Complete Experiments in pseudoscalar meson photoproduction

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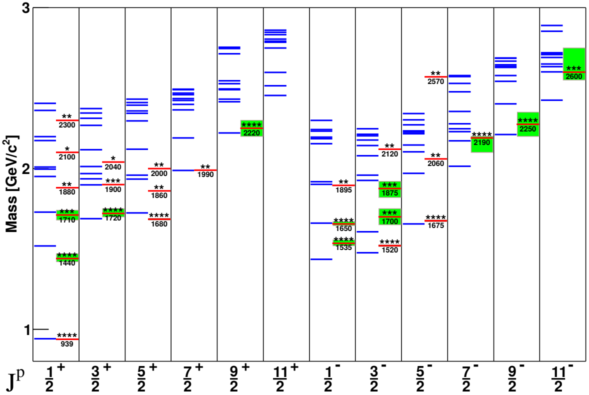
Motivation for photoproduction



- N^* spectrum
- Predictions of the Bonn CQM on the left [Loring et al. (2001)]
- Resonances from [PDG (2014)] on the right

[Andrew Wilson]

∗) Photoproduction is studied at photon facilities in Bonn (CBELSA/TAPS), Mainz (MAMI/A2), Newport News (CLAS), ...

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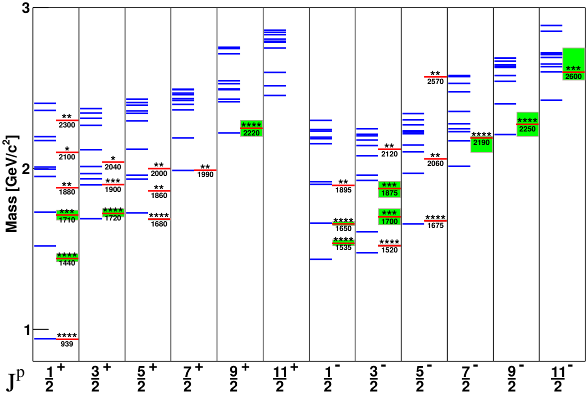

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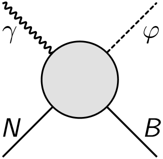
∗) What I do: Identify a sufficient data base to get a unique solution for photoproduction amplitudes (or multipoles) $\rightarrow$ Complete Experiments.

Photoproduction amplitudes

Photoproduction amplitude in the CMS:



$$= \mathcal{T}_{fi} = \mathcal{C}\chi^{\dagger}_{m_{s_f}} \Big[i\vec{\sigma}\cdot\hat{\epsilon}F_1 + \vec{\sigma}\cdot\hat{q}\vec{\sigma}\cdot\left(\hat{k}\times\hat{\epsilon}\right)F_2 + i\vec{\sigma}\cdot\hat{k}\hat{q}\cdot\hat{\epsilon}F_3 + i\vec{\sigma}\cdot\hat{q}\hat{q}\cdot\hat{\epsilon}F_4 \Big] \chi_{m_{s_i}}$$

[Chew, Goldberger, Low & Nambu (1957)]

$\rightarrow$ Process fully described by 4 complex amplitudes $F_i\,(W,\theta)$.

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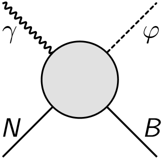

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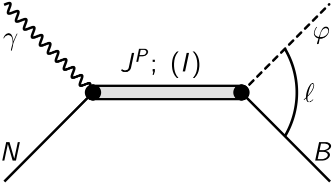
Important concept: expansion of full amplitudes into partial waves:

$$F_1\left(W,\theta\right) = \sum_{\ell=0}^{\infty}\Big\{ \left[\ell M_{\ell+} + E_{\ell+}\right] P^{'}_{\ell+1}\left(\cos\left(\theta\right)\right) + \left[\left(\ell+1\right)M_{\ell-} + E_{\ell-}\right] P^{'}_{\ell-1}\left(\cos\left(\theta\right)\right) \Big\}$$

$$F_2\left(W,\theta\right) = \ldots$$



$*)$ $J = |\ell \pm 1/2|$, $P = (-)^{\ell+1}$.

$*)$ s-chn. resonance $J^P;\ (I)$

$\updownarrow$

multipole $E^{(I)}_{\ell\pm},\ M^{(I)}_{\ell\pm}$

Photoproduction amplitudes

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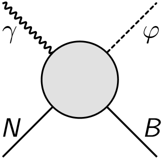

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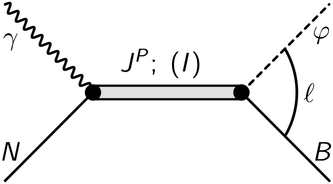

In practice:

Truncate at some finite $\ell_{\max}$

$\rightarrow$ Try to extract the $4\ell_{\max}$ complex multipoles in a fit to the data.

Polarization observables

Generic definition of an observable

$$\Omega = \frac{\beta}{\sigma_0}\left[\left(\frac{d\sigma}{d\Omega}\right)^{(B_1,T_1,R_1)} - \left(\frac{d\sigma}{d\Omega}\right)^{(B_2,T_2,R_2)}\right]$$

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$*$) In total, 16 non-redundant observables

$$\Omega^{\alpha}\left(W,\theta\right) = \frac{1}{2\sigma_0}\sum_{i,j} F_i^{*}\hat{A}_{ij}^{\alpha}F_j, \quad \alpha = 1,\ldots,16$$

can be defined, involving Beam-, Target- and Recoil Polarization.

Beam		Target			Recoil			Target + Recoil			
	-	-	-	-	x'	y'	z'	x'	x'	z'	z'
	-	x	y	z	-	-	-	x	z	x	z
unpolarized	$\left(\frac{d\sigma}{d\Omega}\right)_0 = \sigma_0$		T			P		$T_{x'}$	$L_{x'}$	$T_{z'}$	$L_{z'}$
linear	Σ	H	P	G	$O_{x'}$	T	$O_{z'}$				
circular		F		E	$C_{x'}$		$C_{z'}$				

Observables in the transversity basis

Observable	Transversity representation	Type
σ_0	$\frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\check{\Sigma}$	$\frac{1}{2}\left(- \lvert b_1\rvert^2 - \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	$\mathcal{S}$
$\check{T}$	$\frac{1}{2}\left(\lvert b_1\rvert^2 - \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\check{P}$	$\frac{1}{2}\left(- \lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\check{G}$	$\mathrm{Im}\left[-b_1 b_3^* - b_2 b_4^*\right]$	
$\check{H}$	$-\mathrm{Re}\left[b_1 b_3^* - b_2 b_4^*\right]$	$\mathcal{BT}$
$\check{E}$	$-\mathrm{Re}\left[b_1 b_3^* + b_2 b_4^*\right]$	
$\check{F}$	$\mathrm{Im}\left[b_1 b_3^* - b_2 b_4^*\right]$	
$\check{O}_{x'}$	$-\mathrm{Re}\left[-b_1 b_4^* + b_2 b_3^*\right]$	
$\check{O}_{z'}$	$\mathrm{Im}\left[-b_1 b_4^* - b_2 b_3^*\right]$	$\mathcal{BR}$
$\check{C}_{x'}$	$\mathrm{Im}\left[b_1 b_4^* - b_2 b_3^*\right]$	
$\check{C}_{z'}$	$\mathrm{Re}\left[b_1 b_4^* + b_2 b_3^*\right]$	
$\check{T}_{x'}$	$-\mathrm{Re}\left[-b_1 b_2^* + b_3 b_4^*\right]$	
$\check{T}_{z'}$	$-\mathrm{Im}\left[b_1 b_2^* - b_3 b_4^*\right]$	$\mathcal{TR}$
$\check{L}_{x'}$	$-\mathrm{Im}\left[-b_1 b_2^* - b_3 b_4^*\right]$	
$\check{L}_{z'}$	$\mathrm{Re}\left[-b_1 b_2^* - b_3 b_4^*\right]$	

$*$) Transversity amplitudes: $b_i = \sum_j M_{ij} F_j$.

$*$) Different scheme of spin-quantization:

$$\left< m_{s_f} \right| \mathcal{T} \left| m_{s_i} \right> \updownarrow \left< t_f \right| \mathcal{T} \left| t_i \right>.$$

$t_i(t_f) = \pm\frac{1}{2}$: spin-projection of initial (final) baryon on the normal of the reaction plane.

$*$) Observables simplify:

$$\check{\Omega}^{\alpha} = \frac{1}{2} \sum_{i,j} b_i^* \tilde{\Gamma}^{\alpha}_{ij} b_j.$$

Complete Experiments

$\ast$) Question: How many and which observables $\check{\Omega}^{\alpha}$ have to be measured in order to uniquely extract the full amplitudes (e.g. transversity amplitudes b_i)?.

$\ast$) <u>Mathematical solution:</u> [Chiang & Tabakin, Phys. Rev. C 55, 2054 (1997)]
Utilize b.t.p.-form $\check{\Omega}^{\alpha} = \frac{1}{2} \sum_{i,j} b_i^{\ast} \tilde{\Gamma}^{\alpha}_{ij} b_j$ and the completeness of the $\tilde{\Gamma}^{\alpha}$-matrices ($\tilde{\Gamma}^{\alpha}$ form an orthonormal basis): $\frac{1}{4} \sum_{\alpha} \tilde{\Gamma}^{\alpha}_{ba} \tilde{\Gamma}^{\alpha}_{st} = \delta_{as} \delta_{bt}$

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$$\sum_{\alpha} \tilde{\Gamma}^{\alpha}_{ac} \check{\Omega}^{\alpha} = \frac{1}{2} \sum_{\alpha} \tilde{\Gamma}^{\alpha}_{ac} \sum_{i,j} b_{i}^{*} \tilde{\Gamma}^{\alpha}_{ij} b_{j} = \frac{1}{2} \sum_{i,j} b_{i}^{*} \sum_{\alpha} \tilde{\Gamma}^{\alpha}_{ac} \tilde{\Gamma}^{\alpha}_{ij} b_{j}$$

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$$= 2 \sum_{i,j} b_i^{*} \delta_{ic} \delta_{aj} b_j = 2 b_c^{*} b_a$$

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$$\rightarrow b_i^* b_j = \frac{1}{2} \sum_{\alpha} \left(\tilde{\Gamma}^{\alpha}_{ij} \right)^* \check{\Omega}^{\alpha}$$

$$\rightarrow |b_i| = \sqrt{b_i^* b_i} \;\; \& \;\; e^{\phi_{ij}} = \frac{b_j^* b_i}{|b_j| \, |b_i|}$$

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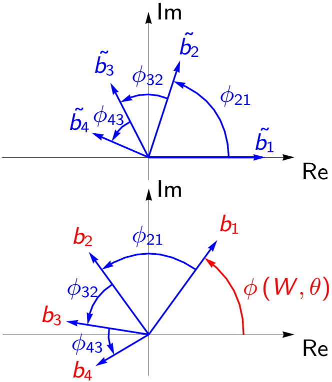

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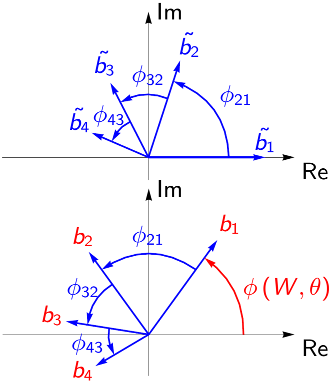
$*$) Use "Fierz-identities" $\check{\Omega}^{\alpha} \check{\Omega}^{\beta} = \mathcal{C}^{\alpha\beta}_{\delta\eta} \check{\Omega}^{\delta} \check{\Omega}^{\eta}$ (with known coefficients $\mathcal{C}^{\alpha\beta}_{\delta\eta}$) to prove:

- 8 observables can yield $\left| b_i \right|$ & ϕ_{ij}.
- Double-polarization obs. with recoil-polarization (type $\mathcal{BR}$ and $\mathcal{TR}$) have to be measured.
- No more than two observables from the same double-polarization class are allowed.
- The phase $\phi(W, \theta)$ remains undetermined.



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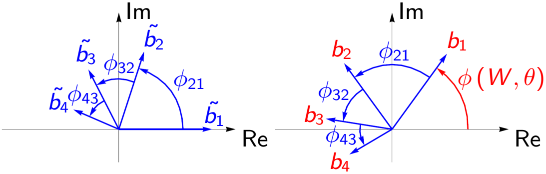

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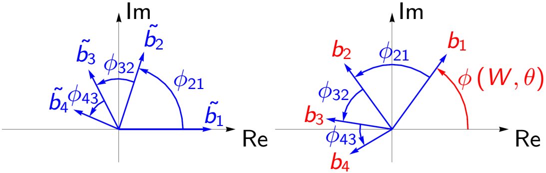

*) Multipoles: consider TPWA truncated at ℓ_{max}

$$\check{\Omega}^{\alpha} \left(W, \theta \right) = \rho \sum_{k=\beta_{\alpha}}^{2\ell_{\mathrm{max}}+\beta_{\alpha}+\gamma_{\alpha}} \left(a_L \right)_k^{\check{\Omega}^{\alpha}} \left(W \right) P_k^{\beta_{\alpha}} \left(\cos \theta \right),$$

$$\left(a_L \right)_k^{\check{\Omega}^{\alpha}} \left(W \right) = \left< \mathcal{M}_{\ell_{\mathrm{max}}} \left(W \right) \right| \left(\mathcal{C}_L \right)_k^{\check{\Omega}^{\alpha}} \left| \mathcal{M}_{\ell_{\mathrm{max}}} \left(W \right) \right>,$$

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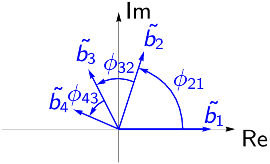

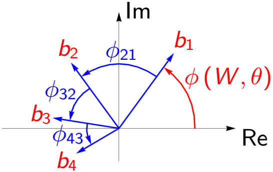

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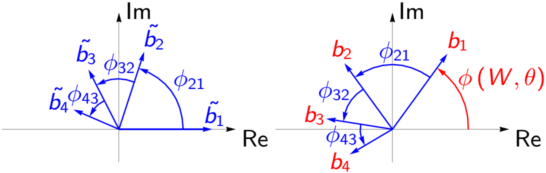

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$\rightarrow$ Instead: [Omelaenko, (1981)] & [Y.W., R. Beck and L. Tiator, (2014)]
Study discrete ambiguities of the group S $\{\sigma_0, \Sigma, T, P\}$

$\Longrightarrow$ "Exact" TPWA can be complete with just 5 observables, e.g. $\{\sigma_0, \Sigma, T, P, F\}$.

Details on the multipole Fit procedure

Two step method:

1. Fit the angular distributions of observables, parametrized by

$$\check{\Omega}^{\alpha}\left(W,\theta\right) = \rho \sum_{k=\beta_{\alpha}}^{2\ell_{\mathrm{max}}+\beta_{\alpha}+\gamma_{\alpha}} \left(a_{L}\right)_{k}^{\alpha}\left(W\right) P_{k}^{\beta_{\alpha}}\left(\cos\theta\right)$$

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2. Minimize the function ("multi-indices" $(i,j)=(\{\alpha,k\},\{\alpha',k'\})$):

$$\chi^{2}=\sum_{i,j}\left[\left(a_{L}^{\mathrm{Fit}}\right)_{i}-\langle\mathcal{M}_{\ell}|\left(C_{L}\right)_{i}|\mathcal{M}_{\ell}\rangle\right]\mathrm{C}_{ij}^{-1}\left[\left(a_{L}^{\mathrm{Fit}}\right)_{j}-\langle\mathcal{M}_{\ell}|\left(C_{L}\right)_{j}|\mathcal{M}_{\ell}\rangle\right],$$

using the MATHEMATICA method

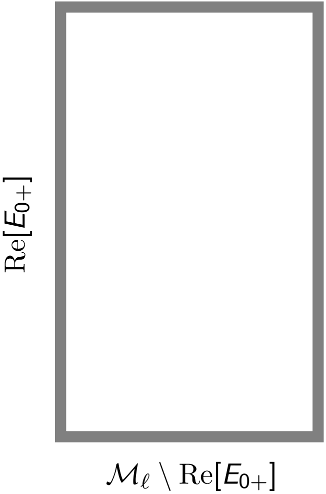
$$\mathrm{FindMinimum}\left[\chi^{2}\left(\mathcal{M}_{\ell}\right),\{\{\mathrm{Re}\left[E_{0+}\right],\left(x_{1}\right)_{0}\},\ldots,\{\mathrm{Im}\left[M_{\ell_{\max}-}\right],\left(y_{n}\right)_{0}\}\}\right]$$

and varying the real and imaginary parts of the (possibly phase constrained) multipoles in the fit.

C_{ij} is the covariance matrix of the Legendre coefficients stemming from step 1.

Generation of start values for FindMinimum

1. The total cross section
   $$\hat{\sigma}\left(W\right) = \sum_{\mathcal{M}_\ell}^{\ell_{\max}} c_{\mathcal{M}_\ell} \left|\mathcal{M}_\ell\right|^2$$
   constrains the $(8\ell_{\max} - 1)$-dimensional multipole space $\mathcal{M}_\ell$.



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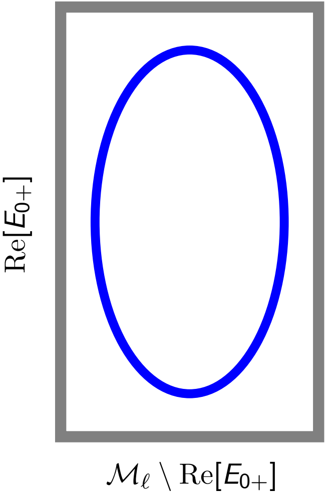
2. $\hat{\sigma}(W)$ defines an $(8\ell_{\max} - 2)$-dimensional ellipsoid in $\mathcal{M}_\ell$.



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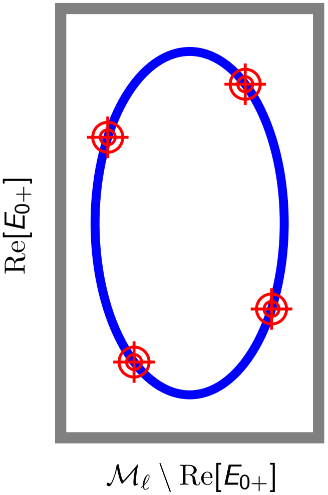
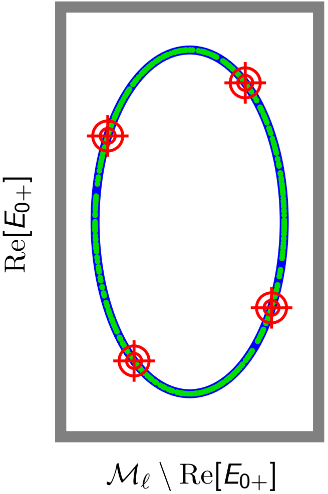
2. $\hat{\sigma}(W)$ defines an $(8\ell_{\mathrm{max}} - 2)$-dimensional ellipsoid in $\mathcal{M}_\ell$.
3. Solutions to the TPWA problem lie on the ellipsoid defined by $\hat{\sigma}(W)$.



Generation of start values for FindMinimum

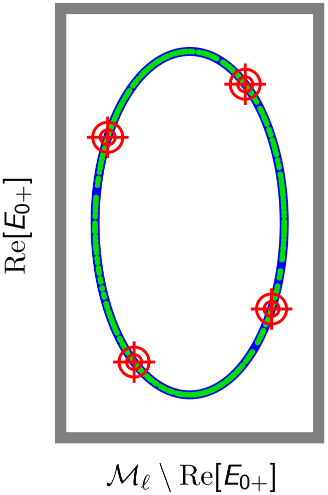
1. The total cross section
$$\hat{\sigma}\left(W\right) = \sum_{\mathcal{M}_{\ell}}^{\ell_{\mathrm{max}}} c_{\mathcal{M}_{\ell}} \left|\mathcal{M}_{\ell}\right|^{2}$$
constrains the $(8\ell_{\mathrm{max}} - 1)$-dimensional multipole space $\mathcal{M}_{\ell}$.
2. $\hat{\sigma}\left(W\right)$ defines an $(8\ell_{\mathrm{max}} - 2)$-dimensional ellipsoid in $\mathcal{M}_{\ell}$.
3. Solutions to the TPWA problem lie on the ellipsoid defined by $\hat{\sigma}\left(W\right)$.
4. The start values for the $\mathrm{FindMinimum}$-Fit are chosen randomly on the $\hat{\sigma}\left(W\right)$-ellipsoid.
   $\Rightarrow$ Monte Carlo sampling of the multipole space.



Generation of start values for FindMinimum

5. A FindMinimum-minimization is performed for each of the randomly generated start configurations.

   $\Rightarrow N_{MC} = \#$ of M.C. start configurations
   $= \#$ of (possibly redundant) solutions



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5. A $\mathrm{FindMinimum}$-minimization is performed for each of the randomly generated start configurations.

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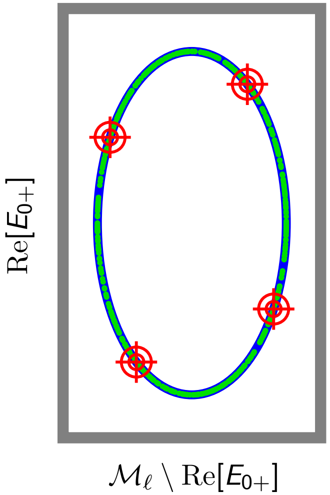
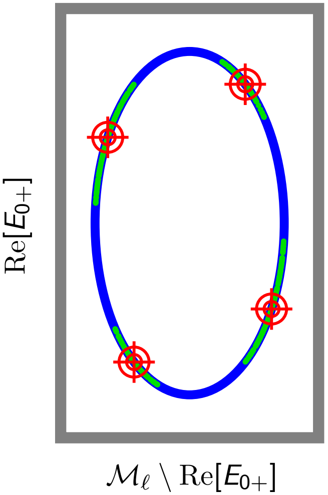
6. Analysis described up to now is fully model-independent.
   <u>However:</u> if wished for or needed, individual partial-wave parameters can be fixed to model-constraints quite freely.



Generation of start values for FindMinimum

6. Analysis described up to now is fully model-independent.
   <u>However:</u> if wished for or needed, individual partial-wave parameters can be fixed to model-constraints quite freely.
7. In this way, map out the global minimum as well as all local minima of the χ^2-function.



Description of the fitted datasets

The following datasets were investigated for $\gamma p \to \pi^0 p$:

I. Data taken at the MAMI facility:
   - σ_0: 266 energy points for $E_\gamma^{\mathrm{LAB}} \in [218, 1573]$ MeV
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$\to$ Datasets overlap on 8 (!) energy-points $E_\gamma^{\mathrm{LAB}} \in [650, 950]$ MeV!

Description of the fitted datasets

The following datasets were investigated for $\gamma p \to \pi^0 p$:

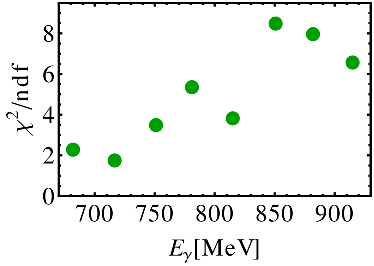
$$\{\sigma_0, \Sigma, T, P, E, G, H\}.$$

From investigations of the angular distributions of the data (and later confirmed by χ^2/ndf in the multipole fit):
$\underline{\ell_{\mathrm{max}} = 2}$ and $\underline{\ell_{\mathrm{max}} = 3}$ truncation approximations can already describe the data.

Results of fully un-constrained analyses

$\underline{\ell_{\max} = 2\text{-fit}}$ ($N_{MC} = 16000$)

I. χ^2-plot:



- There exists a global minimum.

$\underline{\ell_{\max} = 3\text{-fit}}$ ($N_{MC} = 24000$)

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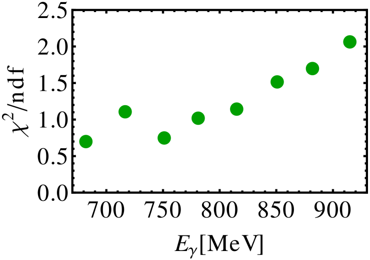

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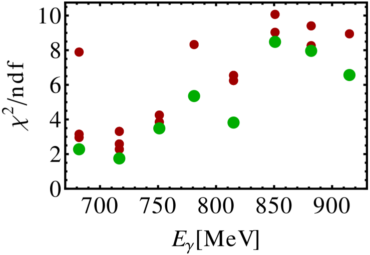
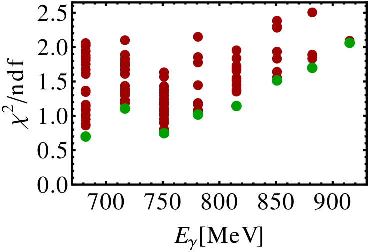


- There exists a global minimum.
- Global min. is well separated from other local minima.

$\underline{\ell_{\max} = \text{3-fit}}$ ($N_{MC} = 24000$)

I. χ^2-plot:

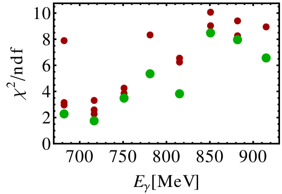


- There exists a global minimum.
- Local minima (ambiguities) exists that have a very similar χ^2 to the global min.

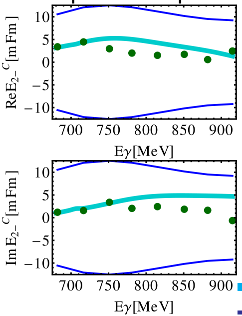
Results of fully un-constrained analyses

$\underline{\ell_{\max} = \text{2-fit}}$ $(N_{MC} = 16000)$

I.



II. Example multipole:



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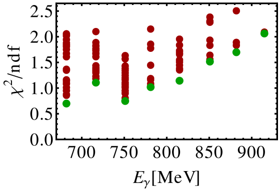
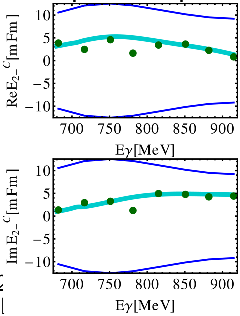
I.



II. Example multipole:



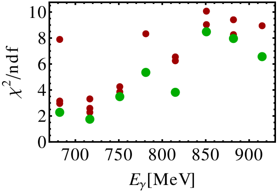
: BnGa 2014-02

: $\hat{\sigma}(W)$-interval

Results of fully un-constrained analyses

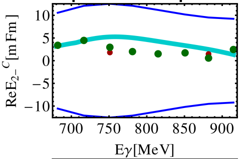
$\underline{\ell_{\max} = 2\text{-fit}}$ ($N_{MC} = 16000$)

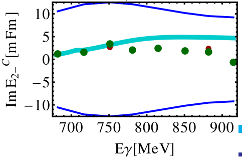
I.



II. Example multipole:

$(\chi^2_{\mathrm{best}}/\mathrm{ndf} + 0.5)$



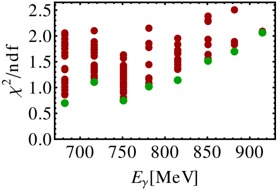


: BnGa 2014-02

: $\hat{\sigma}(W)$-interval

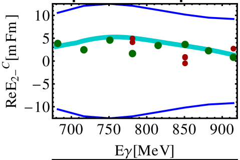
$\underline{\ell_{\max} = 3\text{-fit}}$ ($N_{MC} = 24000$)

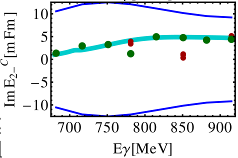
I.



II. Example multipole:

$(\chi^2_{\mathrm{best}}/\mathrm{ndf} + 0.05)$

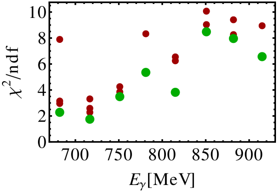




Results of fully un-constrained analyses

$\underline{\ell_{\max} = \text{2-fit}}$ ($N_{MC} = 16000$)

I.



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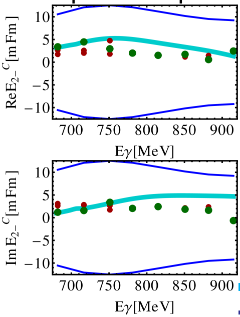
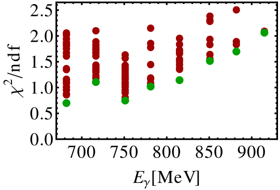


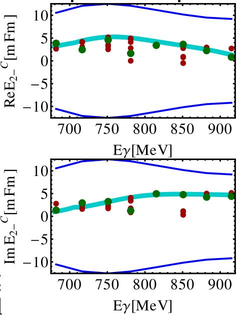
$(\chi^2_{\text{best}}/\text{ndf} + 1.0)$

$\underline{\ell_{\max} = \text{3-fit}}$ ($N_{MC} = 24000$)

I.



II. Example multipole:



$(\chi^2_{\text{best}}/\text{ndf} + 0.3)$

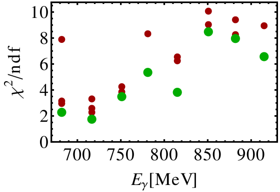
: BnGa 2014-02

: $\hat{\sigma}(W)$-interval

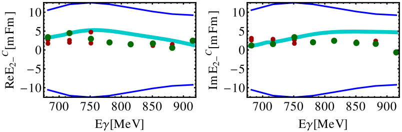
Results of fully un-constrained analyses

$\underline{\ell_{\max} = 2\text{-fit}}$ ($N_{MC} = 16000$)

I.



II.

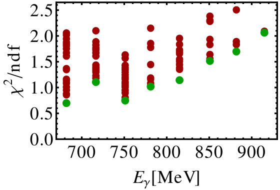


III. Issues:

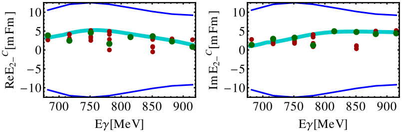
- Under-fitting for higher E_γ
- Missing the $\langle S, F \rangle$-, $\langle P, F \rangle$- and $\langle D, F \rangle$-interferences.

$\underline{\ell_{\max} = 3\text{-fit}}$ ($N_{MC} = 24000$)

I.



II.



III. Issues:

- Over-fitting for lower E_γ
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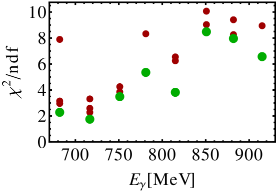
Results of fully un-constrained analyses

$\underline{\ell_{\max} = 2\text{-fit}}$ ($N_{MC} = 16000$)

I.



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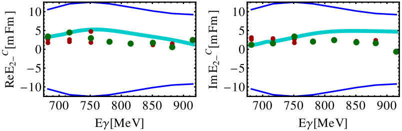

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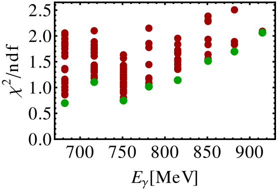
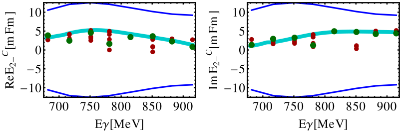
$\underline{\ell_{\max} = 3\text{-fit}}$ ($N_{MC} = 24000$)

I.



II.



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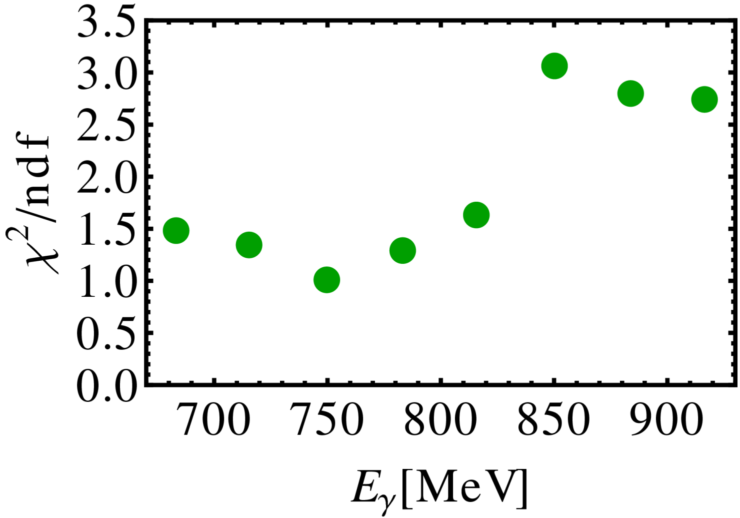
$\rightarrow$ Way out: Introduce model dependence by fixing the F-waves to BnGa 2014-02, letting all other mutlipoles float freely in the fit.

χ^2_{ndf} vs. E_γ for the fit including BnGa-F-waves



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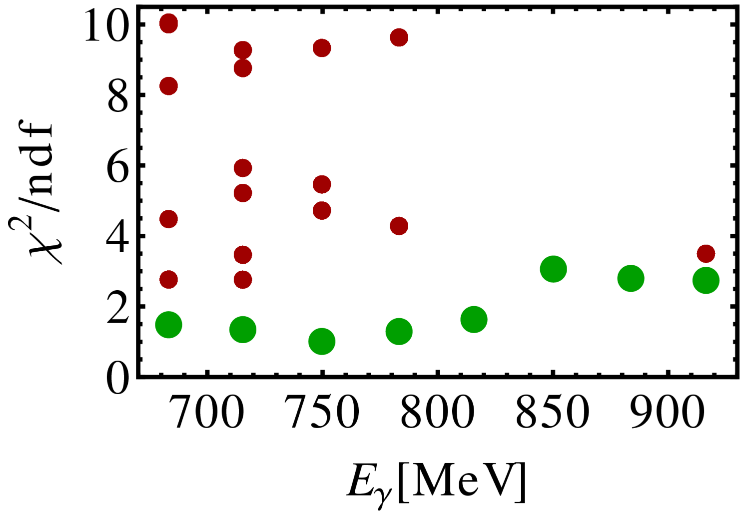
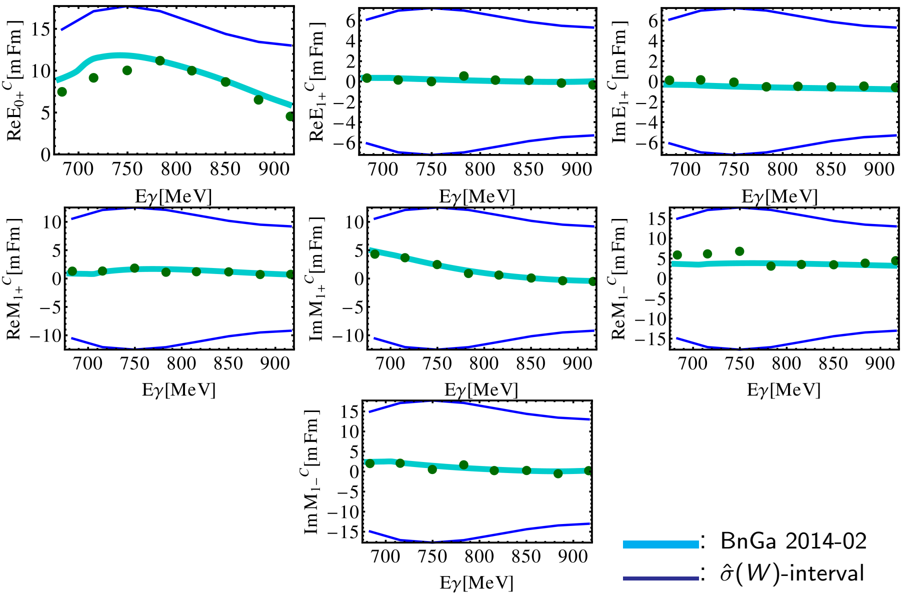


The best solution for S-, P- and D-waves



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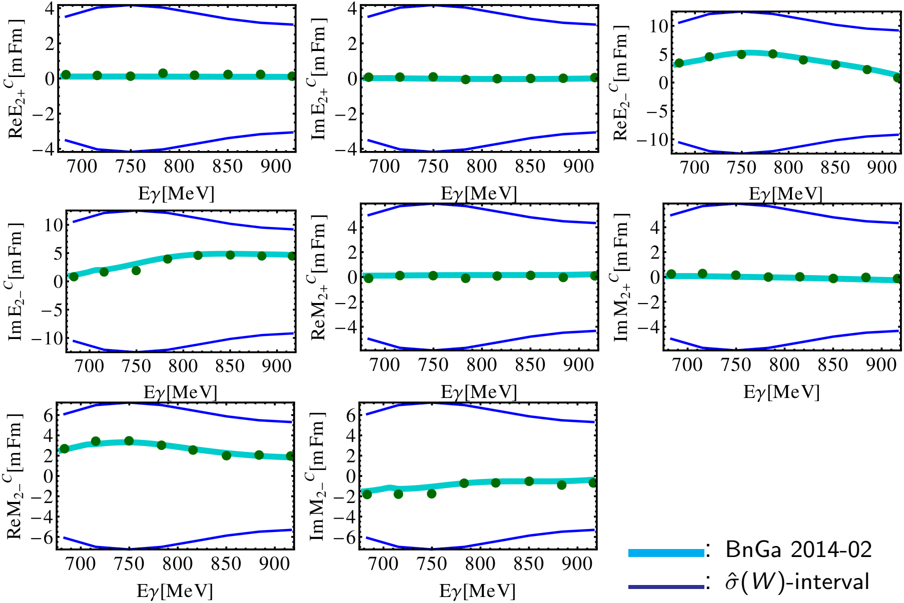
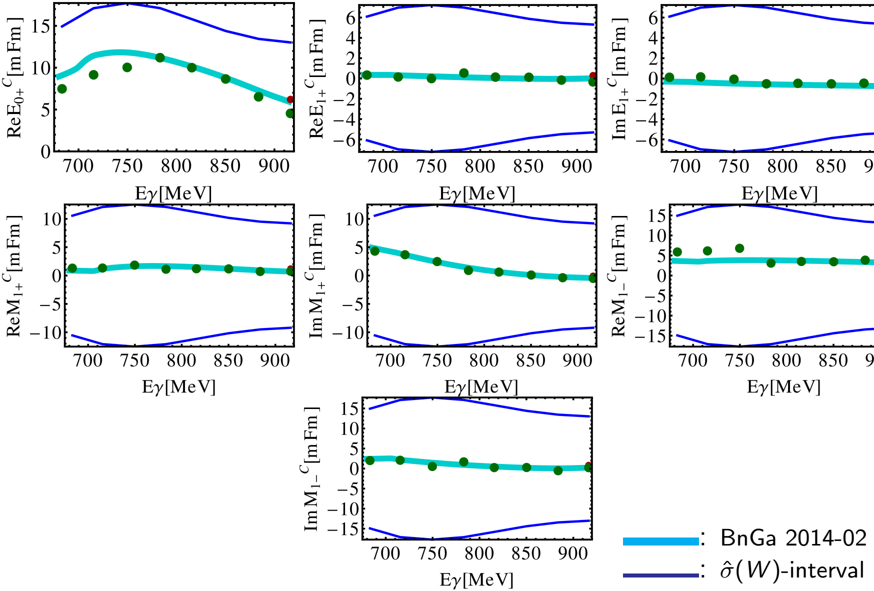


S-, P- and D-waves in the interval $\left(\chi^2_{\mathrm{best}} + 1.0\right)$



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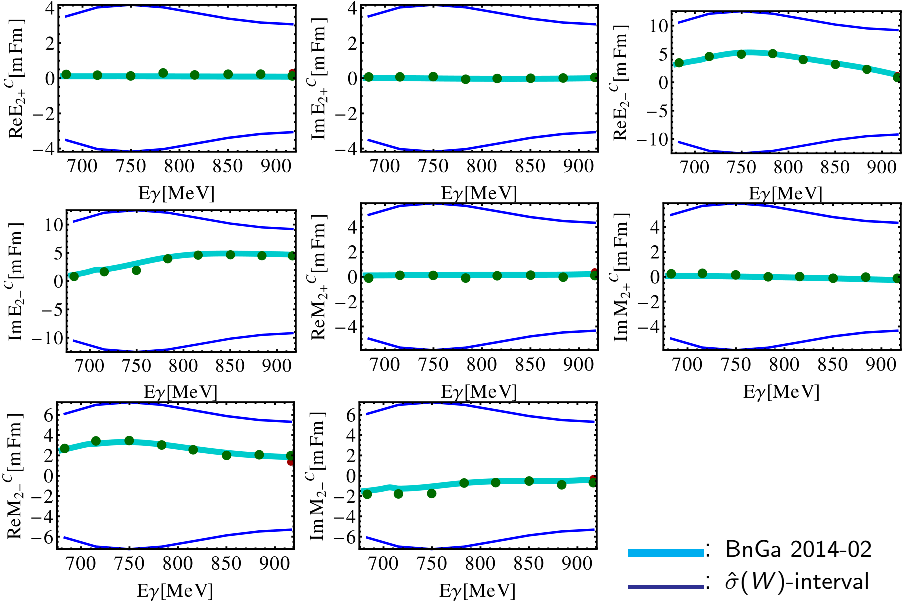
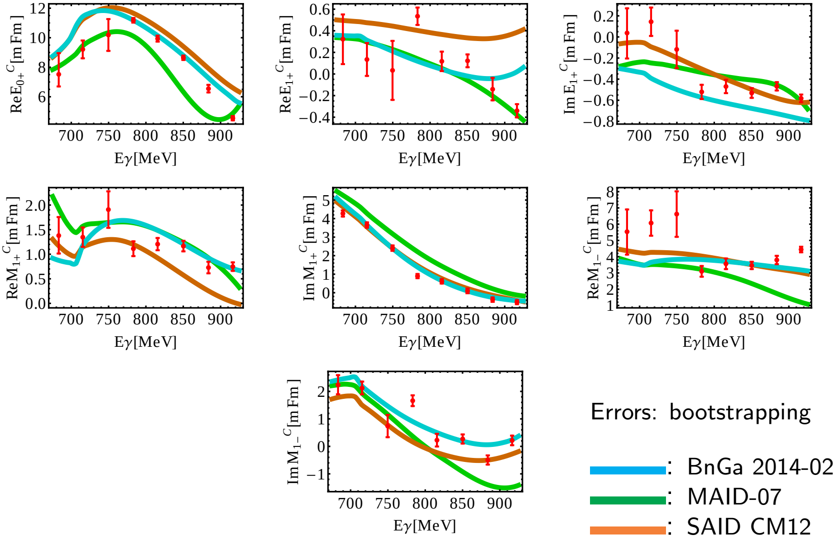


Results for the *S*-, *P*- and *D*-waves



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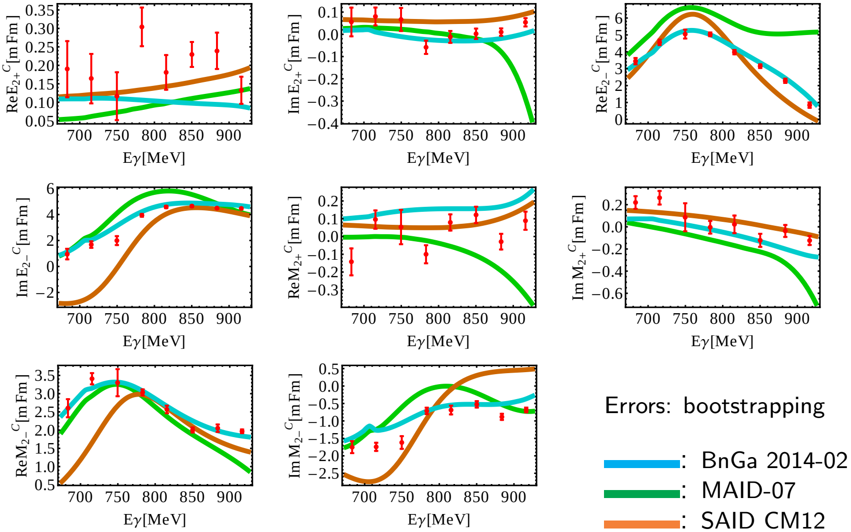

Summary & Outlook

$*)$ A monte-carlo sampling fit method was applied to $\{\sigma_0, \Sigma, T, P, E, G, H\}$-data for $\gamma p \rightarrow \pi^0 p$ in the 2^{nd} resonance region.

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- $\rightarrow$ $\ell_{\mathrm{max}} = 2$ multipole fit: the best solution is "unique" but χ^2 too large (high-low partial wave interferences!)
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$\ast$) What to do with the obtained solution?

- $\rightarrow$ Fitting a model independent pole+background parametrization ("L+P"-method of Alfred Švarc): yields D_{13}-pole-parameters ✓
- $\rightarrow$ Iteration of multipole-fitting with BnGa-code applied to SE-results: under construction ...

Thank You!

Additional Slides

Details on the multipole fit procedure II

Ansatz: Use the total cross section $\hat{\sigma}(W)$. Example: $\ell \leq \ell_{\max} = 1$, phase constraint $\mathrm{Im}\left[E_{0+}^{C}\right] = 0$ & $\mathrm{Re}\left[E_{0+}^{C}\right] > 0$:

$$\hat{\sigma}(W) \approx 4\pi \frac{q}{k}\Big(\mathrm{Re}\left[E_{0+}^{C}\right]^2 + 6\,\mathrm{Re}\left[E_{1+}^{C}\right]^2 + 6\,\mathrm{Im}\left[E_{1+}^{C}\right]^2 + 2\,\mathrm{Re}\left[M_{1+}^{C}\right]^2 + 2\,\mathrm{Im}\left[M_{1+}^{C}\right]^2 + \mathrm{Re}\left[M_{1-}^{C}\right]^2 + \mathrm{Im}\left[M_{1-}^{C}\right]^2\Big)$$

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Ansatz: Use the total cross section $\hat{\sigma}(W)$. Example: $\ell \leq \ell_{\mathrm{max}} = 1$, phase constraint $\mathrm{Im}\left[E_{0+}^{C}\right] = 0$ & $\mathrm{Re}\left[E_{0+}^{C}\right] > 0$:

$$\hat{\sigma}(W) \approx 4\pi \tfrac{q}{k} \Big(\mathrm{Re}\left[E_{0+}^{C}\right]^2 + 6\,\mathrm{Re}\left[E_{1+}^{C}\right]^2 + 6\,\mathrm{Im}\left[E_{1+}^{C}\right]^2 + 2\,\mathrm{Re}\left[M_{1+}^{C}\right]^2 + 2\,\mathrm{Im}\left[M_{1+}^{C}\right]^2 + \mathrm{Re}\left[M_{1-}^{C}\right]^2 + \mathrm{Im}\left[M_{1-}^{C}\right]^2 \Big)$$

$*$) $\hat{\sigma}(W)$ constrains the intervals of the multipoles:

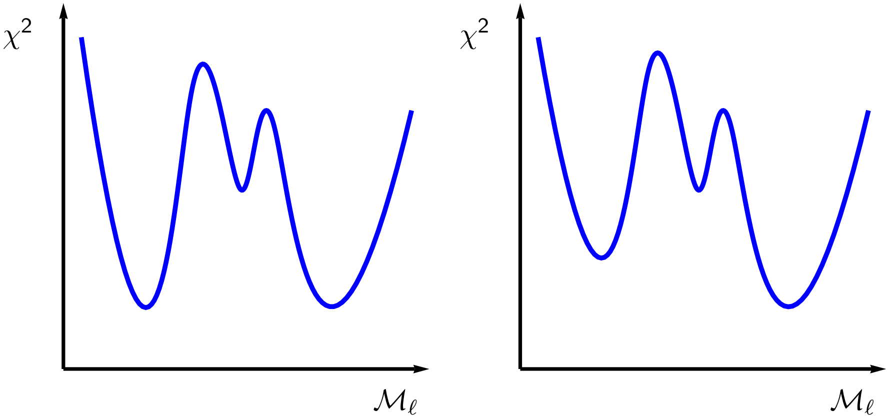
$$\mathrm{Re}\left[E_{0+}^{C}\right] \in \left[0, \sqrt{\tfrac{k}{q}\tfrac{\hat{\sigma}(W)}{4\pi}}\right], \ldots, \mathrm{Im}\left[M_{1-}^{C}\right] \in \left[-\sqrt{\tfrac{k}{q}\tfrac{\hat{\sigma}(W)}{4\pi}}, \sqrt{\tfrac{k}{q}\tfrac{\hat{\sigma}(W)}{4\pi}}\right]$$

$*$) The total cross section, being quadratic form in the multipoles, also defines an ellipsoid in the multipole space.

Cut selections for solution "data"

Cut on solutions χ_j^2 with $\frac{\chi_j^2 - \chi^2_{\mathrm{best}}}{\mathrm{ndf}} < \epsilon$

Mathematical ambiguity | Unique best solution



The χ^2 is defined by the fitted Legendre coefficients $\left(a_L^{\mathrm{Fit}}\right)_k^{\alpha}$.

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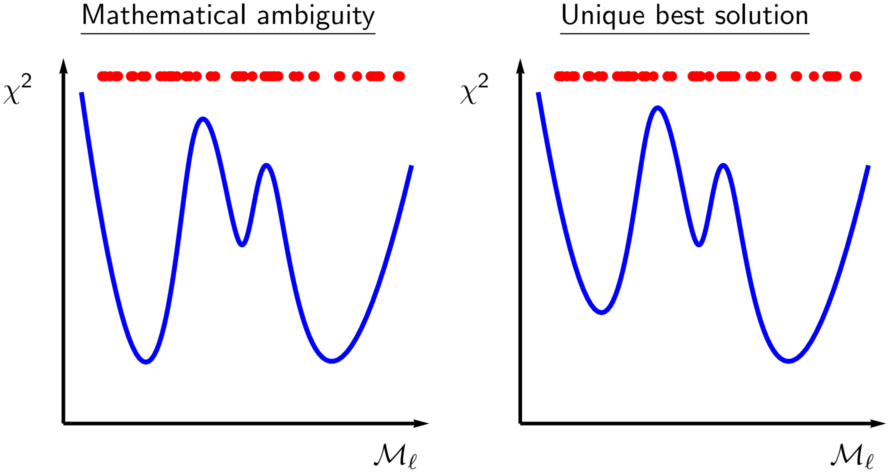

Start values have been distributed on the relevant part of the space $\mathcal{M}_\ell$.

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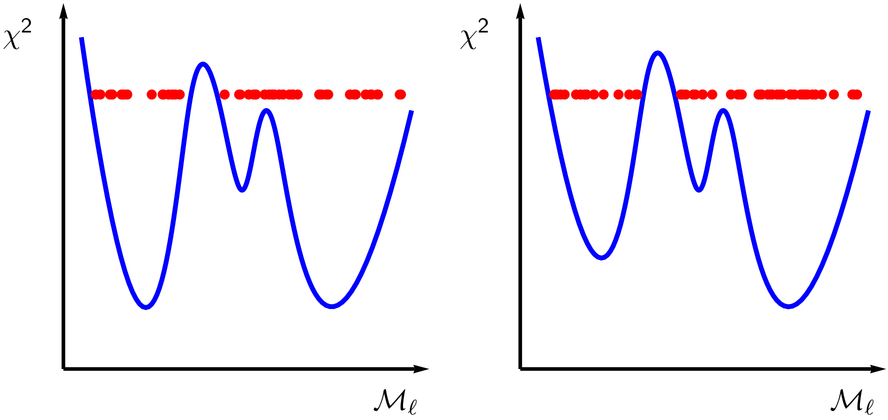

Minimizations of χ^2 converge within several iterations.

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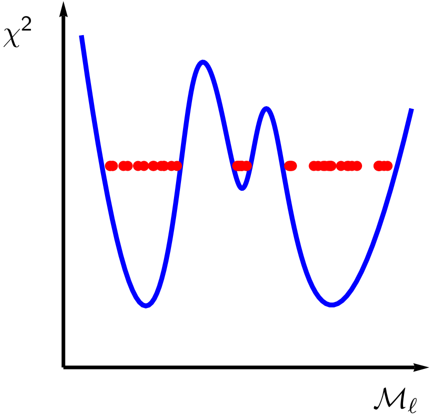

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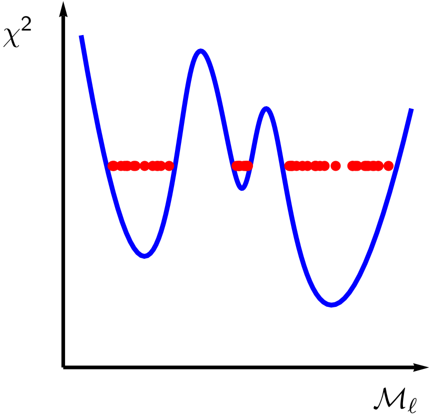

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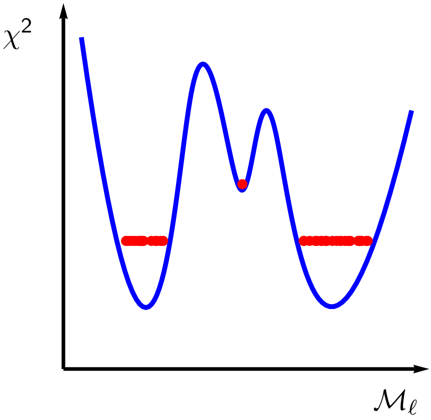

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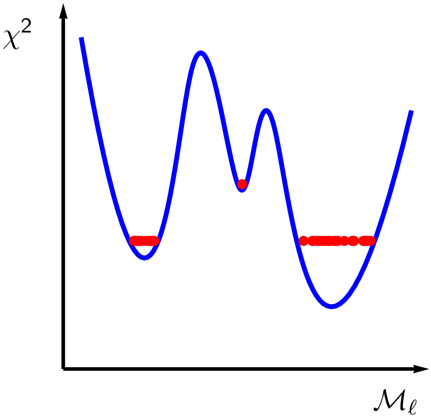

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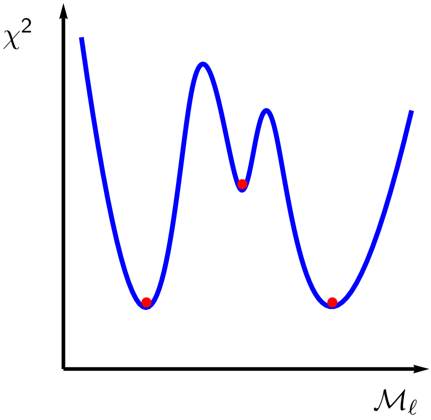

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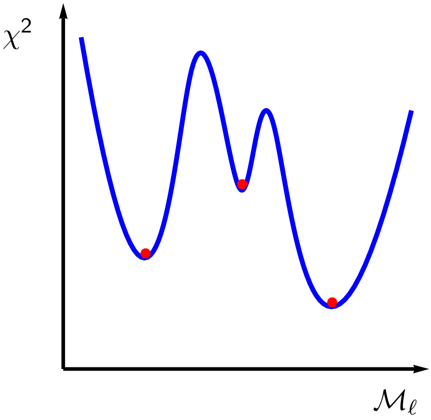

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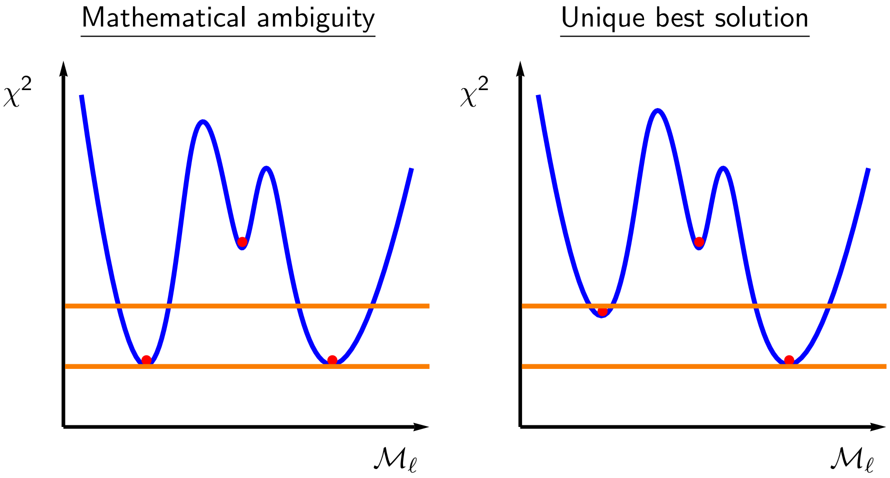

Cut selection using $\epsilon = 1$

Cut selections for solution "data"

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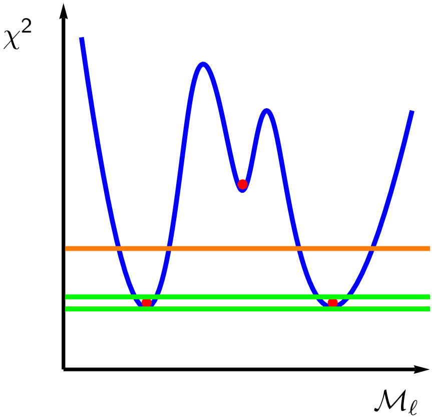

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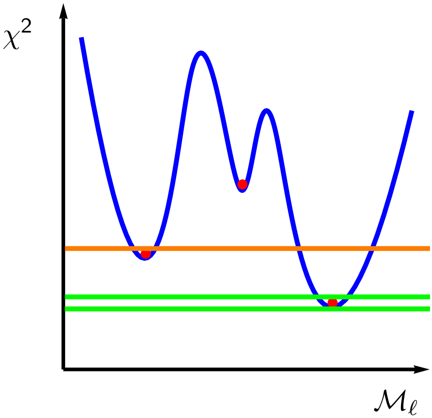

Cut selection using $\epsilon = 1$ / Cut selection using $\epsilon \sim \mathrm{num.\,precision}$

Bootstrapping

$*)$ [B. Efron, The Annals Of Statistics 7 no. 1, 1 (1979)]: Estimate an unknown distribution function of a random variable $R\left(X_1, \ldots, X_n, \hat{F}\right)$, by generating bootstrap random samples $x_b = (x_1^*, \ldots, x_n^*)$ from the data $(x_1, \ldots, x_n)$ and approximating the R-distribution-fct. by $R_b\left(x_b, \hat{F}\right)$.

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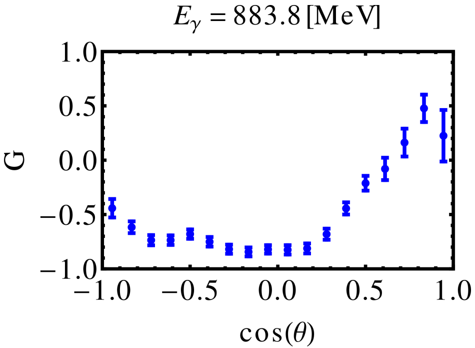

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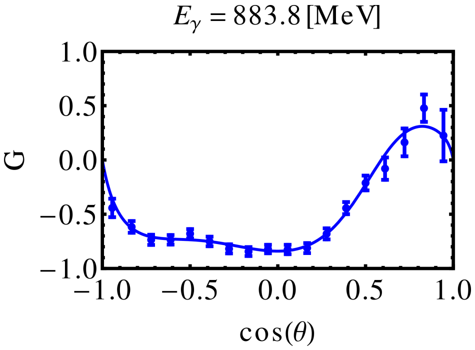

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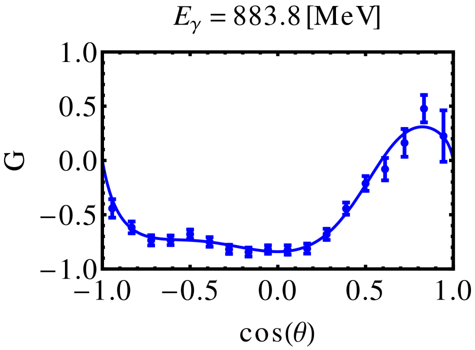

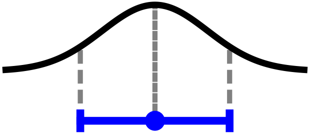

- Use gaussian distribution function centered at Ω^α, with $\sigma = \Delta\Omega^\alpha$.

$\rightarrow$ Resample data with replacement.

$\rightarrow$ Ensemble of $(1 + N_{\mathrm{Ens.}})$ bootstrap samples. Do TPWA for each.

$\rightarrow$ Histogram results for each multipole-fit-parameter.

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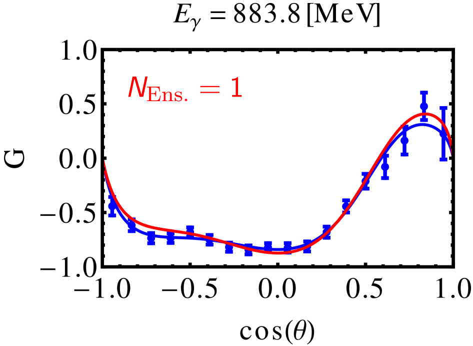

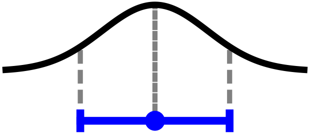

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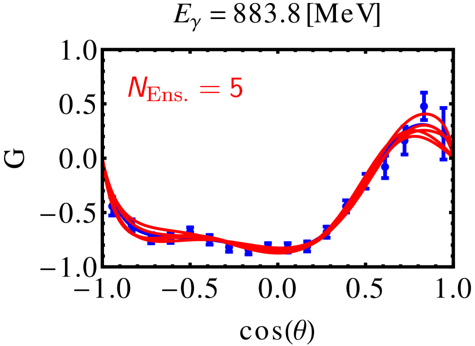

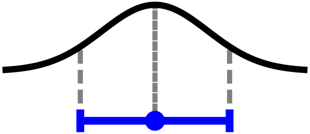

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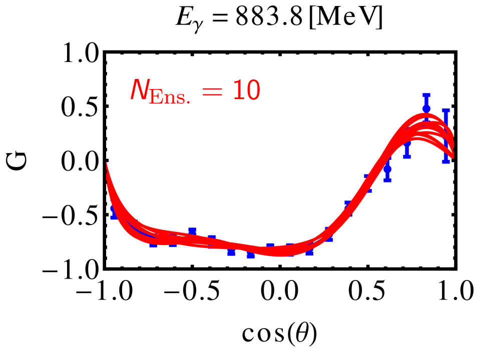

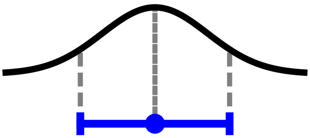

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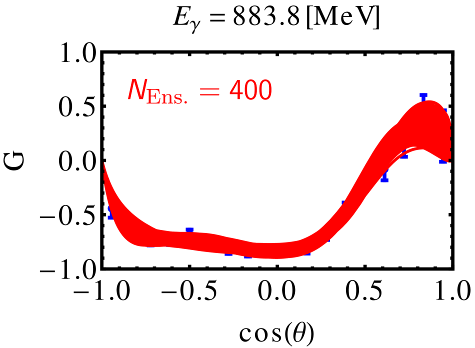

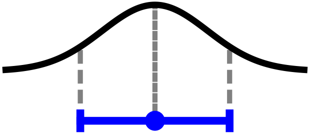

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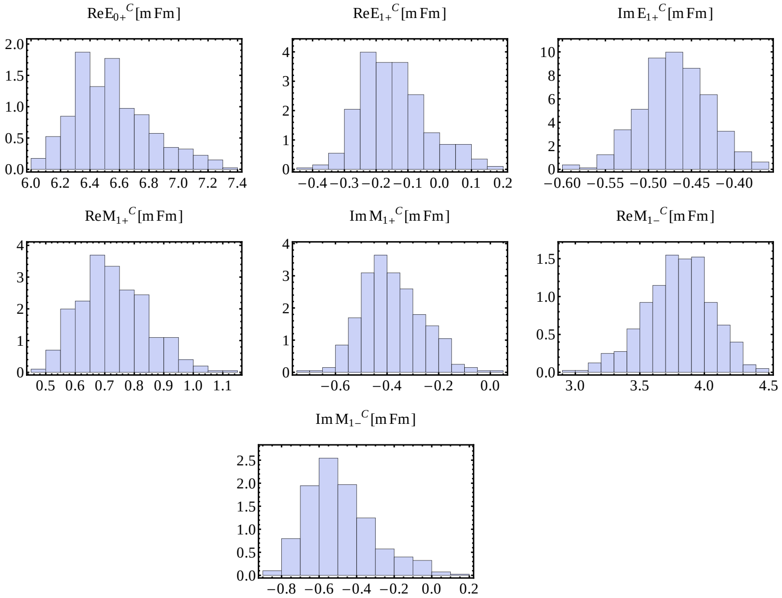
Bootstrap histograms for $E_\gamma = 883.8$ MeV



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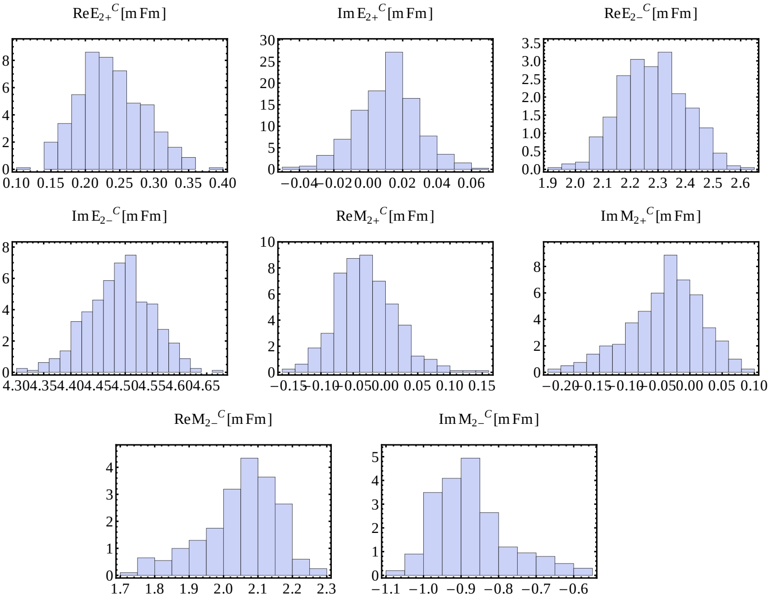

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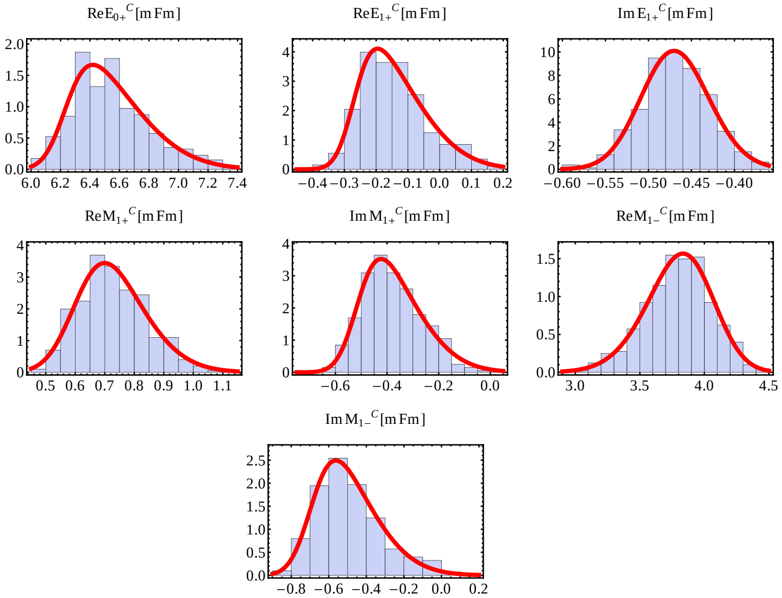

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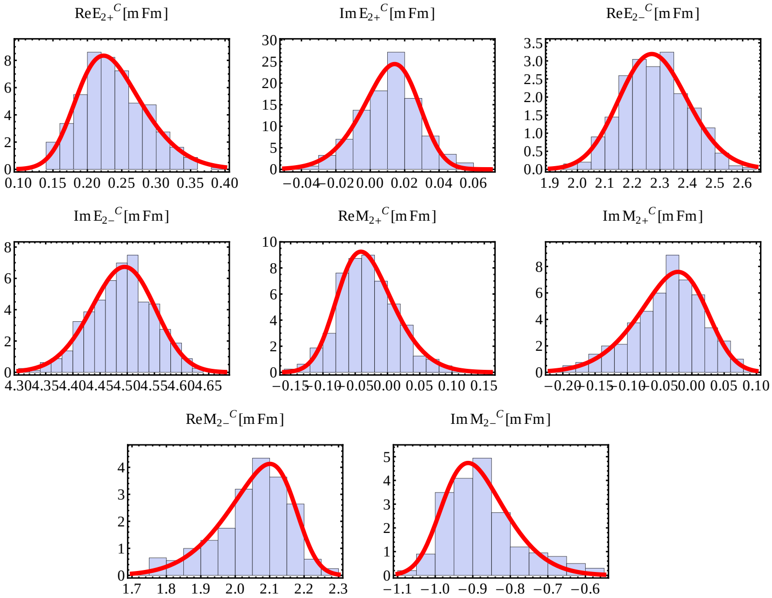

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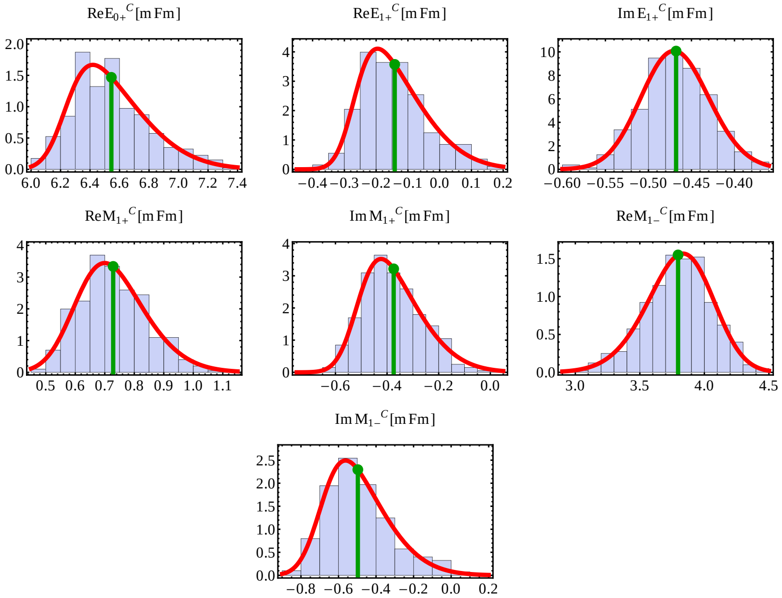

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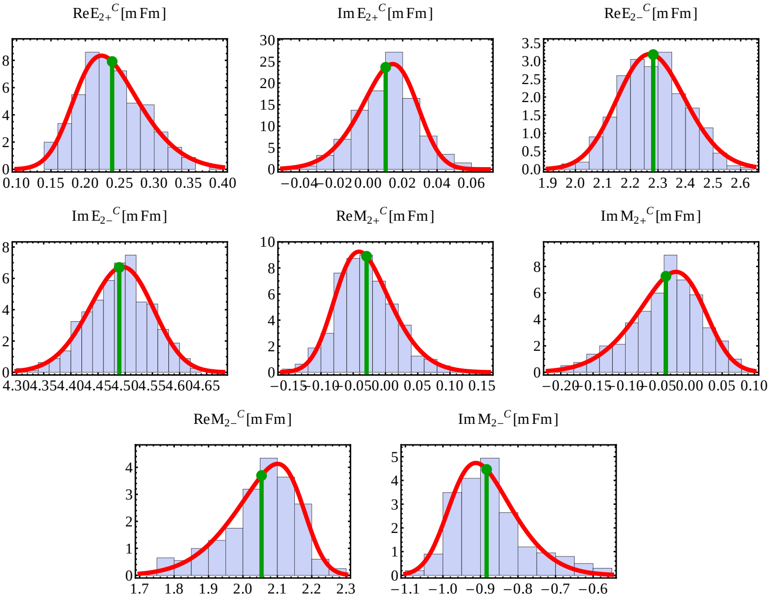

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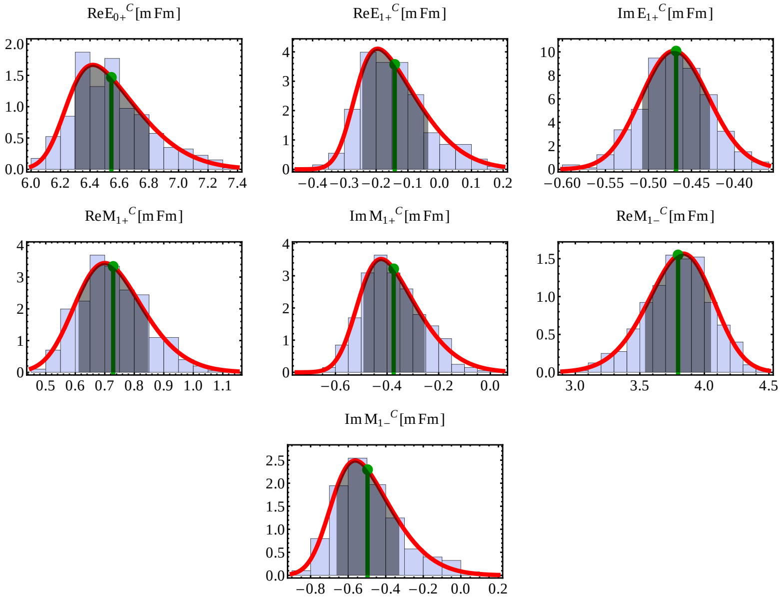

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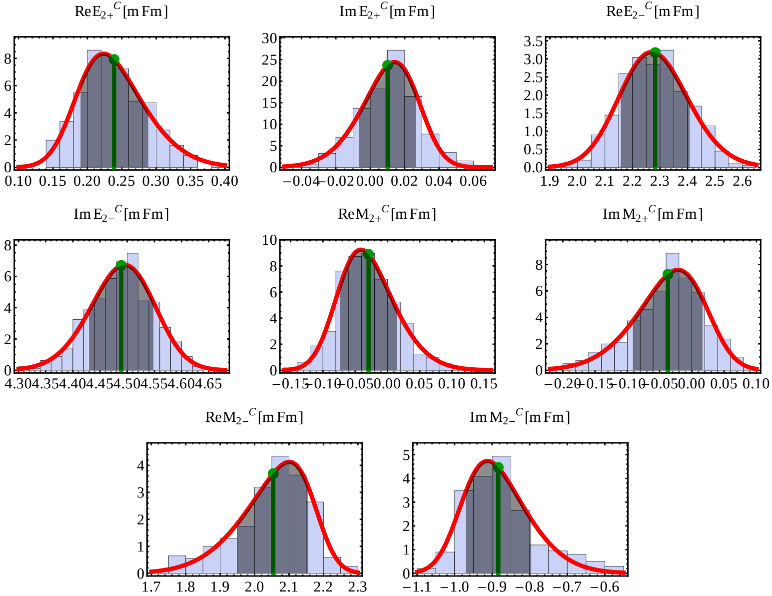

Polarization Observables - more detail

Problem: 4 complex amplitudes $F_i\left(W,\theta\right) \equiv 8$ real numbers
$\Rightarrow$ 1 observable $\left(\frac{d\sigma}{d\Omega}\right)_0$ insufficient to determine the amplitudes!

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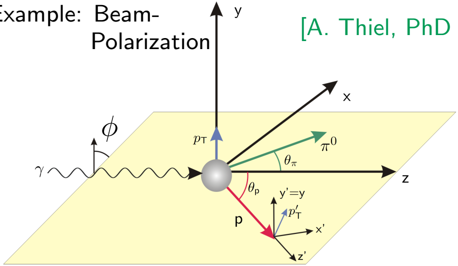
Solution: Utilize the polarization degrees of freedom of the reaction

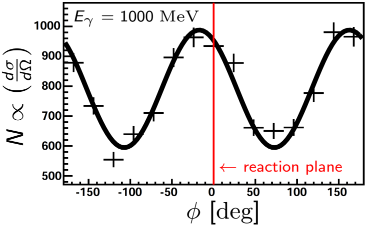
Example: Beam-Polarization

[A. Thiel, PhD (2012)]



$\Rightarrow$

$-0.44 < \cos\theta \leq -0.33$



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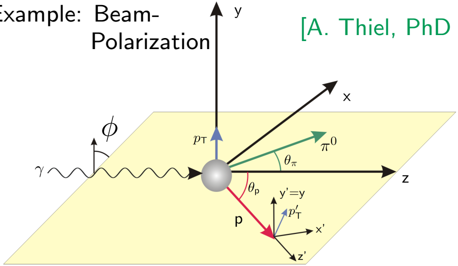

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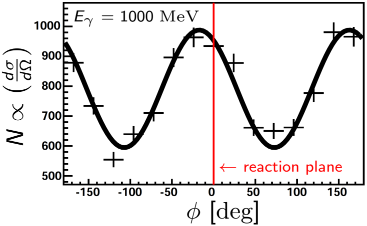

The observable Σ appears as amplitude of the ϕ-modulation

$$\left(\frac{d\sigma}{d\Omega}\right)(\theta,\phi) = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(1 - \epsilon_L \Sigma \cos(2\phi)\right).$$

Σ is an asymmetry between different polarization states:

$$\Sigma = \frac{1}{2\left(\frac{d\sigma}{d\Omega}\right)_0}\left[\left(\frac{d\sigma}{d\Omega}\right)^{(\perp,0,0)} - \left(\frac{d\sigma}{d\Omega}\right)^{(\parallel,0,0)}\right].$$

Complete Experiment: truncated partial wave analysis

$\ast$) Question: How many and which observables are needed if multipoles $\{E_{\ell\pm}, M_{\ell\pm}\}$ are the goal in a PWA truncated at some ℓ_{max}?.

Complete Experiment: truncated partial wave analysis

$\ast)$ Important hint: [A.S. Omelaenko, Sov. J. Nucl. Phys. 34, 406 (1981).]
Study of discrete ambiguities in a TPWA using the following trick: switch $\cos(\theta) \leftrightarrow t := \tan(\theta/2)$, use $b_4(\theta) = b_3(-\theta)$, $b_2(\theta) = b_1(-\theta)$ and do a linear factor decomposition of b_2 and b_4:

$$b_2(\theta) = -\mathcal{C} a_{2\ell_{\max}} \frac{\exp\left(i\frac{\theta}{2}\right)}{(1+t^2)^{\ell_{\max}}} \left[(t-\beta_1)(t-\beta_2)\ldots(t-\beta_{2\ell_{\max}})\right]$$

$$b_4(\theta) = \mathcal{C} a_{2\ell_{\max}} \frac{\exp\left(i\frac{\theta}{2}\right)}{(1+t^2)^{\ell_{\max}}} \left[(t-\alpha_1)(t-\alpha_2)\ldots(t-\alpha_{2\ell_{\max}})\right]$$

$\rightarrow$ A set of Omelaenko-roots $\{\alpha_k, \beta_k\}$ is fully equivalent to a multipole-solution $\{E_{\ell\pm}, M_{\ell\pm}\}$.

Complete Experiment: truncated partial wave analysis

∗) Important hint: [A.S. Omelaenko, Sov. J. Nucl. Phys. 34, 406 (1981).]

$$b_2\left(\theta\right) = -\mathcal{C} a_{2\ell_{\mathrm{max}}} \frac{\exp\left(i\frac{\theta}{2}\right)}{(1+t^2)^{\ell_{\mathrm{max}}}} \left[\left(t-\beta_1\right)\left(t-\beta_2\right)\ldots\left(t-\beta_{2\ell_{\mathrm{max}}}\right)\right]$$

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∗) Complex conjugation of all $\left\{\alpha_k, \beta_k\right\}$, or some subset of them, leaves $\left|b_i\right|^2$ invariant and therefore also the group S

$$\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\} \equiv \check{\Omega}^{\alpha s} = \frac{1}{2}\left(\pm\left|b_1\right|^2 \pm \left|b_2\right|^2 \pm \left|b_3\right|^2 + \left|b_4\right|^2\right)$$

Complete Experiment: truncated partial wave analysis

*) Important hint: [A.S. Omelaenko, Sov. J. Nucl. Phys. 34, 406 (1981).]

$$b_2(\theta) = -\mathcal{C} a_{2\ell_{\max}} \frac{\exp\left(i\frac{\theta}{2}\right)}{(1+t^2)^{\ell_{\max}}} \left[(t-\beta_1)(t-\beta_2)\ldots(t-\beta_{2\ell_{\max}})\right]$$

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*) Complex conjugation of all $\{\alpha_k, \beta_k\}$, or some subset of them, leaves $|b_i|^2$ invariant and therefore also the group S

$$\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\} \equiv \check{\Omega}^{\alpha s} = \frac{1}{2}\left(\pm|b_1|^2 \pm |b_2|^2 \pm |b_3|^2 + |b_4|^2\right)$$

*) Omelaenko's constraint $\boxed{\prod_k \alpha_k = \prod_{k'} \beta_{k'}}$ has to be fulfilled.

Complete Experiment: truncated partial wave analysis

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$$b_4(\theta) = \mathcal{C} a_{2\ell_{\max}} \frac{\exp\left(i\frac{\theta}{2}\right)}{(1+t^2)^{\ell_{\max}}} \left[(t-\alpha_1)(t-\alpha_2)\ldots(t-\alpha_{2\ell_{\max}})\right]$$

∗) Complex conjugation of $\{\alpha_k, \beta_k\}$ leaves $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$ invariant.

∗) Omelaenko's constraint $\prod_k \alpha_k = \prod_{k'} \beta_{k'}$.

→ 2 kinds of symmetries/ambiguities:

double ambiguity

- $\alpha_k \to \alpha_k^*$, $\beta_k \to \beta_k^*$ for all k
- One has always $\prod_k \alpha_k = \prod_{k'} \beta_{k'}$ $\to \prod_k \alpha_k^* = \prod_{k'} \beta_{k'}^*$

→ Mathematically exact symmetry

- Is resolved by F, G, as well as any $\mathcal{BR}$ and $\mathcal{TR}$ observable.

accidential ambiguities

- Conjugation of subset of $\{\alpha_k, \beta_k\}$ $\to \left\{\tilde{\alpha}_k, \tilde{\beta}_k\right\}$
- Very (very) likely $\prod_k \tilde{\alpha}_k \simeq \prod_{k'} \tilde{\beta}_{k'}$

→ Manifest as approximate symmetry

- Is resolved by in principle any observable.

Which $\ell_{\max}$ to choose? $\rightarrow$ "LFit-method"

$*$) Utilize the parametrization of the angular distributions of polarization observables $\check{\Omega}^{\alpha}$ as expansions into $P_{\ell}^{m}(\cos\theta)$ for fixed energy:

$$\check{\Omega}^{\alpha}(W,\theta) = \sum_{k=\beta_{\alpha}}^{2\ell_{\max}+\beta_{\alpha}+\gamma_{\alpha}} (a_L)_k^{\alpha}(W)\, P_k^{\beta_{\alpha}}(\cos\theta)\,.$$

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$\to$ Raise truncation order by 1 and do new fit until $\left(\chi^2/\mathrm{ndf}\right) \approx 1$.

$\to$ Hint for dominant partial wave by the order $\ell_{\max}$ at which this procedure terminates.

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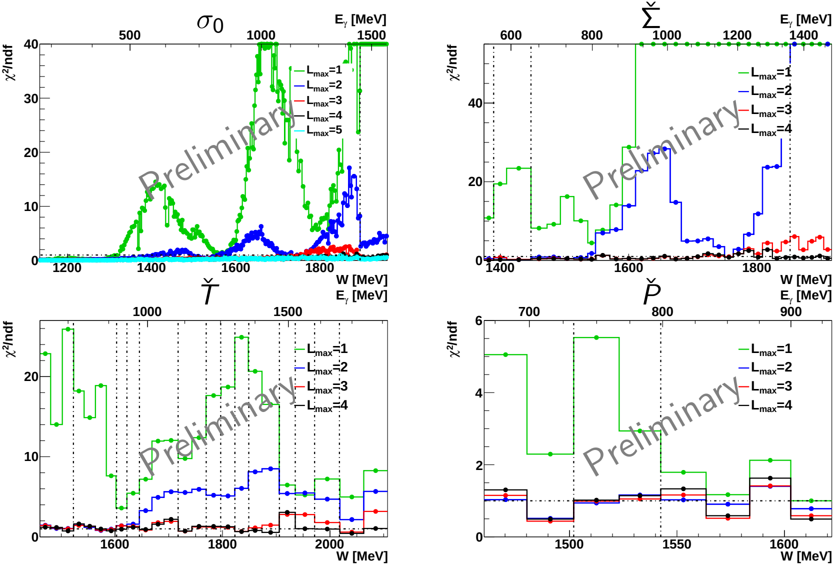
$\rightarrow$ Hint for dominant partial wave by the order $\ell_{\max}$ at which this procedure terminates.

$*$) <u>Nice:</u> Procedure is simple, model-independent and furthermore reliably reflects the capability of the data to give infomation on higher partial wave contributions.

LFits to $\{\sigma_0, \Sigma, T, P, E, G, H\}$

To be published in [Y. W., F. Afzal, A. Thiel and R. Beck, (2016)]



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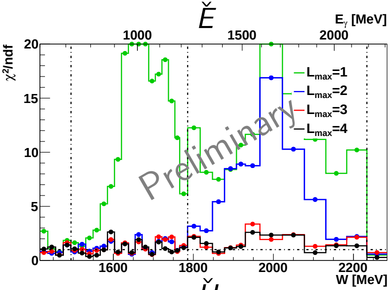
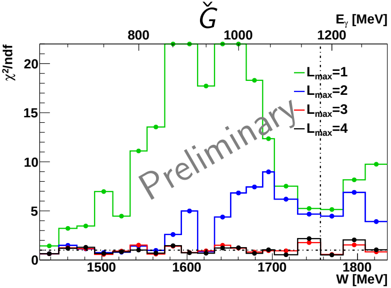
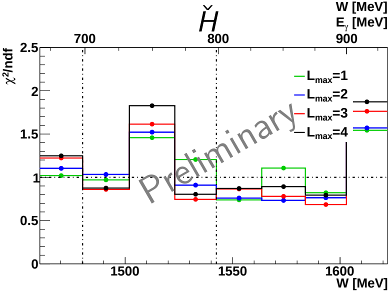




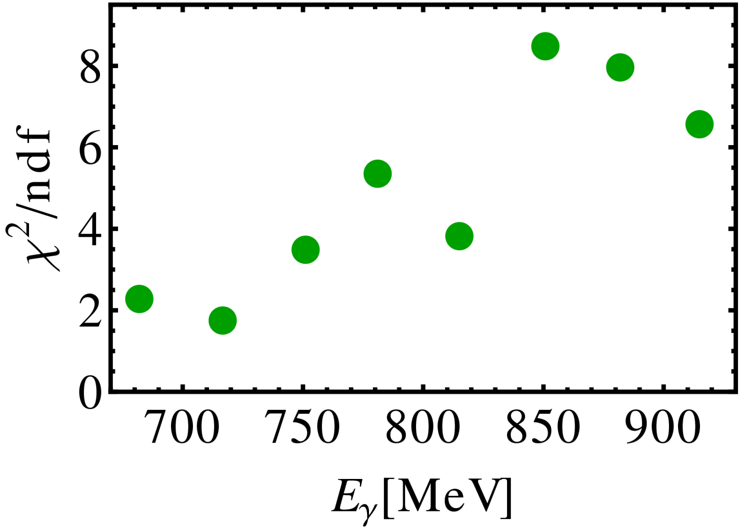


Overall, $\ell_{\mathrm{max}} = 2$ should be OK in all energy bins

$$E_{\gamma}^{\mathrm{LAB}} \in [650, 950] \text{ MeV}$$

except maybe the last 2 bins.

χ^2_{best} vs. E_γ for the $\ell_{\text{max}} = 2$-fit



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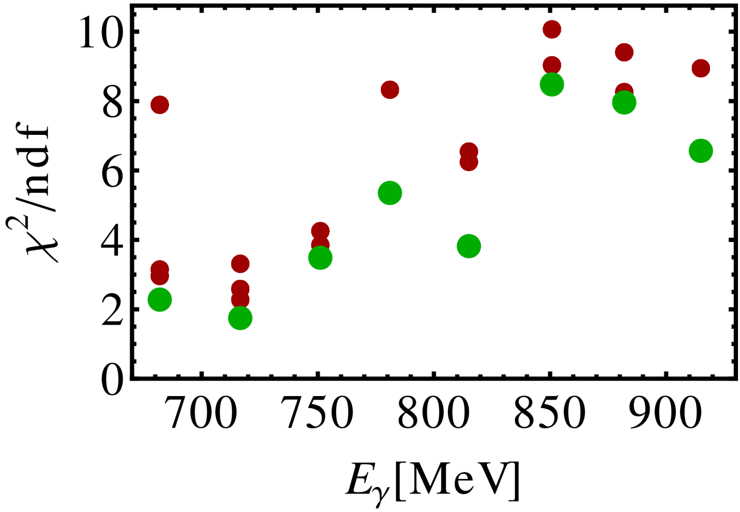
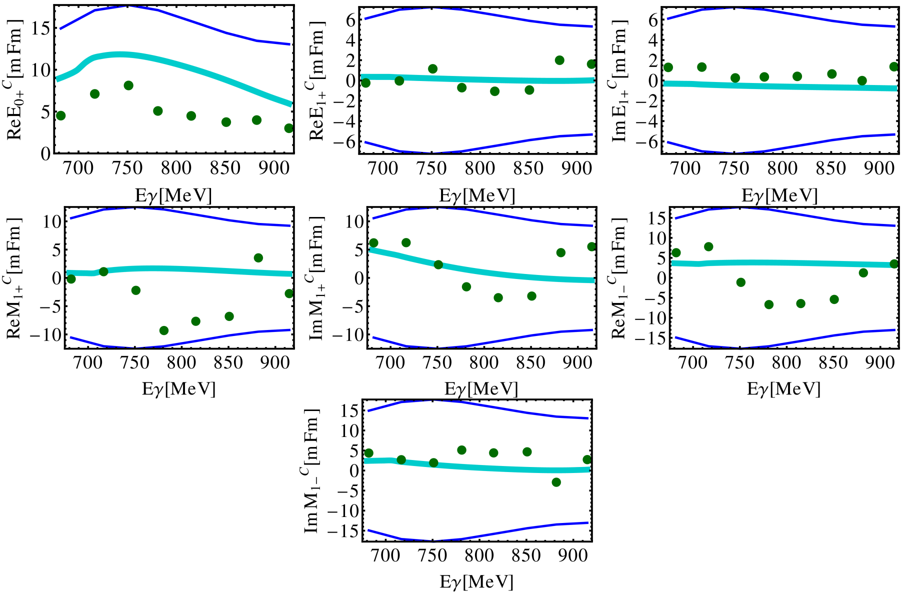


The best solution for S-, P- and D-waves



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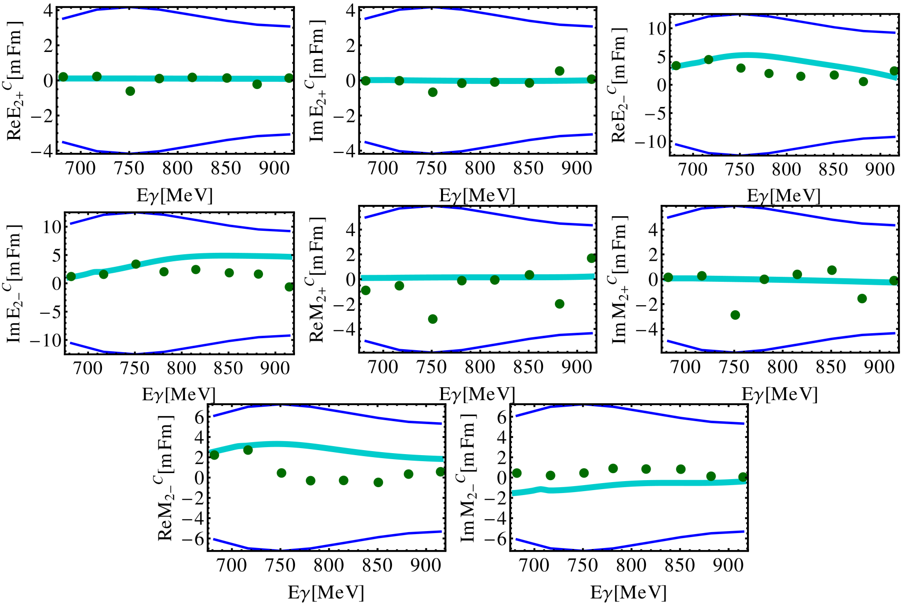
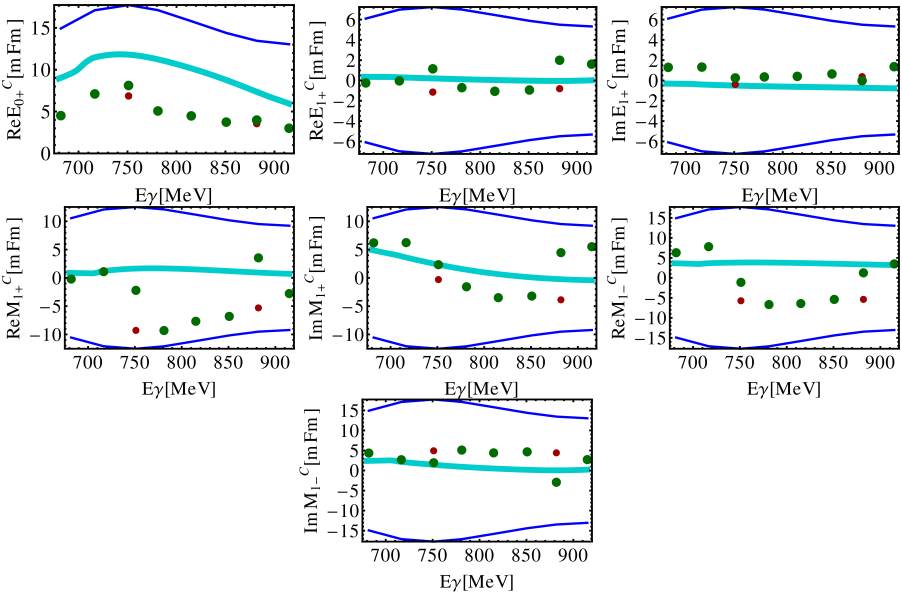


S-, P- and D-waves in the interval $\left(\chi^2_{\mathrm{best}} + 0.5\right)$



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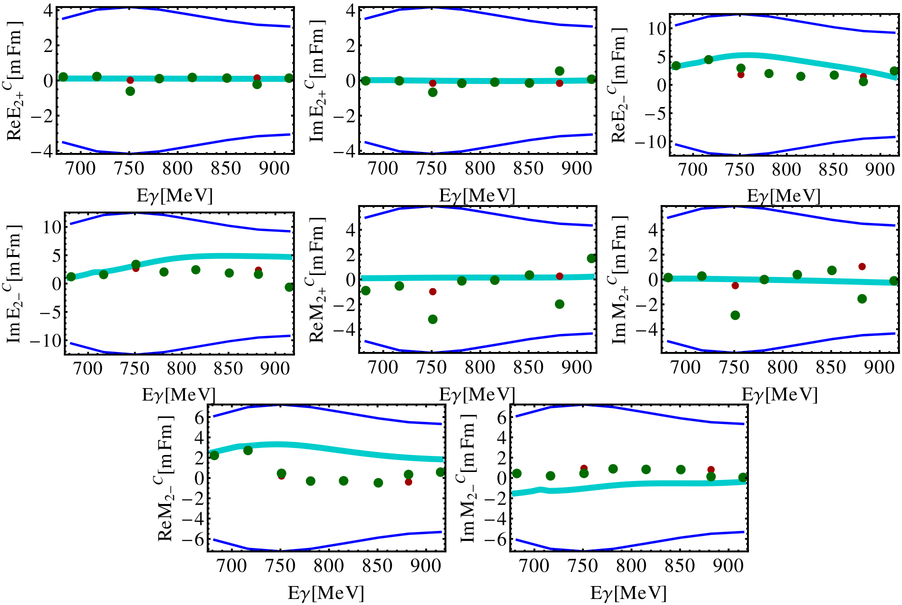
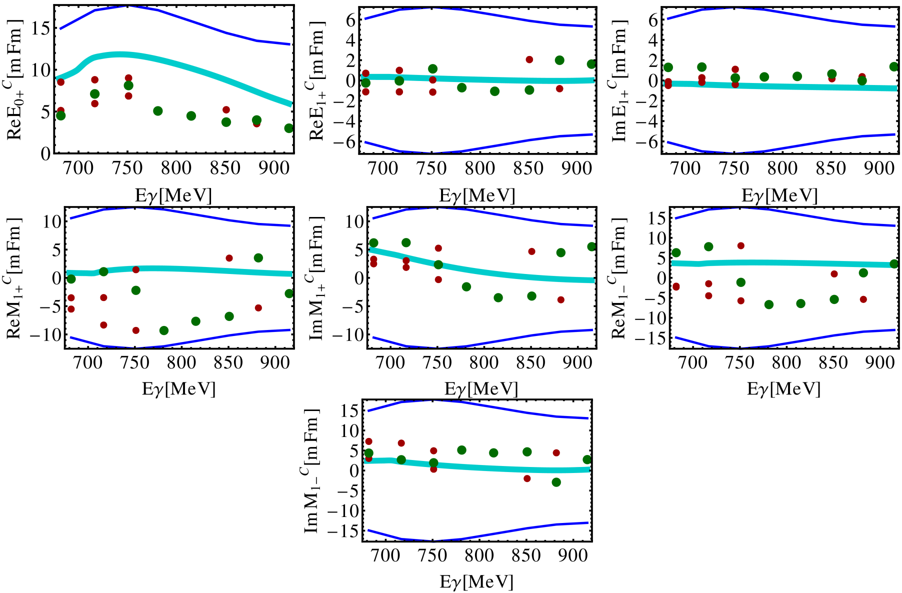
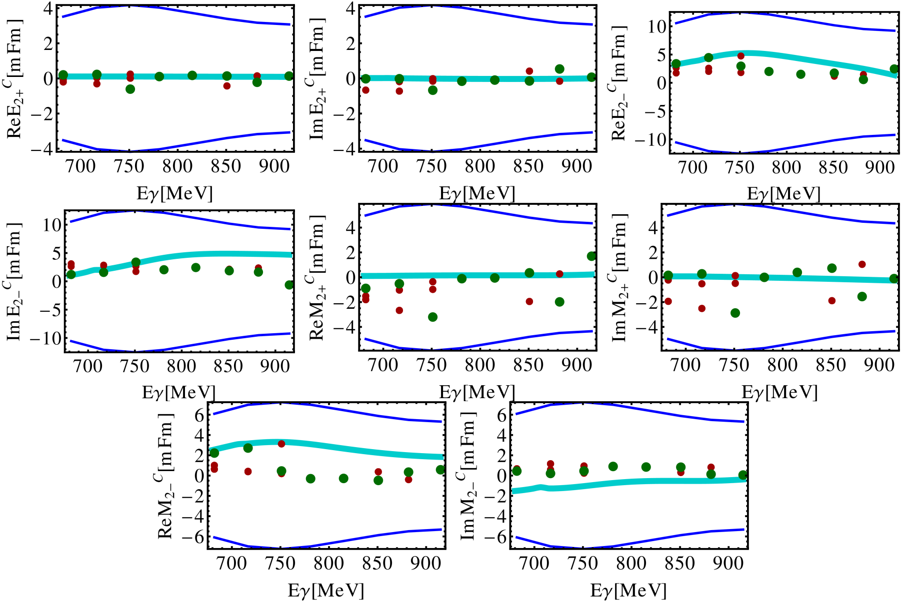


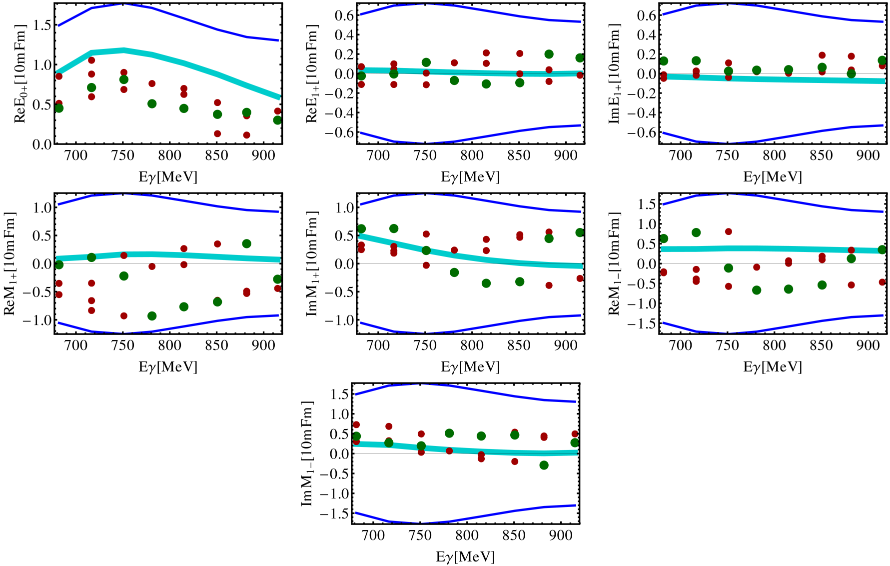
S-, P- and D-waves in the interval $\left(\chi^2_{\text{best}} + 1.0\right)$



S-, P- and D-waves in the interval $\left(\chi^2_{\mathrm{best}} + 1.0\right)$



S-, P- and D-waves in the interval $\left(\chi^2_{\mathrm{best}} + 4.0\right)$



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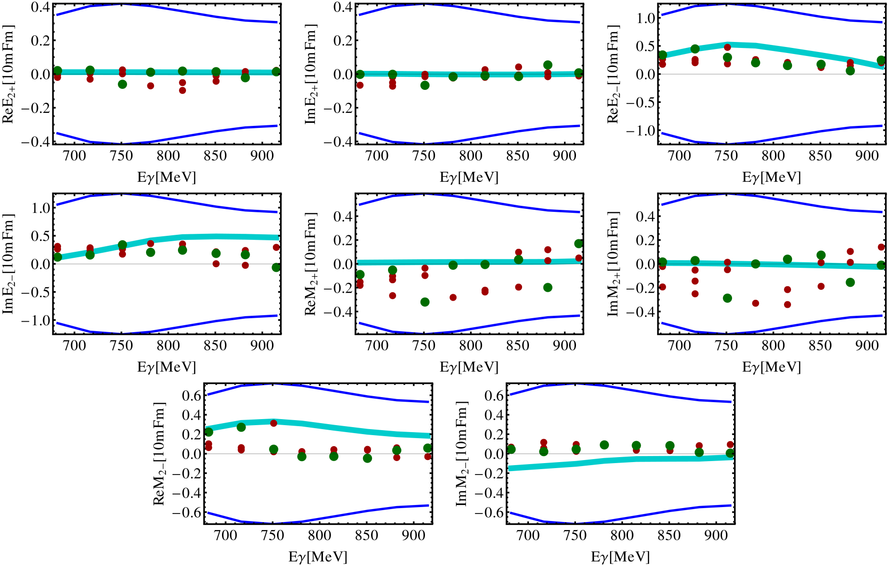

Problems with the $\ell_{\mathrm{max}} = 2$ multipole fit

There exists a unique, well-separated (in χ^2) solution for $\ell_{\mathrm{max}} = 2$,

<u>however:</u>

(i) χ^2/ndf is too large for all energy bins except the first 2-4.

(ii) Solution does not make sense compared to models (more precisely, to BnGa 2014-02).

Partial wave interferences in Legendre coefficients

$$
(a_L)_k^{\alpha} = \left[\; \mathcal{M}^{*}_{\ell \leq \ell_{\mathrm{max}}} \; \middle| \; \mathcal{M}^{*}_{\ell > \ell_{\mathrm{max}}} \; \right] \left[\begin{array}{c|c} \left(\mathcal{C}_L\right)_k^{\alpha} & \left(\tilde{\mathcal{C}}_L\right)_k^{\alpha} \\ \hline \left[\left(\tilde{\mathcal{C}}_L\right)_k^{\alpha}\right]^{\dagger} & \left(\hat{\mathcal{C}}_L\right)_k^{\alpha} \end{array} \right] \left[\begin{array}{c} \mathcal{M}_{\ell \leq \ell_{\mathrm{max}}} \\ \hline \mathcal{M}_{\ell > \ell_{\mathrm{max}}} \end{array} \right]
$$

$*$) In the $(a_L)_k^{\alpha}$, partial waves with $\ell_{\mathrm{max}} \geq 3$ may interfere with those having $\ell_{\mathrm{max}} \leq 3$ but the LFits may only hint at this, or <u>not</u> show this at all!

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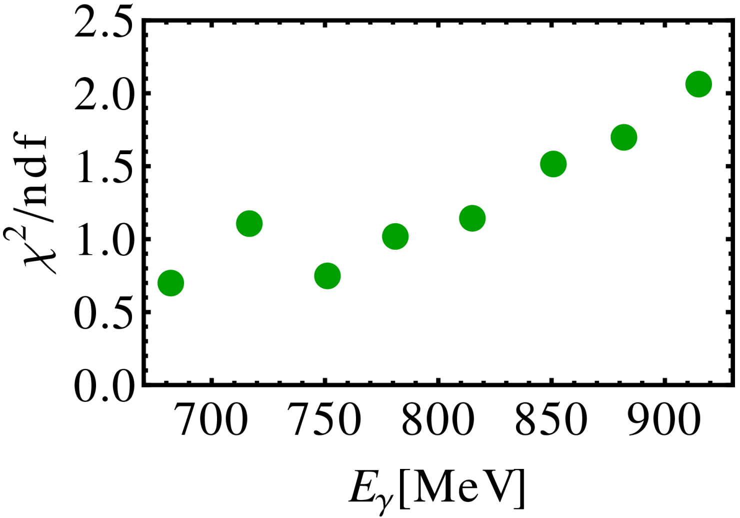
- $*$) In case the multipole fit has all partial waves $\mathcal{M}_\ell$ with $\ell \geq 3$ set equal to zero, it has no chance to take into account the interferences and modify the results for S-, P-, and D-waves accordingly.
- $\rightarrow$ One has to at least take into account F-waves into the fitting in some way!
- $\rightarrow$ Fit a truncation at $\ell_{\max} = 3$ and let the F-waves run freely in the fit.

χ^2_{best} vs. E_γ for the $\ell_{\max} = 3$-fit



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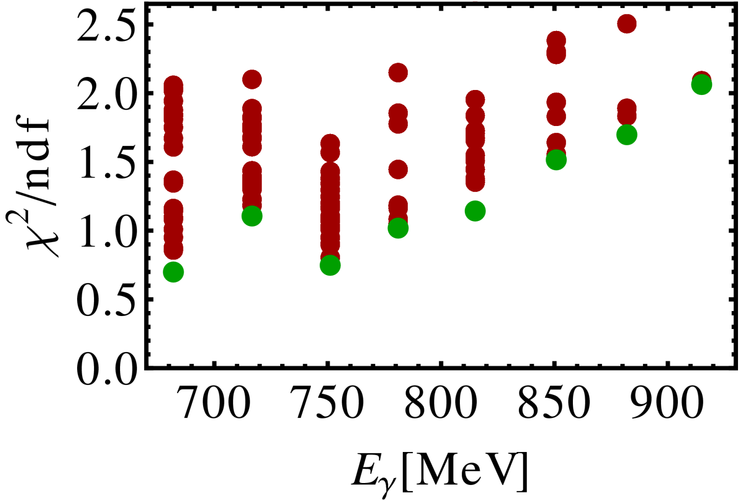
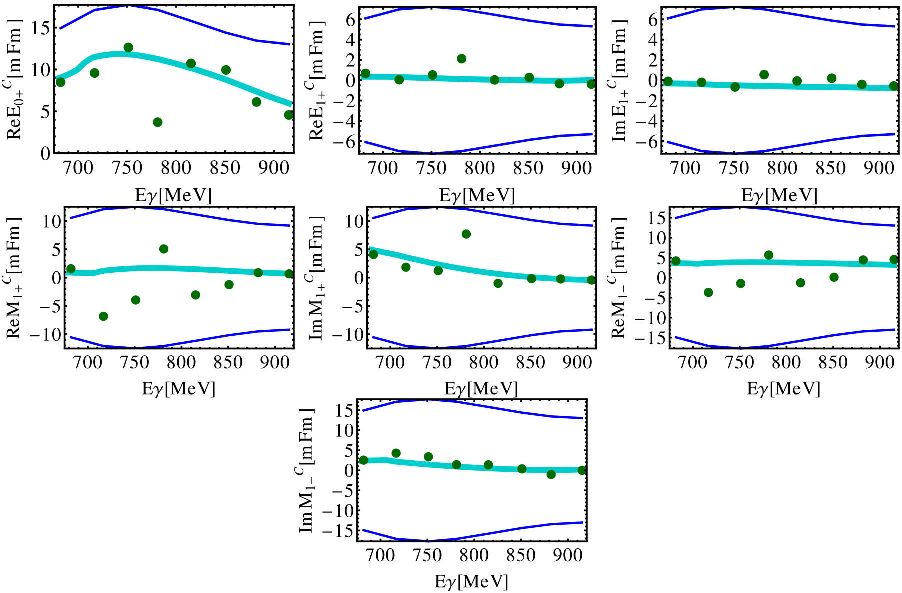
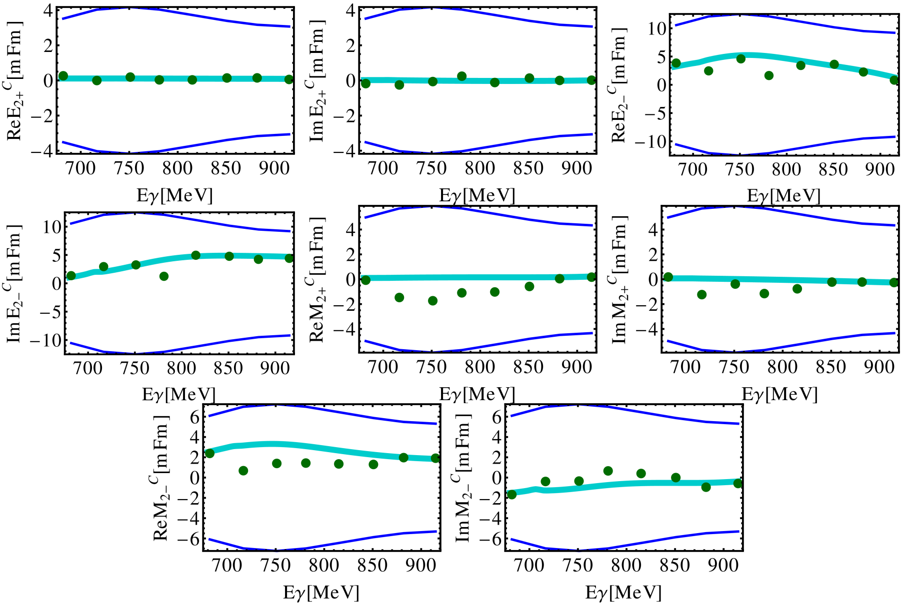


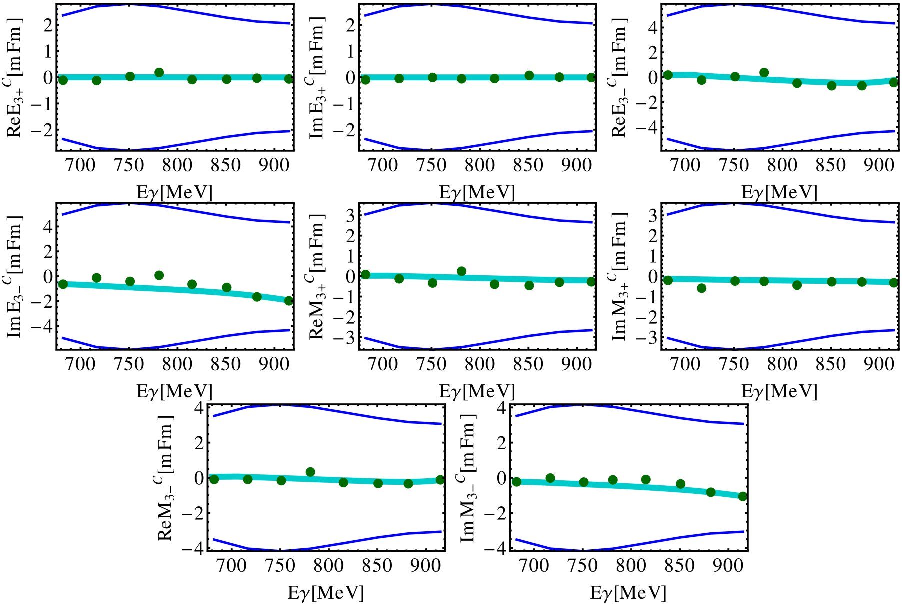
The best solution for S-, P-, D- and F-waves



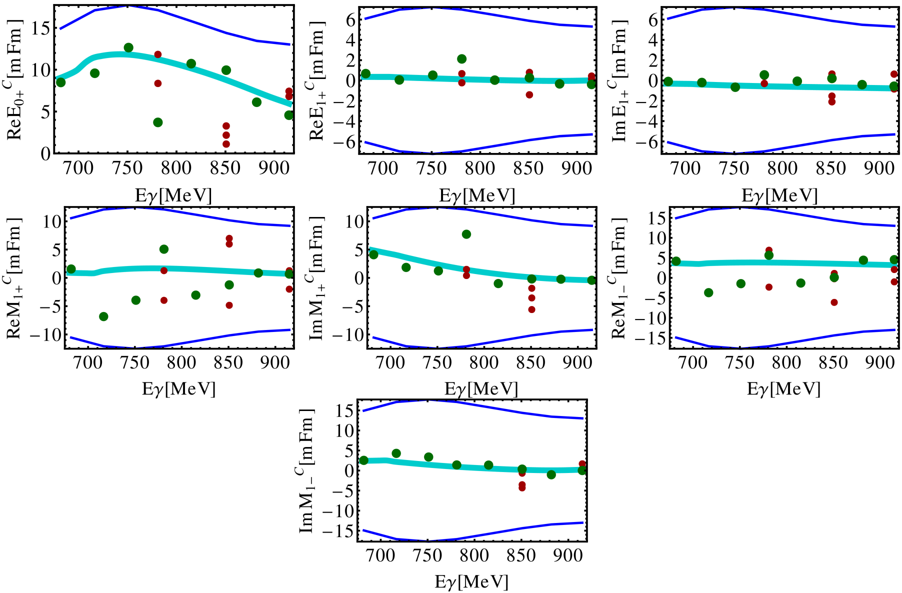
The best solution for *S*-, *P*-, *D*- and *F*-waves



The best solution for S-, P-, D- and F-waves



S-, P-, D- and F-waves in the interval $\left(\chi^2_{\mathrm{best}} + 0.05\right)$



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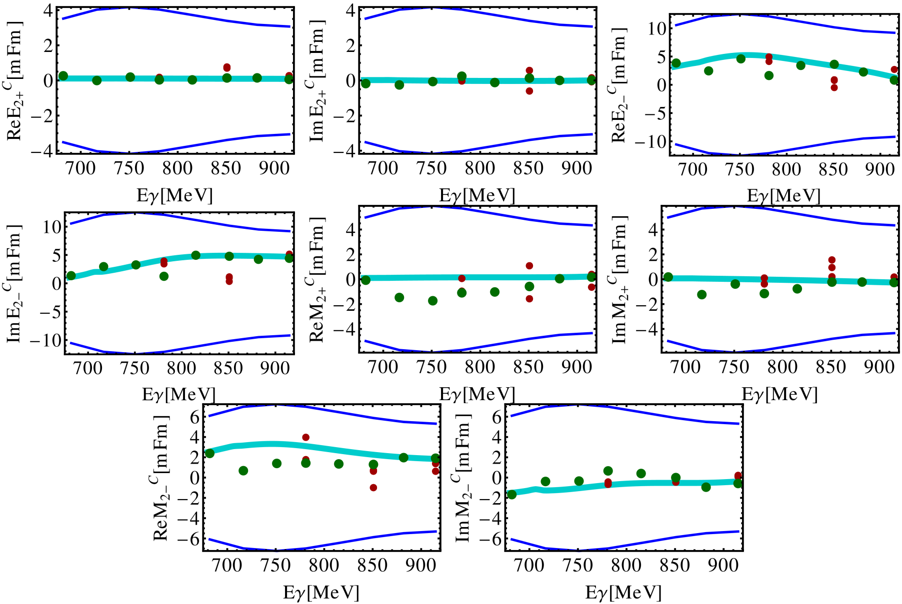

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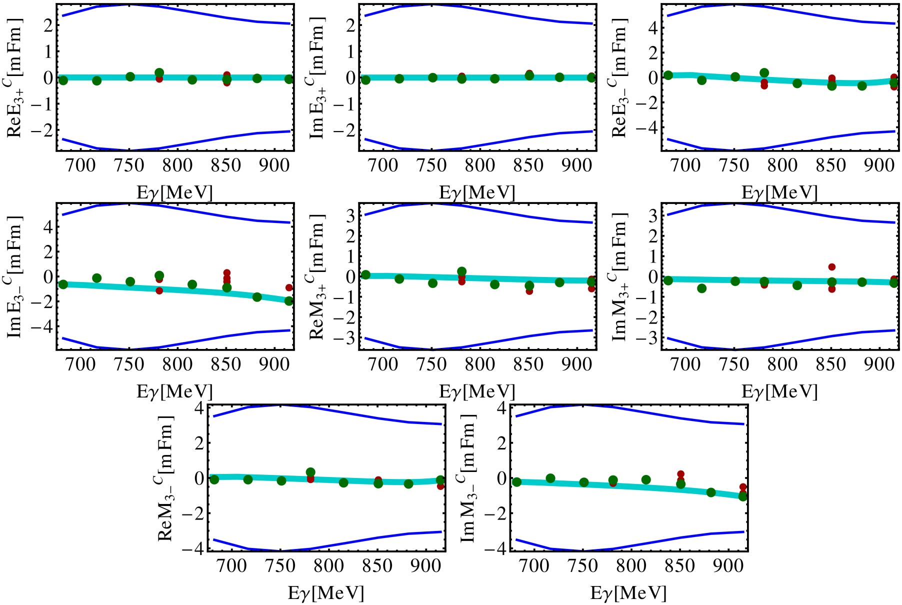
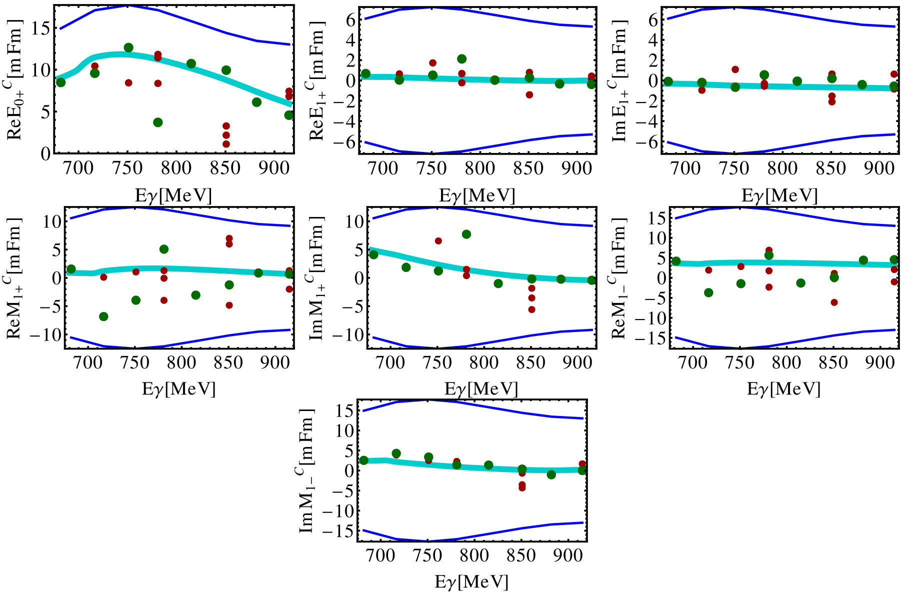
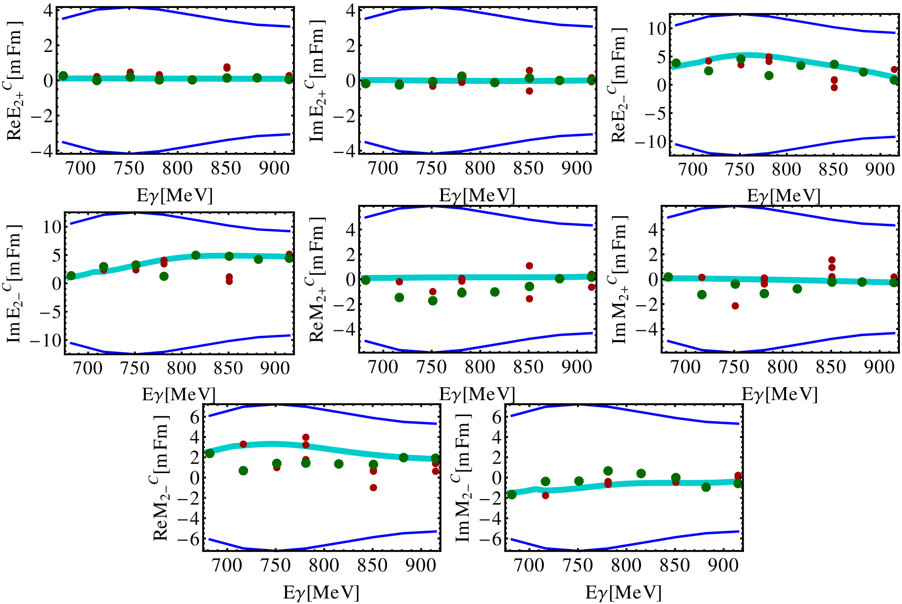


S-, P-, D- and F-waves in the interval $\left(\chi^2_{\text{best}} + 0.1\right)$



S-, *P*-, *D*- and *F*-waves in the interval $\left(\chi^2_{\mathrm{best}} + 0.1\right)$



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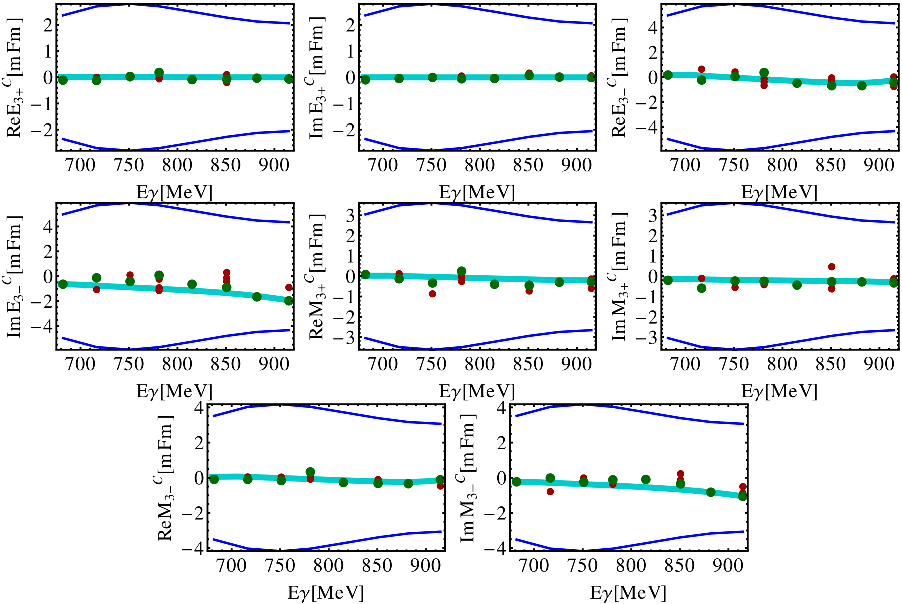
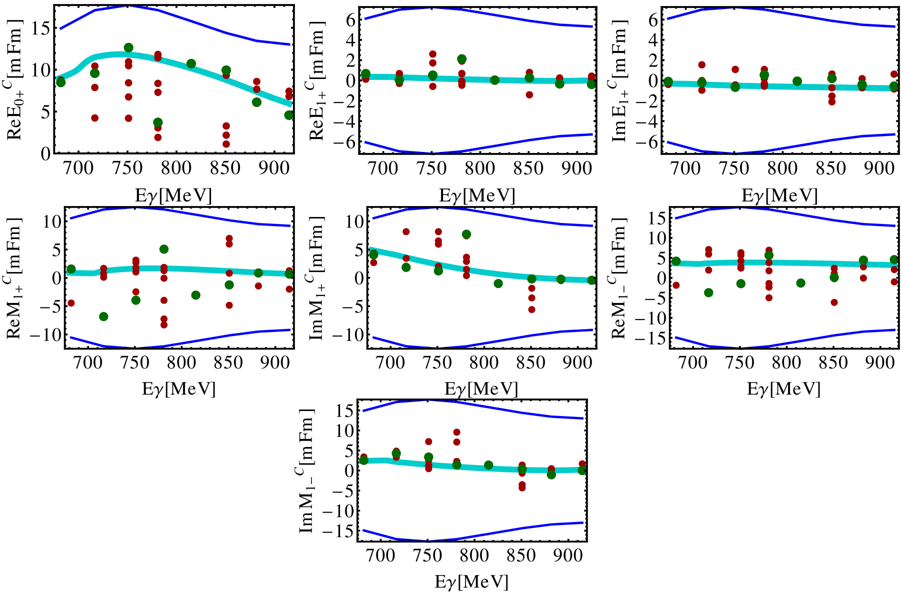
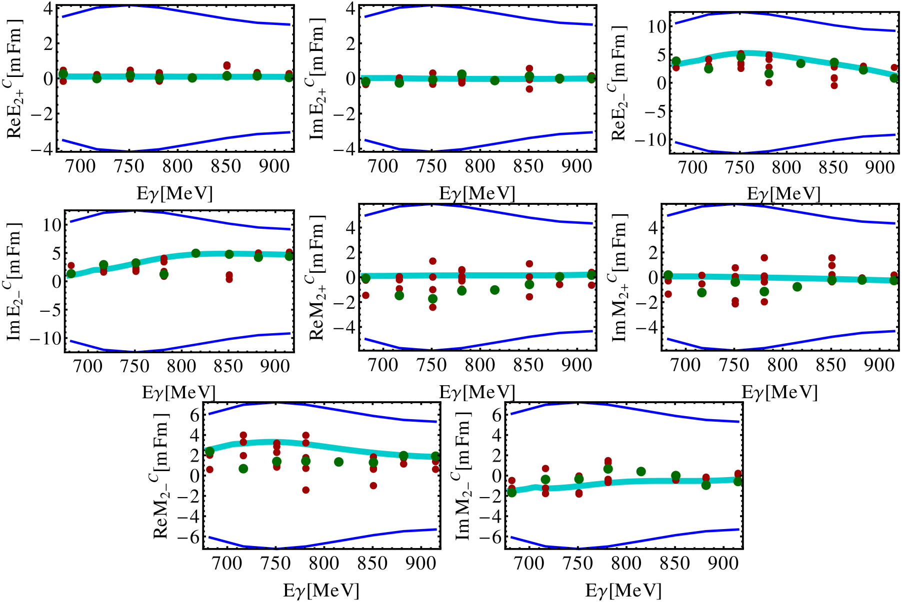


S-, P-, D- and F-waves in the interval $\left(\chi^2_{\text{best}} + 0.2\right)$



S-, P-, D- and F-waves in the interval $\left(\chi^2_{\mathrm{best}} + 0.2\right)$



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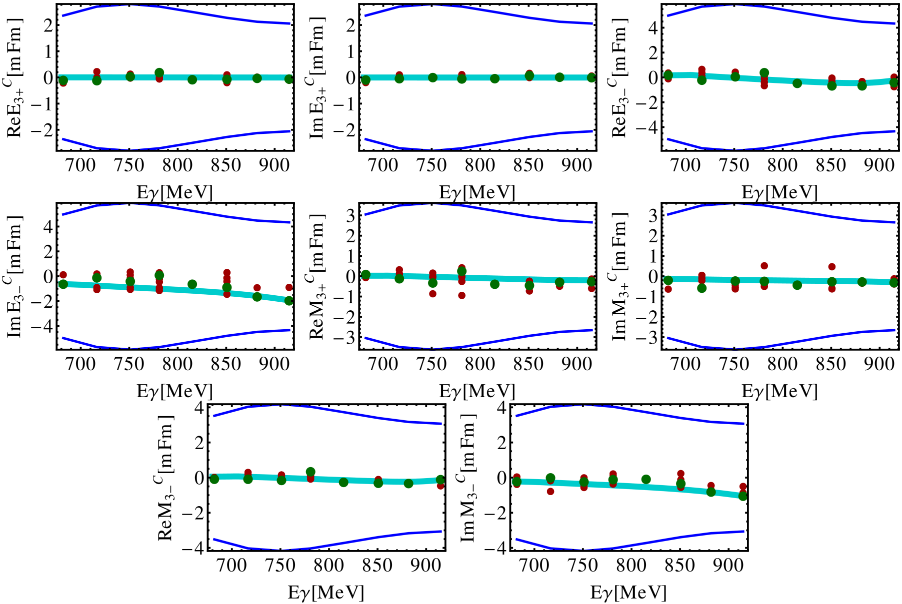

Problems with the $\ell_{\mathrm{max}} = 3$ multipole fit

There exists a global minimum, which is however not well separated from the other local minima of χ^2!

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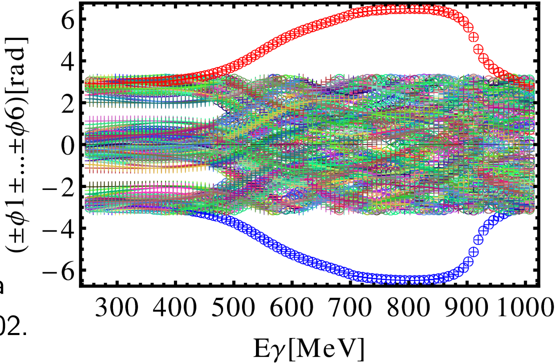
(ii) There exist

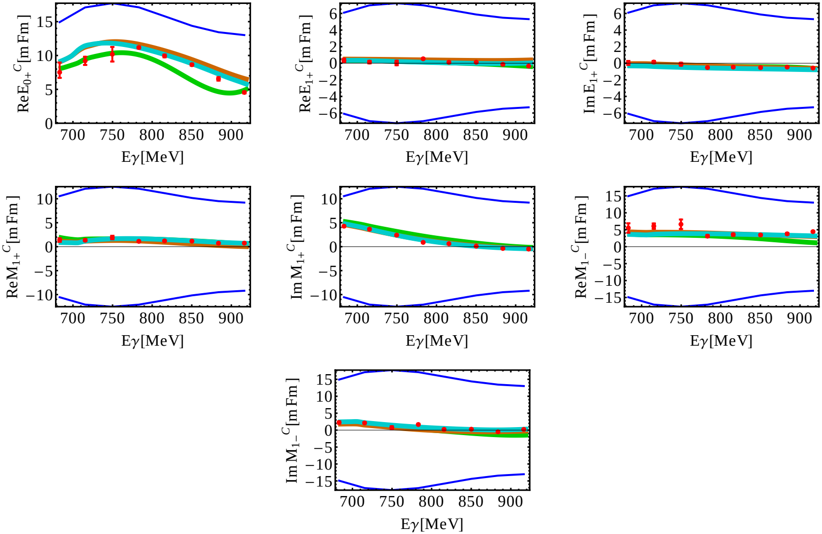
$$\frac{1}{2}\left(4^{2\ell_{\mathrm{max}}} - 2\right) = 2047$$

candidates for accidential ambiguities for $\ell_{\mathrm{max}} = 3$!

Way out: fix F-waves to a model, here: BnGa 2014-02.



Bootstrap results for the S-, P- and D-waves - Whole plot interval



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