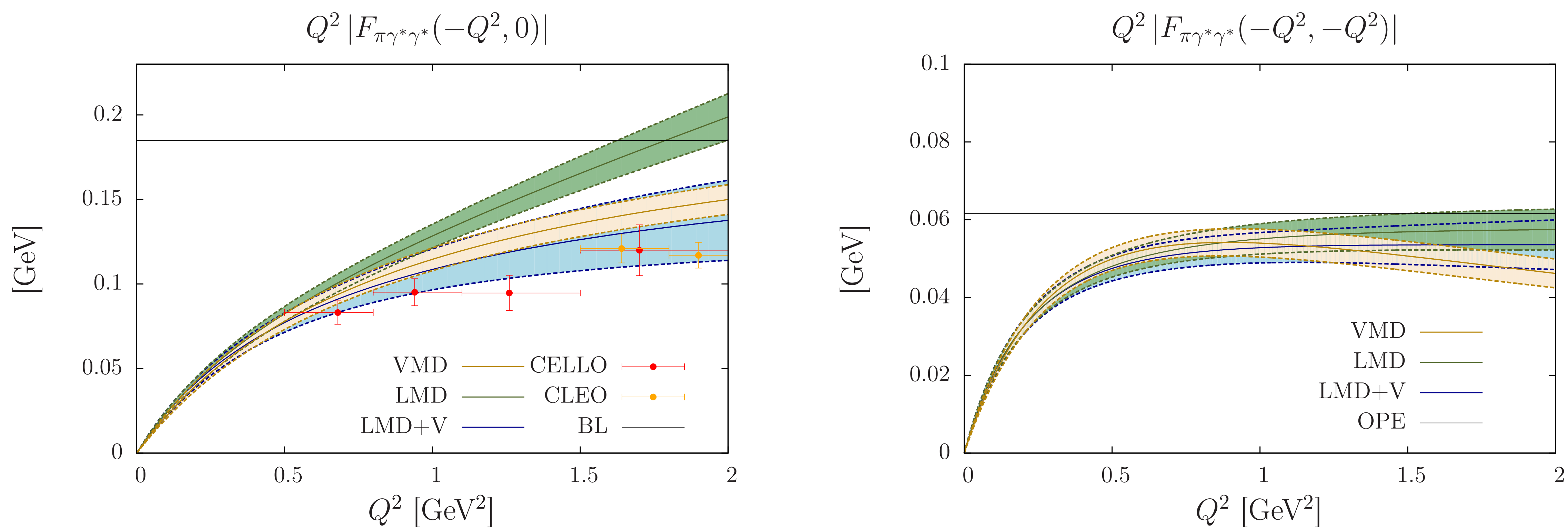
$$\sum_{\vec{x},\vec{z}} \left\langle T\left\{J_\nu(\vec{0}, \tau + t_\pi) J_\mu(\vec{z}, t_\pi) P(\vec{x}, 0)\right\}\right\rangle \mathrm{e}^{i\vec{p}\cdot\vec{x}} \mathrm{e}^{-i\vec{q}_1\cdot\vec{z}}$$

* Compute connected and disconnected contributions

Transition form factor $\pi^0 \longrightarrow \gamma^*\gamma^*$

- Fit VMD, LMD, LMD-V models, e.g.

$$\mathcal{F}^{\rm LMD}_{\pi^0\gamma^*\gamma^*} = \frac{\alpha M_{\rm V}^4 + \beta(q_1^2+q_2^2)}{(M_{\rm V}^2-q_1^2)(M_{\rm V}^2-q_2^2)}$$

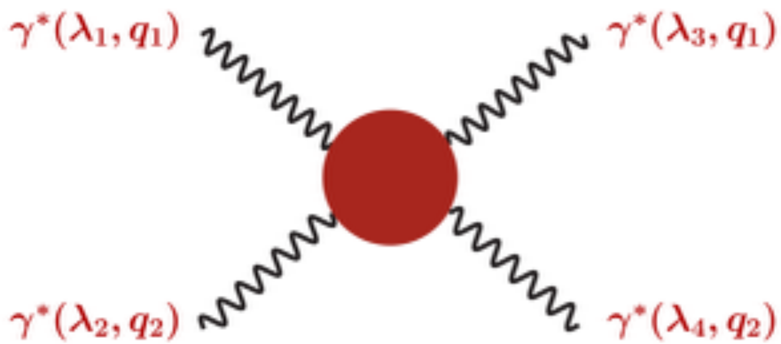


- Results for π^0 contribution to hadronic light-by-light scattering:

$$(a_\mu^{\rm hlbl})_{\pi^0} = (65.0 \pm 8.3)\cdot 10^{-11} \quad \text{(LMD+V)}$$ (stat. error only)

[Gérardin, Meyer, Nyffeler, Phys Rev D94 (2016) 074507]

Light-by-light forward scattering amplitude

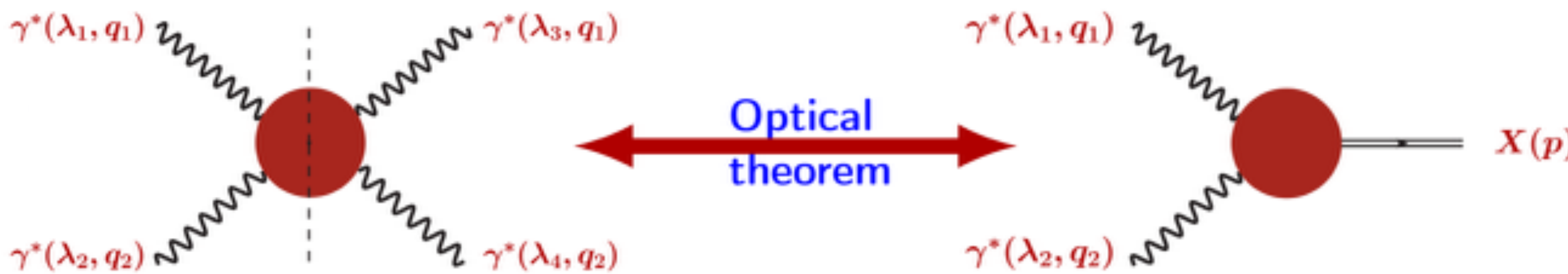


Eight independent amplitudes:

$$M_{TT}, M^t_{TT}, M^a_{TT}, M_{TL}, M_{LT}, M^a_{TL}, M^t_{TL}, M_{LL}$$

$$Q_i^2 = -q_i^2 > 0, \quad i = 1, 2, \quad \nu = -Q_1 \cdot Q_2$$

- Relate forward amplitudes to two-photon fusion cross sections:

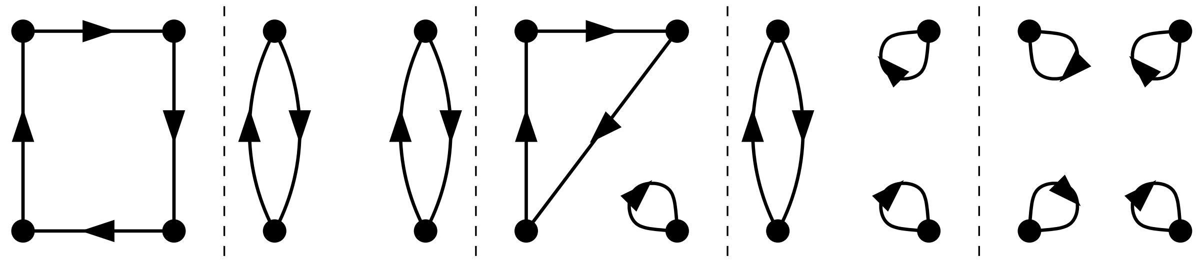


$$\overline{M}_\alpha(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \, \frac{\sqrt{X'}\,\sigma_\alpha(\nu')/\tau_\alpha(\nu')}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}$$

- Expect main contributions from mesons
- Constrain form factors used to estimate a_μ^{hlbl}

Light-by-light forward scattering amplitude

* Four-point correlator of one local and three conserved vector currents



* Fully connected contribution with summed fixed kernels:

$$\Pi^{\mathrm{pos}\prime}_{\mu_1\mu_2\mu_3\mu_4}(x_4; f_1, f_2) = \sum_{x_1,x_2} f(x_1)\, f(x_2)\, \Pi^{\mathrm{pos}}_{\mu_1\mu_2\mu_3\mu_4}(x_1, x_2, 0, x_4)$$

* Euclidean four-point function in momentum space:

$$\Pi^{E}_{\mu_1\mu_2\mu_3\mu_4}(p_4; p_1, p_2) = \sum_{x_4} \mathrm{e}^{-ip_4\cdot x_4}\, \Pi^{\mathrm{pos}\prime}_{\mu_1\mu_2\mu_3\mu_4}(x_4; p_1, p_2)$$

* Forward scattering of transversely polarised virtual photons:

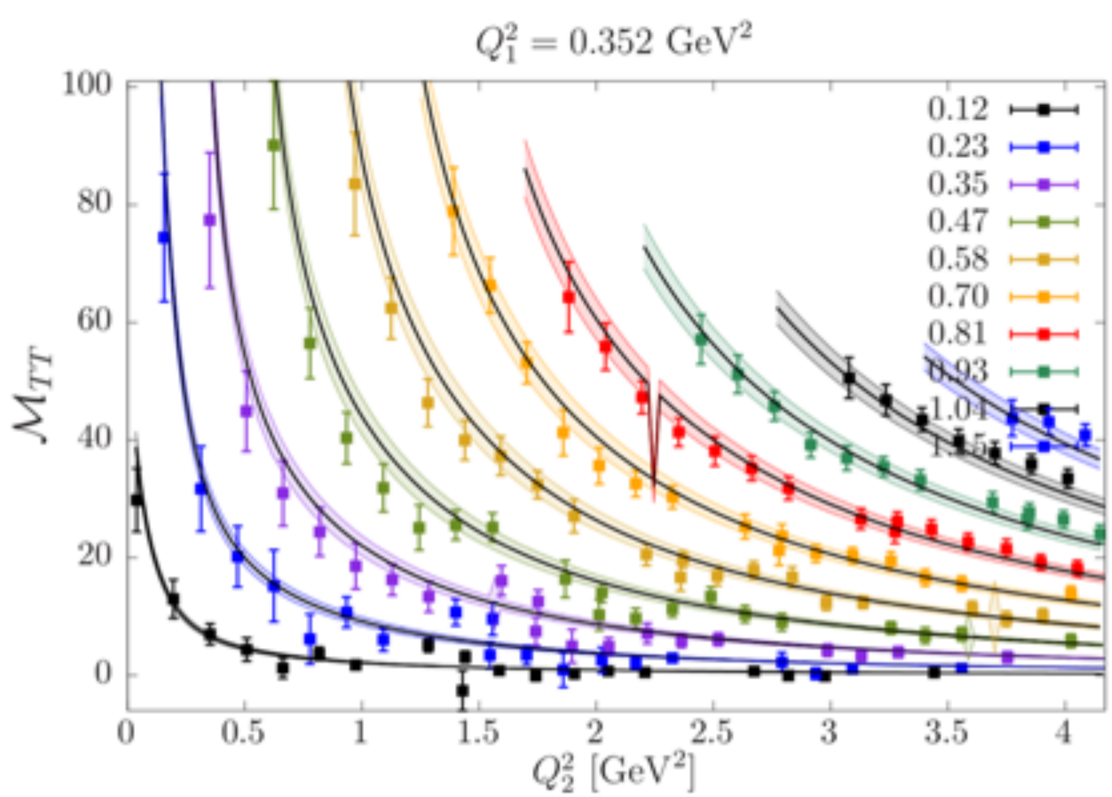
$$M_{TT}(-Q_1^2, -Q_2^2, \nu) = \frac{e^4}{4} R_{\mu_1\mu_2} R_{\mu_3\mu_4}\, \Pi^{E}_{\mu_1\mu_2\mu_3\mu_4}(-Q_2; -Q_1, Q_1)$$

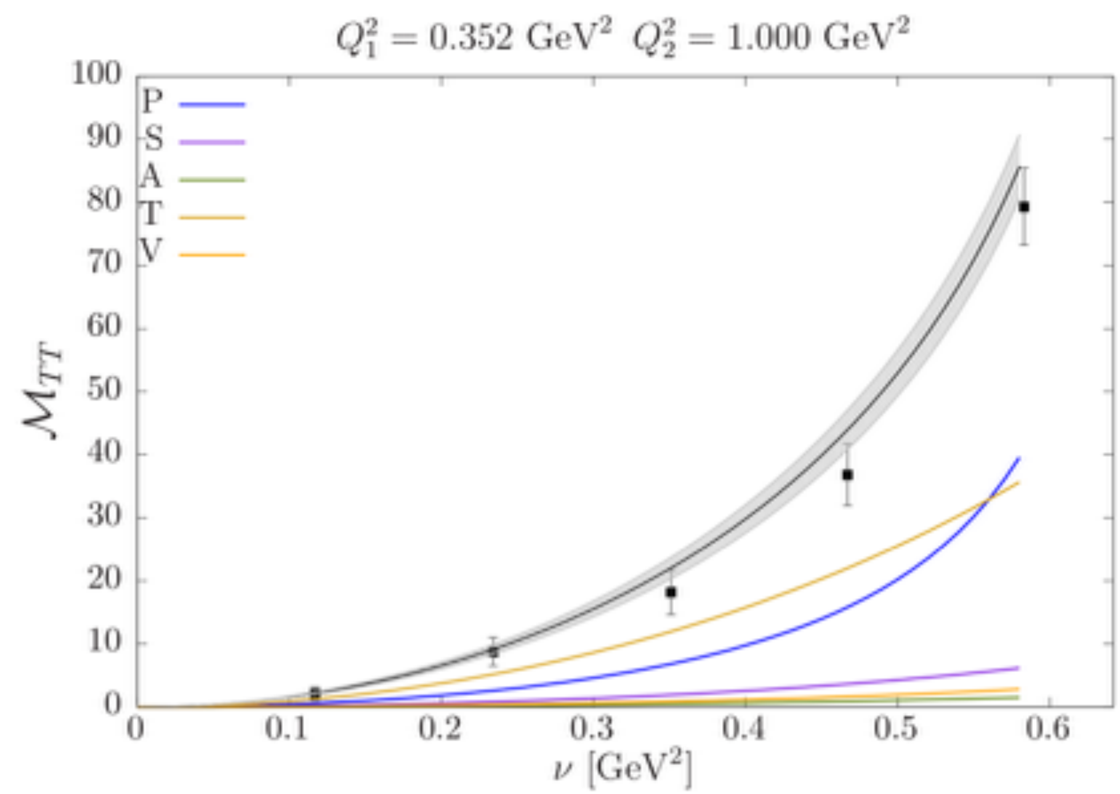
Light-by-light forward scattering amplitude

- Example: contributions to $\mathcal{M}_{TT}$ from different mesonic channels

$$\overline{\mathcal{M}}_{TT}(\nu) = \frac{4\nu^2}{\pi}\int_{\nu_0}^{\infty} d\nu' \, \frac{\sigma_{TT}(\nu')}{\nu'(\nu'^2-\nu^2-i\epsilon)}, \qquad \sigma_{TT} \propto \left[\frac{\mathcal{F}_{P\gamma^*\gamma^*}(Q_1^2,Q_2^2)}{\mathcal{F}_{P\gamma^*\gamma^*}(0,0)}\right]^2$$

Use monopole/dipole *ansatz* for S, A, T and V form factors





[Gérardin @ Lattice 2017]

Summary and Outlook

* Hadronic vacuum polarisation: focus on refinements
  - Include physical pion mass
  - Determine timelike pion form factor
  - Include isospin breaking
* Hadronic light-by-light scattering
  - QED kernel can be combined with direct lattice calculation or hadronic model for four-point function
  - Four-point function can be determined directly — serves to test hadronic models
  - Complementary approach: transition form factors