Uncertainties in the RGE Evolution of α_s

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α_s satisfies the Renormalization Group Equation

$$\mu^2 \frac{d}{d\mu^2}(a_s) = \beta(a_s) = -\beta_0 a_s^2 (1 + b_1 a_s + b_2 a_s^2 + b_3 a_s^3 + \dots) \quad (1)$$

$$a_s \equiv \frac{\alpha_s}{\pi} = \frac{g}{4\pi^2}, \qquad b_N \equiv \frac{\beta_N}{\beta_0}$$

- β_0 and β_1 are renormalization scheme independent.
- In the $\overline{\text{MS}}$ β_2 [Tarasov et al., 1980, Larin and Vermaseren, 1993] and β_3 [van Ritbergen et al., 1997] are known.

Using the n_f^4 obtained by [Gracey, 1996]

$$E_4 = -\frac{\left(\frac{4}{3}(214 + 288\zeta_3) + 3(-229 + 480\zeta_3)\right)}{995328} \approx -0.0018 \quad (2)$$

[Ellis et al., 1998] was able to predict using Padé Approximants the β_4 coefficient which we use in a conservative way $\beta_4 = 0 \pm \beta_4^{\text{WAPAP}}$

$$\beta_4^{\text{WAPAP}} = (A_4 + B_4 N_f + C_4 N_f^2 + D_4 N_f^3 + E_4 N_f^4) \quad (3)$$

$$A_4 = 476.6, \quad B_4 = -152.3, \quad C_4 = 16.02 \quad D_4 = -0.05908$$

- By truncating the beta function is possible to find iterative and analytical solutions as well ([Prosperi et al., 2007] for a review.)
- The iterative can be parametrized in terms of Λ_{QCD} or α_0.
- We choose α_0 to avoid the redefinition at each loop of the parameter.
- The well known way of obtaining the iterative solution is summarized in an algorithm

$$\begin{aligned} K_n(x) &= \beta_0 x + b_1 a_0 \ln[1 + K_{n-1}(x)] + \sum_{k=2} \frac{C_k a_0^k}{k-1}\left[1 - \left(\frac{1}{1+K_{n-1}(x)}\right)^{k-1}\right] \\ & \qquad n > 1 \\ &\vdots \\ K_1(x) &= \frac{\beta_0 x}{1 + b_1 a_0} \\ K_0(x) &= 0, \qquad x = a_0 \ln(\mu^2/\mu_0^2) \end{aligned} \tag{4}$$

where the C_k's are easily calculated in terms of the inverse series of the beta function, or equivalently in terms of the PA

$$\begin{aligned} \frac{1}{\beta(a)} &= \frac{1}{\beta_0 a^2}\left(-1 + C_1 a + C_2 a^2 + C_3 a^3 + ... + C_N a^N\right) \\ &= -\frac{1}{\beta_0 a^2}\frac{1}{[0/N]} \end{aligned} \tag{5}$$

$C_1 = b_1$, $C_2 = -b_1^2 + b_2$, $C_3 = b_1^3 - 2b_1 b_2 + b_3$, etc.

$$\begin{aligned} a_{\text{it}}^{(n)}(x) = a_0 \Bigg[& 1 + \beta_0 x + a_0 b_1 \ln\left(1 + K_{n-1}(x)\right) \\ & + \sum_{k=2}^{n-1} a_0^k \frac{C_k}{k-1}\left(1 - \left(\frac{1}{1 + K_{n-1}(x)}\right)^{k-1}\right)\Bigg]^{-1} \end{aligned} \tag{6}$$

- In order to diminish non-linear effects we evolve in N small steps with size $\delta_k = (\mu - \mu_0)/N$
- We observe that the result changes 0.052% with respect to the 1 step evolution when we evolve in $N = 100$ and 0.053% whe the evolution has $N = 1000$ steps, we evolve in 200 steps.
- We can estimate in a simple way the uncertainty due to truncation of the beta function as

$$\Delta_\beta^1 = |\alpha_{\text{it}}^{(5)}(\delta_k) - \alpha_{\text{it}}|_{\beta_4=0}^{(5)}(\delta_k)|, \tag{7}$$

$$\Delta_\beta^2 = |\alpha_{\text{it}}^{(5)}(\delta_k) - \alpha_{\text{it}}^{(4)}(\delta_k)|, \tag{8}$$

  where we use the β_4^{WAPAP} as an input for the 5-loop solution.
- The estimation difference when we evolve from μ_1 to μ_2 and then back from μ_2 to μ_1 should be small (less than 1%)in comparison with a single evolution.

- The one loop solution resumms the Leading Logarithms LL, the second loop resumms the NLL, and so on.
- We expand the 4-loop and 5-loop solutions in powers of a_0 to find explicitly at each order the difference between the expansions through the transformation

$$b_N \to b_N \pm a_0 \Delta b_N \tag{9}$$

with N the number of loops, in an analogous fashion as in [Erler, 2000] where it is proposed

$$\beta_3 \to \beta_3 \pm a_0 \beta_4 \tag{10}$$

- We found

$$\Delta b_1 = s_1 b_1^2 + s_2 b_2 \qquad \text{(2-loop)}$$
$$\Delta b_2 = t_1 b_1^3 + t_2 b_1 b_2 + t_3 b_3 \qquad \text{(3-loop)}$$
$$\Delta b_3 = u_1 b_1^4 + u_2 b_1^2 b_2 + u_3 b_2^2 + u_4 b_1 b_3 + u_5 b_4 \qquad \text{(4-loop)}$$

a_0^n	s_1	s_2	t_1	t_2	t_3	u_1	u_2	u_3	u_4	u_5
a_0^5	-1.2	1.2	2	-1.00	1	-	-	-	-	-
a_0^6	-1.44	1.44	2.67	-1.00	1.17	1	2.00	-1.00	-1.00	1
a_0^7	-1.55	1.55	3.39	-1.00	1.33	1.71	2.29	-1.14	-1.00	1.14
a_0^8	-1.72	1.72	4.16	-1.00	1.50	2.50	2.63	-1.29	-1.02	1.29
a_0^9	-1.88	1.88	4.97	-1.00	1.67	3.35	3.00	-1.44	-1.06	1.44
a_0^{10}	-2.03	2.03	5.82	-1.00	1.83	4.26	3.40	-1.60	-1.10	1.60
a_0^{11}	-2.18	2.18	6.70	-1.00	2.00	5.22	3.82	-1.76	-1.15	1.76
a_0^{12}	-2.33	2.33	7.60	-1.00	2.17	6.23	4.25	-1.92	-1.21	1.92
a_0^{13}	-2.47	2.47	8.54	-1.00	2.33	7.28	4.69	-2.08	-1.27	2.08
a_0^{14}	-2.61	2.61	9.50	-1.00	2.50	8.37	5.14	-2.24	-1.33	2.24
a_0^{15}	-2.75	2.75	10.5	-1.00	2.67	9.50	5.60	-2.40	-1.40	2.40

The uncertainty is obtained as follows

$$Y = b_3 \pm a_0 \sigma_\beta(Y) \tag{11}$$

so that

$$\sigma_\beta(Y) = \sum_i \eta_i \sigma(u_i), \tag{12}$$

where the η_i's are the combinations of b_N and the $\sigma(u_i)$ are obtained using standard estimators, then

$$\Delta_\beta = \left| \frac{d\alpha_s}{dY} \sigma_\beta(Y) \right| \tag{13}$$

or in small steps

$$\Delta_\beta^3 = \left| \frac{d\alpha_s(\delta_k)}{dY} \sigma_\beta(Y) \right| . \tag{14}$$

The other ingredients to the evolution are the matching conditions at flavor thresholds...

- The [Appelquist and Carazzone, 1975] theorem does not hold in its naive form when a mass-independent renormalization scheme is employed.
- The solution comes in the form of matching conditions obtained by explicitly decoupling using the methods of effective field theory [Weinberg, 1980], for a_s

$$a'_s = a_s\Big(1 + \sum_{n=1}^{\infty} k_n(l) a_s^n\Big) \tag{15}$$

- k_2 [Larin et al., 1995],k_3 [Chetyrkin et al., 1997], k_4 [Kniehl et al., 2006](fully analytical)
- For consistency, 4-loop running is accompanied by 3-loop matching, we take the last coefficient used to match as a systematic uncertainty
- The quark masses carry another source of uncertainty

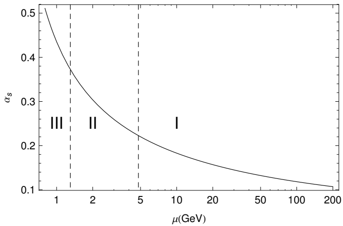
$$\begin{aligned} m_b(m_b) &= 4.199 \pm 0.024 \quad \text{GeV} \\ m_c(m_c) &= 1.274\ {}^{+0.030}_{-0.035} \quad \text{GeV} \end{aligned} \tag{16}$$

There are two main options in the literature to evolve the coupling with m_c and m_b threshold [Rodrigo et al., 1998] when the evolution goes from $m_\tau = 1.7768$GeV [Beringer et al., 2012] to $M_Z = 91.1876$GeV [Schael et al., 2006]

$$\alpha_3(m_\tau) \to \alpha_4(m_\tau) \to \alpha_4(m_b) \to \alpha_5(m_b) \to \alpha_5(M_Z) \tag{17}$$

or we can run to m_c, implement matching conditions and turn back to m_τ, i.e.,

$$\alpha_3(m_\tau) \to \alpha_3(m_c) \to \alpha_4(m_c) \to \alpha_4(m_\tau) \to \alpha_4(m_b) \to \alpha_5(m_b) \to \alpha_5(M_Z) \tag{18}$$



- We evolve from m_τ to M_Z fixing $\alpha_s(m_\tau) = 0.3300$ without the uncertainties in the extraction.
- We choose the second option using one particular estimation for the beta function truncation

$$\begin{aligned}\alpha_{(nf=5)}(M_Z) = 0.11998 &\pm (1.919\times 10^{-4})^1_\beta \pm (0.303\times 10^{-4})^{m_c}_{\mathsf{th}} \\ &\pm (0.151\times 10^{-4})^{m_b}_{\mathsf{th}} + ({}^{+0.451}_{-0.527}\times 10^{-4})_{m_c} \\ &\pm (0.103\times 10^{-4})_{m_b} \pm (0.317\times 10^{-4})_{\mathsf{it}}\end{aligned} \quad (19)$$

Using the other possibilities we get a Table the final result reads

$$\begin{aligned}\alpha_{(nf=5)}(M_Z) = 0.11998 &\pm (1.679\times 10^{-4})_{\mathsf{tot}} + ({}^{+0.451}_{-0.527}\times 10^{-4})_{m_c} \\ &\pm (0.103\times 10^{-4})_{m_b}\end{aligned} \quad (20)$$

		n_f=3		n_f=4		n_f=5
μ		m_τ	m_c	m_τ	m_b	M_Z
$\alpha_s^{(n_f)}(\mu)$		0.33	0.41562	0.33672	0.23221	0.11998
$\alpha_s^{(n_f+1)}(\mu)$			0.41381		0.23196	
$\Delta_\beta \times 10^{-4}$						
	$\beta_4 = 100$					
	Δ_β^1		5.440	2.854	4.862	1.919
	Δ_β^2		5.388	2.950	3.928	1.575
	Δ_β^3		1.946	0.756	3.299	1.260
	β_4^{WAPAP}					
	Δ_β^1		3.184	1.330	4.875	1.968
	Δ_β^2		3.133	1.424	3.942	1.625
	Δ_β^3		0.922	0.067	3.317	1.287

$\Delta_{\text{th}}\ \times 10^{-4}$					
	$\Delta_{\text{th}}(m_c)$	6.911	4.316	1.911	0.303
	$\Delta_{\text{th}}(m_b)$	2.148	1.341	0.594	0.151
$\Delta_{\text{m}}\ \times 10^{-4}$					
	Δ_{m_c}	$^{+6.422}_{-7.492}$	$^{+4.008}_{-4.676}$	$^{+1.775}_{-2.071}$	$^{+0.451}_{-0.527}$
	Δ_{m_b}	1.442	0.914	0.405	0.103
$\Delta_{\text{it}}\ \times 10^{-4}$		1.285	0.923	1.073	0.317
$\Delta_{\text{tot}}\ \times 10^{-4}$		± 8.009	± 4.853	± 4.675	± 1.679
		$^{+6.422}_{-7.492}$	$^{+4.008}_{-4.676}$	$^{+1.775}_{-2.071}$	$^{+0.451}_{-0.527}$
		± 1.442	± 0.914	± 0.405	± 0.317

Thanks!

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