Geometry of BPS vortex-antivortex moduli spaces

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Outline

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Vortex moduli spaces

The gauged $\mathbb{P}^1$-model: BPS vortices and antivortices

The case $\Sigma = \mathbb{C}$

Geometry of $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ from GLSMs

The case $\Sigma = S^2_R$

Supersymmetry, topology, nonabelions

Main collaborator in today's talk:

- Martin Speight (Leeds)

Related work with other collaborators:

- Marcel Bökstedt (Aarhus)
- Christian Wegner (Bonn)

cf. `arXiv:1410.2429`, `arXiv:1605.07921` (MB+NR)

Gauged sigma-models

Ingredients:

- $(\Sigma, j_\Sigma, \omega_\Sigma)$ Kähler structure on an oriented surface (*base*)
- (X, j_X, ω_X) another Kähler manifold (*target*)

- G compact Lie group with invariant metric
  $\mathfrak{g} := \mathrm{Lie}(G)$
- $\sharp : \mathfrak{g}^* \to \mathfrak{g}$ 'musical' isomorphism

- G-action on X: holomorphic, Hamiltonian
- $\mu^\sharp : X \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{\sharp} \mathfrak{g}$ moment map

Gauged sigma-models

Fields: $(A, \phi) \quad \in \quad \mathcal{A}(P) \times \Gamma(\Sigma, P^X)$

- A connection in principal G-bundle $P \to \Sigma$
- ϕ section of associated bundle $P^X := P \times_G X \to \Sigma$
  i.e. G-equivariant map $\phi : P \to X$

Topological charge:

$[\phi]_2^G := ((\tilde{f} \times \phi)/G)_*[\Sigma] \ \in \ H_2^G(X; \mathbb{Z}) \quad$ for $P = f^*\mathrm{E}G$

The GSM action and Bogomol'nyi trick:

$$\begin{aligned} E(A, \phi) \ &:= \ \frac{1}{2}\int_\Sigma \left(|F_A|^2 + |\mathrm{d}^A\phi|^2 + |\mu \circ \phi|^2\right) \\ &= \ \langle [\omega_X - \mu]_G^2, [\phi]_2^G \rangle + \frac{1}{2}\int_\Sigma \left(\left| *F_A + \mu^\sharp \circ \phi\right|^2 + 2|\bar{\partial}^A\phi|^2\right) \end{aligned}$$

Vortex moduli spaces

The (symplectic) vortex equations:

$$\text{(V1)} \quad \bar{\partial}^A \phi = 0$$

$$\text{(V2)} \quad *F_A + \mu^\sharp \circ \phi = 0$$

Invariant under action of gauge group $\mathcal{G} = \mathrm{Aut}_\Sigma(P) \ni g$:

$$(A, \phi) \quad \mapsto \quad (\mathrm{Ad}_g \circ A - g^{-1}\mathrm{d}g, g \cdot \phi)$$

NB: Same can be done for "antivortices" s.t. $\partial^A \phi = 0$ etc.
But they don't live with vortices in BPS configurations.

We'll see in a moment how to implement coexistence of vortices and antivortices in another sense.

Vortex moduli spaces

Fix $\mathbf{h} \in H_2^G(X;\mathbb{Z})$. Moduli spaces defined:

$$\mathcal{M}_{\mathbf{h}}^X(\Sigma) := \left\{ (A,\phi) \;\middle|\; \begin{array}{c} (V1),(V2) \\ \text{and } [\phi]_2^G = \mathbf{h} \end{array} \right\} / \mathcal{G}$$

Can recast this quotient in terms of Kähler reduction:

- (V1) is invariant under complexification $\mathcal{G}^{\mathbb{C}}$.
- RHS of (V2) interpreted as $\mathcal{G}$-moment map.

Thus (the smooth part of) $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$ receives a *Kähler structure*.

Vortex moduli spaces

Tangent spaces:

$$\begin{aligned} T_A \mathcal{A}(P) &= \Omega^1(\Sigma; P \times_{Ad} \mathfrak{g}) \\ T_\phi \Gamma(\Sigma, P^X) &= \Gamma(\Sigma, \phi^* TX/G) \end{aligned}$$

Complex structure:

$$(\dot{A}, \dot{\phi}) \mapsto (*\dot{A}, (\phi^* j_\Sigma)\dot{\phi})$$

L^2-metric:

$$(\dot{A}_1, \dot{\phi}_1) \cdot (\dot{A}_2, \dot{\phi}_2) := \int_\Sigma \left(\frac{1}{2} \langle \dot{A}_1 \overset{\wedge}{,} *\dot{A}_2 \rangle + (\phi^* g_X)(\dot{\phi}_1, \dot{\phi}_2) \omega_\Sigma \right)$$

Vortex moduli spaces

An interesting setting:

- X Kähler toric manifold,
- $G = T \subset T^{\mathbb{C}} \subset X$ its (real) torus

Then for X, Σ compact we have a good description of $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$:

THEOREM (M Bökstedt + NR):
Suppose X is constructed as Fan_Δ for a Delzant polytope Δ, $\mathbf{h} \in {}^T\mathrm{BPS}_Y^X$ with $a_{\mathbb{R}}^* \circ \mathbf{h}([\omega_\Sigma]^\vee) \in \mathrm{int}\,\Delta$ and

$$k_\rho = \langle c_1^T(D_\rho), \mathbf{h}\rangle \qquad \text{for } \rho \in \mathrm{Fan}_\Delta(1).$$

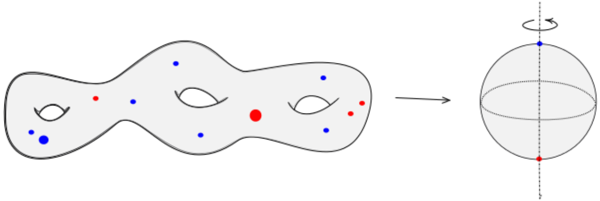
Then $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$ is the smooth manifold

$$\begin{aligned} \mathcal{M}_{\mathbf{h}}^X(\Sigma) &= \mathrm{Div}_+^{\mathbf{k}}(\Sigma; (\partial\Delta)^\vee) \subset \prod_{\rho \in \mathrm{Fan}_\Delta(1)} \mathrm{Sym}^{k_\rho}(\Sigma) \\ &=: \left\{ \mathbf{d} : [\lambda_0, \cdots, \lambda_\ell] \neq (\partial\Delta)^\vee \Rightarrow \bigcap_{i=0}^{\ell} \mathrm{supp}(d_{\lambda_\rho}) = \emptyset \right\} \end{aligned}$$

The gauged $\mathbb{P}^1$-model: BPS vortices and antivortices

- For today's talk we will take $X = \mathbb{P}^1 \cong S^2$, $T = \mathrm{U}(1) \cong S^1$.
- In this situation, $\mathcal{M}^{S^2}_{(k_0,k_1)}(\Sigma)$ was already well understood as a complex manifold: (Mundet, Sibner-Sibner-Yang, Baptista)

$$\mathcal{M}^{S^2}_{(k_+,k_-)}(\Sigma) = \mathrm{Sym}^{k_+}(\Sigma) \times \mathrm{Sym}^{k_-}(\Sigma) \setminus \Delta_{(k_+,k_-)}$$



These spaces have a boundary even if Σ is compact.

The gauged $\mathbb{P}^1$-model: BPS vortices and antivortices

Analysis: work on (dense) trivialising $U \subset \Sigma$, and use a (stereographic) coordinate on target $\mathbb{P}^1$.

Moment map:

$$\mu(x) = \frac{|x|^2 - 1}{|x|^2 + 1} - \tau, \qquad \tau \text{ real}$$

Integrating (V2) over Σ yields a constraint

$$-(1+\tau)\mathrm{Vol}(\Sigma) \leq 2\pi(k_+ - k_-) \leq (1-\tau)\mathrm{Vol}(\Sigma)$$

usually called 'Bradlow's bound'.
Here, $(k_+ - k_-) = \deg P$.

Energy bound: $E(A, \phi) \geq 2\pi(1-\tau)k_+ + 2\pi(1+\tau)k_-$

The case $\Sigma = \mathbb{C}$

- Topology is different for $\Sigma = \mathbb{C}$: P trivial, $\partial\Sigma = S^1_\infty$
- $\mu(\phi) = \frac{|\phi|^2-1}{|\phi|^2+1}$, $\quad \phi$ maps S^1_∞ to equator of $\mathbb{P}^1$
- $\int_\Sigma F_A = \int_{S^1_\infty} A = 2\pi \deg(\phi|_{S^1_\infty}) = 2\pi(k_+ - k_-)$

  $k_{+/-} = \sharp\{$ signed preimages of zeros/poles $\}$
- Total energy $= 2\pi(k_+ + k_-)$
- Theorem (Y Yang):

$$\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\mathbb{C}) = \mathrm{Sym}^{k_+}\mathbb{C} \times \mathrm{Sym}^{k_-}\mathbb{C} \setminus \Delta_{(k_+,k_-)}$$

The case $\Sigma = \mathbb{C}$

- Stereographic coordinate on target,

$$h := \log|\phi|^2$$

- (V1)+(V2) $\quad\Rightarrow\quad$ Taubes' equation:

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\sum_{r=1}^{k_-}\delta_{z_r^+}(z) - \sum_{r=1}^{k_-}\delta_{z_r^-}(z)\right)$$

The case $\Sigma = \mathbb{C}$

- Stereographic coordinate on target $h := \log|\phi|^2$
- Taubes' equation: $(k_+, k_-) = (1, 1)$

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\delta_{z^+}(z) - \delta_{z^-}(z)\right)$$

The case $\Sigma = \mathbb{C}$

- Stereographic coordinate on target $h := \log|\phi|^2$
- Taubes' equation: $z_+ = -z_- = \varepsilon$

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\delta_\varepsilon(z) - \delta_{-\varepsilon}(z)\right)$$

The case $\Sigma = \mathbb{C}$

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- Regularize: $h = \log\left|\frac{z-\varepsilon}{z+\varepsilon}\right|^2 + \tilde{h}$

The case $\Sigma = \mathbb{C}$

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$$\nabla^2 \tilde{h} - 2\,\frac{|z-\varepsilon|^2 \mathrm{e}^{\tilde{h}} - |z+\varepsilon|^2}{|z-\varepsilon|^2 \mathrm{e}^{\tilde{h}} + |z+\varepsilon|^2} = 0$$

The case $\Sigma = \mathbb{C}$

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- Rescale: $z =: \varepsilon w, \qquad \hat{h}(w) = \tilde{h}(\varepsilon w)$

The case $\Sigma = \mathbb{C}$

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- Regularize: $h = \log \left| \frac{z-\varepsilon}{z+\varepsilon} \right|^2 + \hat{h}$
- Rescale: $z =: \varepsilon w$

$$\nabla_w^2 \hat{h} - 2\varepsilon^2 \frac{|w-1|^2 \mathrm{e}^{\hat{h}} - |w+1|^2}{|w-1|^2 \mathrm{e}^{\hat{h}} + |w+1|^2} = 0$$

- Solve with boundary condition $\hat{h} \xrightarrow{|w|\to\infty} 0.$

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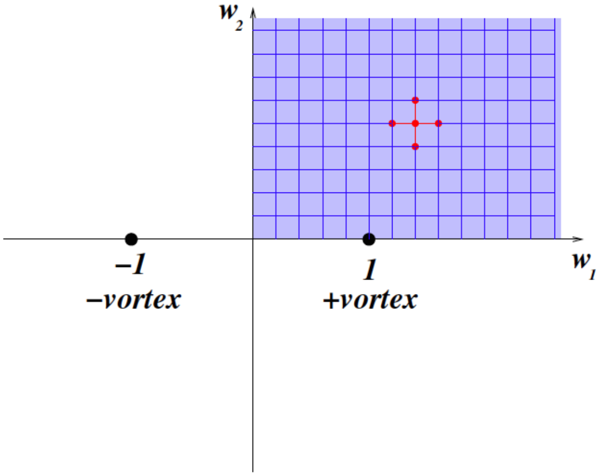
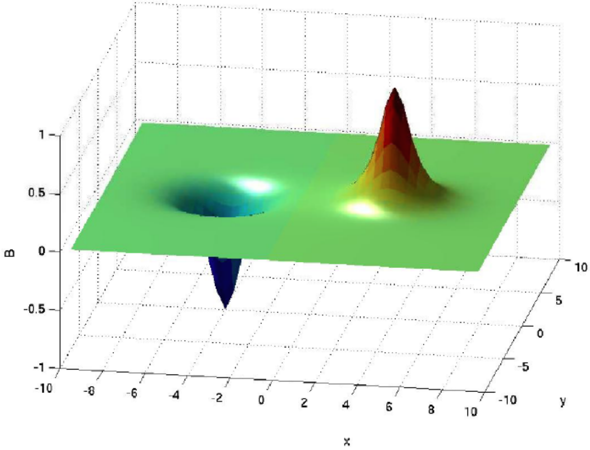
The case $\Sigma = \mathbb{C}$



The case $\Sigma = \mathbb{C}$



$\varepsilon = 4$

The case $\Sigma = \mathbb{C}$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A}, \dot{\phi})\|_{L^2}^2 = \frac{1}{2}\int_\Sigma \left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

- Assume all vortex positions remain distinct
- $\phi =: \mathrm{e}^{\frac{1}{2}h + \mathrm{i}\chi}$ and $\dot{\phi} =: \phi\eta$, so $\eta = \frac{1}{2}\dot{h} + \mathrm{i}\dot{\chi}$
- $\dot{h}$ satisfies the linearisation of Taubes' equation
- $\dot{\chi}$ from Gauß's law: L^2-orthogonality to $\mathcal{G}$-orbits

$$\left(\nabla^2 - \mathrm{sech}^2\frac{h}{2}\right)\eta = 4\pi\left(\sum_{r=1}^{k_0} \dot{z}_r^+ \delta_{z_r^+}(z) - \sum_{s=1}^{k_-} \dot{z}_s^- \delta_{z_s^-}(z)\right)$$

$$\Rightarrow \quad \eta = \sum_{r=1}^{k_0} \dot{z}_r^+ \frac{\partial h}{\partial z_r^+} + \sum_{s=1}^{k_-} \dot{z}_s^- \frac{\partial h}{\partial z_s^-} \qquad \text{is (unique) solution}$$

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The case $\Sigma = \mathbb{C}$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A}, \dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_\Sigma \left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

- Assume all vortex positions remain distinct
- $\phi =: \mathrm{e}^{\frac{1}{2}h + \mathrm{i}\chi}$ and $\dot{\phi} =: \phi\eta$, so $\eta = \frac{1}{2}\dot{h} + \mathrm{i}\dot{\chi}$
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The case $\Sigma = \mathbb{C}$

Calculation of the metric: **Strachan–Samols localisation**

$$
\begin{aligned}
\|(\dot{A}, \dot{\phi})\|_{L^2}^2 &= \frac{1}{2}\int_{\mathbb{C}} \left(|\dot{A}|^2 + |\dot{\phi}|^2\right) \\
&= \frac{1}{2}\int_{\mathbb{C}} \left(4\,\partial_z \bar{\eta} \partial_{\bar{z}} \eta + \operatorname{sech}^2 \frac{h}{2} \bar{\eta}\eta\right)
\end{aligned}
$$

The case $\Sigma = \mathbb{C}$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A},\dot{\phi})\|^2_{L^2} \quad = \quad \frac{1}{2}\int_{\mathbb{C}} \left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

$$= \quad \frac{1}{2}\lim_{\epsilon\to 0}\int_{\mathbb{C}\setminus\cup_{r,s}B_\epsilon(z^{\pm}_{r,s})} \left(4\,\partial_z\bar{\eta}\partial_{\bar{z}}\eta + \mathrm{sech}^2\frac{h}{2}\bar{\eta}\eta\right)$$

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Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A},\dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_{\mathbb{C}} \left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

$$= \mathrm{i}\lim_{\epsilon\to 0} \sum_{r,s=1}^{k_+,k_-} \oint_{\partial B_\epsilon(z^{\pm}_{r,s})} \bar{\eta}\,\bar{\partial}\eta$$

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$$= \quad \mathrm{i}\lim_{\epsilon\to 0}\sum_{r,s=1}^{k_+,k_-}\oint_{\partial B_\epsilon(z^\pm_{r,s})}\bar{\eta}\,\bar{\partial}\eta$$

$$\eta = \frac{\mp\dot{z}^\pm_{r,s}}{z - z^\pm_{r,s}} + o(1) \quad \text{as } z \to z^\pm_{r,s}$$

The case $\Sigma = \mathbb{C}$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A},\dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_{\mathbb{C}}\left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

$$= \pi\left(\sum_{r,s=1}^{k_+,k_-}|\dot{z}^{\pm}_{r,s}|^2 + \sum_{p,q}^{+\text{ or }-}\frac{\partial b_q}{\partial z_p}\dot{z}_p\dot{\bar{z}}_q\right)$$

Asymptotics near vortex cores, $z = z^{\pm}_{r,s}$:

$$+h = \log|z-z_r^+|^2 + a_r^+ + \frac{\bar{b}_r^+}{2}(z-z_r^+) + \frac{b_r^+}{2}(\bar{z}-\bar{z}_r^+) + O(|z-z_r^+|^2) \quad \text{as } z\to z_r^+$$

The case $\Sigma = \mathbb{C}$

Calculation of the metric: **Strachan–Samols localisation**

$$\begin{aligned} \|(\dot{A}, \dot{\phi})\|_{L^2}^2 &= \frac{1}{2}\int_{\mathbb{C}} \left(|\dot{A}|^2 + |\dot{\phi}|^2\right) \\ &= \pi \left(\sum_{r,s=1}^{k_+,k_-} |\dot{z}_{r,s}^{\pm}|^2 + \sum_{p,q}^{+\text{ or }-} \frac{\partial b_q}{\partial z_p} \dot{z}_p \dot{\bar{z}}_q \right) \end{aligned}$$

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The case $\Sigma = \mathbb{C}$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A},\dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_{\mathbb{C}}\left(|\dot{A}|^2+|\dot{\phi}|^2\right)$$

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Asymptotics near vortex cores, $z = z^{\pm}_{r,s}$:

$$-h = \log|z-z_s^-|^2+a_s^-+\frac{\bar{b}_s^-}{2}(z-z_s^-)+\frac{b_s^-}{2}(\bar{z}-\bar{z}_s^-)+O(|z-z_s^-|^2) \quad \text{as } z\to z_s^-$$

The case $\Sigma = \mathbb{C}$

$\mathcal{M}^{\mathbb{P}^1}_{(1,1)}(\mathbb{C}) \cong \mathbb{C}_{\mathrm{cm}} \times \mathbb{C}^*$

Aim: understand metric for centred $(+,-)$-pairs (i.e. on $\mathbb{C}^*$)

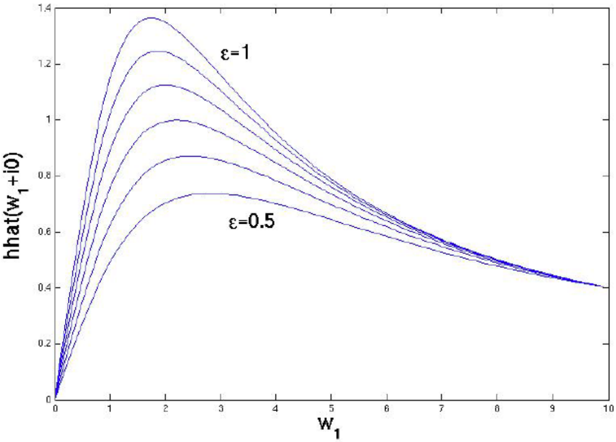
$z_+ = -z_+ + - = \varepsilon \mathrm{e}^{\mathrm{i}\vartheta}$

$$g^{(0)} = F(\varepsilon)(\mathrm{d}\varepsilon^2 + \varepsilon^2 \mathrm{d}\vartheta^2)$$

where:

- $F(\varepsilon) = 2\pi \left(2 + \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\varepsilon b(\varepsilon))\right)$
- $b(\varepsilon) := b_1^+(\varepsilon, -\varepsilon)$
- $\varepsilon b(\varepsilon) = \left.\frac{\partial \hat{h}}{\partial\,(\mathrm{Re}\, w)}\right|_{w=1} - 1$

The case $\Sigma = \mathbb{C}$



The case $\Sigma = \mathbb{C}$

Self-similarity conjecture: consider $\varepsilon << 1$.

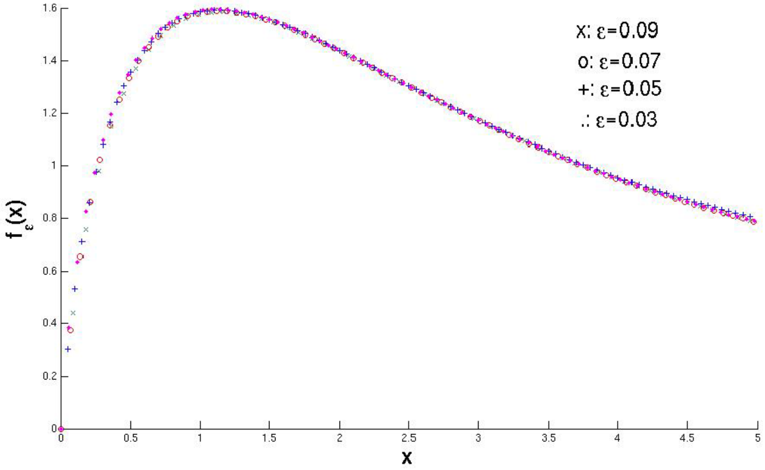
- Results suggest

$$\hat{h}_\varepsilon(w) \approx \varepsilon f_*(\varepsilon w) \qquad \text{for } f_* \text{ fixed !?}$$

- To test this: study

$$f_\varepsilon(z) := \frac{1}{\varepsilon}\hat{h}\left(\frac{z}{\varepsilon}\right)$$

The case $\Sigma = \mathbb{C}$



The case $\Sigma = \mathbb{C}$

- Results suggest

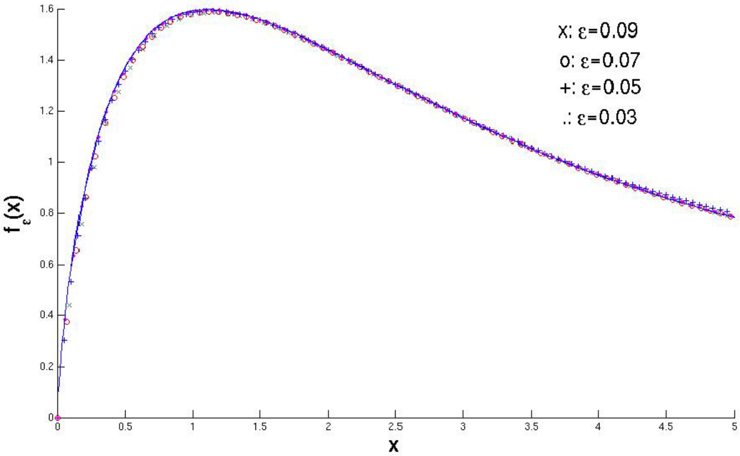
$$\hat{h}_\varepsilon(w) \approx \varepsilon f_*(\varepsilon w) \qquad \text{for } f_* \text{ fixed !?}$$

- To test this: study

$$f_\varepsilon(z) := \frac{1}{\varepsilon}\hat{h}\left(\frac{z}{\varepsilon}\right) \quad \approx \quad f_*(z)$$

- Now plug this into Taubes' equation to obtain PDE for f_*
- Take formal limit as $\varepsilon \to 0$;
  obtain screened Poisson equation with a simple source.
- Solve equation to obtain asymptotics of metric $g^{(0)}$.

The case $\Sigma = \mathbb{C}$



The case $\Sigma = \mathbb{C}$

Asymptotics of conformal factor:

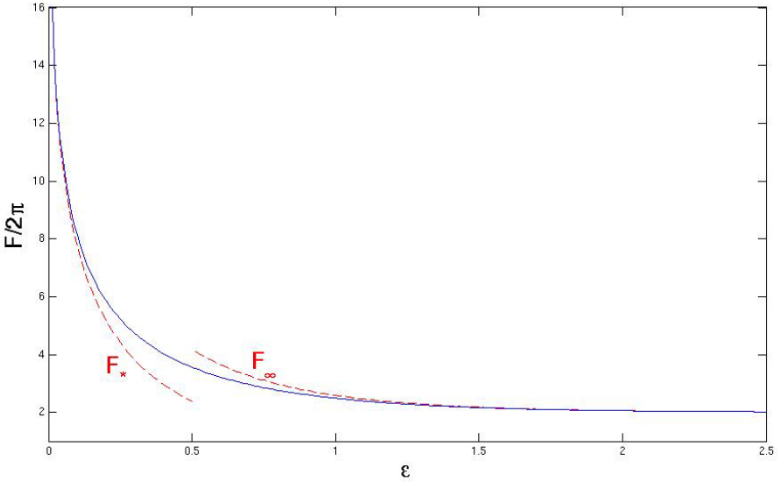
- As $\varepsilon \to 0^+$:

$$F_*(\varepsilon) = 2\pi(2 + 4K_0(\varepsilon) - 2\varepsilon K_1(\varepsilon)) \sim -8\pi \log \varepsilon$$

- As $\varepsilon \to \infty$ (different argument, cf. Manton & Speight):

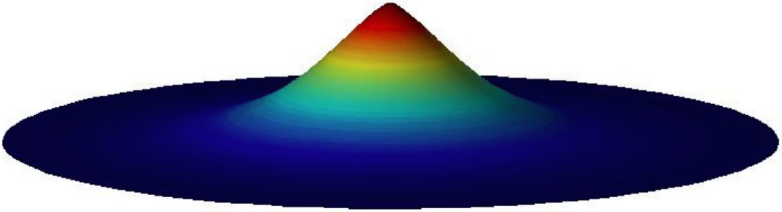
$$F_\infty(\varepsilon) = 2\pi \left(2 + \frac{q^2}{\pi^2} K_0(2\varepsilon)\right), \qquad q \approx -7.1388$$

The case $\Sigma = \mathbb{C}$



The case $\Sigma = \mathbb{C}$

Our formula for F_* implies incompleteness with unbounded positive Gauß curvature as $\varepsilon \to 0$.



Geometry of $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ from GLSMs

Consider a parent model with Σ compact, $X = \mathbb{C}^2$, $T^2 = \mathrm{U}(1)^2$.
Representation with weights $Q^j_\pm \in \mathbb{Z}$,

$$(x_+, x_-) \mapsto \left(\lambda_1^{Q^1_+} \lambda_2^{Q^2_+} x_+, \lambda_1^{Q^1_-} \lambda_2^{Q^2_-} x_- \right)$$

The vortex equations on (A_1, A_2) and $\varphi = (\varphi_+, \varphi_-)$ are:

$$\bar\partial^{(A_1,A_2)} \varphi_\pm := \left(\bar\partial - i \sum_{j=1}^{2} Q^j_\pm A_j \right) \varphi_\pm = 0$$

$$*F_{A_j} = -\mu_j^\sharp \circ \phi = \frac{e_j^2}{2} \left(Q^j_+ |\varphi_+|^2 + Q^j_- |\varphi_-|^2 - \tau_j \right) \quad j = 1, 2$$

Geometry of $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ from GLSMs

1) Integrating the latter over Σ, with $(k_1, k_2) = \deg P$, get

$$\frac{2\pi k_j}{e_j^2} = \frac{\tau_j}{2}\mathrm{Vol}(\Sigma) - \frac{1}{2}\left(Q_j^+\|\varphi_+\|^2_{L^2} + Q_j^-\|\varphi_-\|^2_{L^2}\right)$$

which gives as Bradlow's bound: (τ_1, τ_2) must lie in the affine cone with vertex $\frac{4\pi}{\mathrm{Vol}(\Sigma)}(\frac{k_1}{e_1^2}, \frac{k_2}{e_2^2}) \in Lie(T)$ spanned by Q_+, Q_-.

2) Will consider the $\mathbb{P}^1$-model as a formal limit as $e_1^2 \to \infty$, $e_2^2 = 1$
Get a constraint:

$$Q^1_+\|\varphi_+\|^2 + Q^1_-\|\varphi_-\|^2 = \tau_1 + O(e_1^{-2}) \qquad \text{uniformly on } \Sigma$$

Model $\mathbb{P}^1 = S^3/S^1 = \{\text{this hypersurface}\}/\mathrm{U}(1)_1$, by setting $Q^1_\pm = 1$. Then take $Q^2 = (1, 0)$ and

$$\phi = [\varphi_+ : \varphi_-], \qquad A = \mathrm{ext}_{T^2 \to T^2/\mathrm{U}(1)_1}(A_1, A_2) = A_2$$

Geometry of $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ from GLSMs

3) Thus get $\bar{\partial}^A\phi = 0$ i.e. (V1) and also

$$*F_{A_2} = -\frac{1}{2}\left(\frac{(\tau_1-\tau_2)|\phi|^2-\tau_2}{1+|\phi|^2}\right) + O(e_1^{-2})$$

which matches (V2) iff $(\tau_1, \tau_2) = (4, 2-\tau)$.
Bradlow's bounds also match as $e_1^2 \to \infty$.

Proposition 1 (NR + M Speight):

Suppose $k_\pm := \sum_{j=1}^2 Q_\pm^j > \max\{2g-2, 0\}$ with $Q_\pm$ as above, and $\tau_1 - \tau_2 > \frac{4\pi}{\mathrm{Vol}(\Sigma)}\left(\frac{k_1}{e_1^2} - \frac{k_2}{e_2^2}\right), \tau_2 > \frac{4\pi k_2}{e_2^2 \mathrm{Vol}(\Sigma)}$. Then

$$\mathcal{M}^{\mathbb{C}^2}_{(k_1,k_2)}(\Sigma) \cong \mathrm{Sym}^{k_+}(\Sigma) \times \mathrm{Sym}^{k_-}(\Sigma)$$

Geometry of $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ from GLSMs

Proposition 2 (NR + M Speight):

Under the same conditions, the Kähler class of the L^2-metric for the GLSM vortex moduli space is

$$[\omega_{L^2}] = \sum_{s=\pm}(C_s\eta_s + D_s\theta_s)$$

where $\eta_\pm = c_1(P^{\mathbb{C}}_{Q_\pm})$, $\theta_\pm = \mathrm{AJ}^*_\pm(\Theta\text{-class})$ and

$$\begin{aligned}
C_+ &:= \pi\tau_2 \mathrm{Vol}(\Sigma) - \frac{2\pi^2(k_+ - k_-)}{e_2^2},\\
C_- &:= \pi(\tau_1 - \tau_2)\mathrm{Vol}(\Sigma) - 2\pi^2\left(\frac{k_-}{e_1^2} - \frac{k_+ - k_-}{e_2^2}\right),\\
D_+ &:= \frac{2\pi^2}{e_2^2},\\
D_- &:= 2\pi^2\left(\frac{1}{e_1^2} - \frac{1}{e_2^2}\right).
\end{aligned}$$

Geometry of $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ from GLSMs

This leads to the result

$$\mathrm{Vol}\left(\mathcal{M}^{\mathbb{C}^2}_{(k_+,k_-)}(\Sigma)\right) = \prod_{s=\pm}\sum_{i_s=0}^{g} \frac{g!\, C_s^{n_s-i_s} D_s^{i_s}}{(n_s-i_s)!(g-i_s)!i_s!}$$

as well as to the

Conjecture:

$$\mathrm{Vol}\left(\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)\right) = \lim_{e_1^2\to\infty} \mathrm{Vol}\left(\mathcal{M}^{\mathbb{C}^2}_{(k_+,k_-)}(\Sigma)\right)_{e_2^2=1}$$

$$= \frac{(2\pi)^{k_++k_-}}{k_+!k_-!}(\mathrm{Vol}(\Sigma)+\pi(k_+-k_-))^{k_+}(\mathrm{Vol}(\Sigma)-\pi(k_--k_+))^{k_-}$$

(Similarly, get conjectural formula for total scalar curvature.)

The case $\Sigma = S^2_R$

Round metric: $g_\Sigma = \frac{4R^2}{(1+|z|^2)^2}\mathrm{d}z\mathrm{d}\bar{z} =: \Omega(|z|)\,\mathrm{d}z\mathrm{d}\bar{z}$

Taubes' equation:

$$\nabla^2 h - 2\Omega(|z|)\tanh\frac{h}{2} = 4\pi\left(\sum_{r=1}^{k_+}\delta_{z_r^+}(z) - \sum_{r=1}^{k_-}\delta_{z_r^-}(z)\right)$$

L^2-metric on $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(S^2_R)$:

$$\begin{aligned} g_{L^2} \quad &= \quad 2\pi\left(\sum_{r=1}^{k_+}\Omega(|z_r^+|)|\mathrm{d}z_r^+|^2 + \sum_{r=1}^{k_-}\Omega(|z_r^-|)|\mathrm{d}z_r^-|^2\right. \\ &\quad + \sum_{r,s=1}^{k_+}\frac{\partial b_s^+}{\partial z_r^+}\mathrm{d}z_r^+\mathrm{d}\bar{z}_s^+ + \sum_{r,s=1}^{k_-}\frac{\partial b_s^-}{\partial z_r^-}\mathrm{d}z_r^-\mathrm{d}\bar{z}_s^- \\ &\quad \left. + \sum_{r=1}^{k_+}\sum_{s=1}^{k_-}\frac{\partial b_s^-}{\partial z_r^+}\mathrm{d}z_r^+\mathrm{d}\bar{z}_s^- + \quad + \sum_{r=1}^{k_-}\sum_{s=1}^{k_+}\frac{\partial b_s^+}{\partial z_r^-}\mathrm{d}z_r^-\mathrm{d}\bar{z}_s^+\right) \end{aligned}$$

The case $\Sigma = S^2_R$

On $\mathcal{M}^{(0)}_{1,1}(S^2_R)$, L^2-metric given by

$$g^{(0)}_{L^2} = 2\pi\left(2\Omega(\varepsilon) + b'(\varepsilon) + \frac{b(\varepsilon)}{\varepsilon}\right)(\mathrm{d}\varepsilon^2 + \varepsilon^2\mathrm{d}\psi^2)$$

where b is determined through

$$\varepsilon b(\varepsilon) = \left.\frac{\partial\hat{h}}{\partial w_1}\right|_{w=(1,0)} - 1$$

from $\hat{h}$ (regularising h) solving

$$\nabla^2_w\hat{h} - \frac{8R^2\varepsilon^2}{(1+\varepsilon^2|w|^2)^2}\frac{|w-1|^2\mathrm{e}^{\hat{h}} - |w+1|^2}{|w-1|^2\mathrm{e}^{\hat{h}} + |w+1|^2} = 0.$$

The case $\Sigma = S^2_R$

Numerical evidence suggests:

$$\lim_{\varepsilon \to 0} \varepsilon b(\varepsilon) = -1$$

which implies the heuristic volume formula above.

Less clear (but still suggested?) is whether

$$\lim_{\varepsilon \to 0} \frac{\mathrm{d}(\varepsilon b(\varepsilon))}{\mathrm{d}\varepsilon} > 0,$$

which would imply $F(\varepsilon) \sim \frac{1}{\varepsilon}$ and *in*completeness of $g_{L^2}^{(0)}$.

The case $\Sigma = S^2_R$

LEMMA:
Any $\mathrm{SO}(3)\times\mathbb{Z}_2$-invariant Kähler metric on $\mathcal{M}_{(1,1)} = S^2\times S^2 \setminus S^2_\Delta$ has the form

$$g = -\frac{Q'(\varepsilon)}{\varepsilon}(\mathrm{d}\varepsilon^2+\varepsilon^2\sigma_3^2) + Q(\varepsilon)\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\sigma_1^2 + \frac{1+\varepsilon^2}{1-\varepsilon^2}\sigma_2^2\right)$$

for some smooth decreasing $Q:(0,1]\to\mathbb{R}^+$, where σ_j are left-invariant forms on $\mathrm{SO}(3)$.
Regularity at $\varepsilon\to 1$ requires $Q(1)=0$.

This has $\qquad \mathrm{Vol}(\mathcal{M}_{(1,1)}, g) = (2\pi)^2 Q(0)^2.$

The vortex-antivortex L^2-metric has

$$Q(\varepsilon) = -2\pi\left(\varepsilon b(\varepsilon) - \frac{4R^2}{1+\varepsilon^2} + 2R^2 + 1\right).$$

The case $\Sigma = S^2_R$

THEOREM (NR + J Speight):

$$\mathrm{Vol}\left(\mathcal{M}^{\mathbb{P}^1}_{(1,1)}(S^2_R)\right) = (2\pi)^2(4\pi R^2)^2$$

PROOF: It amounts to establishing that

$$\left\|\frac{\partial \hat{h}}{\partial w_1}\right\|_{C^0(B_{1/2}(1))} \leq \left\|\frac{\partial \hat{h}}{\partial w_1}\right\|_{H^0(B_{1/2}(1))} \overset{!}{<} C\varepsilon$$

This involves doing several elliptic estimates for the PDE satisfied by $\hat{h}$, a little Sobolev theory, and various other tricks.

Supersymmetry, topology, nonabelions

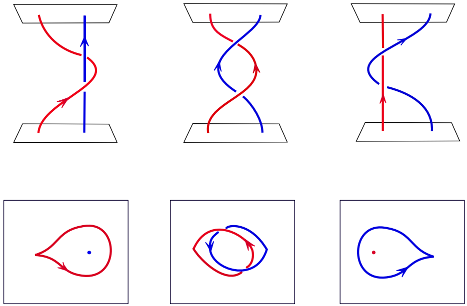
- GSM admits local $\mathcal{N} = (2,2)$ susy extension
- A-twist yields global $\mathcal{N} = (2,0)$ susy gauged sigma-model. (J Baptista)
- Semiclassical approximation to QFT: susy QM on $\mathcal{M}_{\mathbf{h}}^{X}(\Sigma)$
- Waveforms taking values in local systems
- Counting ground states amounts to computation of analytic L^2-Betti numbers for covers $\left(\widetilde{\mathcal{M}}_{\mathbf{h}}^{X}(\Sigma), \tilde{g}_{L^2}\right)$
- Quite generally, obtain infinite nonabelian fundemental groups.

Supersymmetry, topology, nonabelions

E.g.

$$0 \to \mathbb{Z}_{\gcd(k_+,k_-)} \to \pi_1\left(\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)\right) \to H_1(\Sigma;\mathbb{Z})^{\oplus 2} \to 0$$

Nonabelian statistics may arise from braiding:



Relating analytic L^2-Betti numbers to more easily computable invariants relies on understanding the asymptotics of L^2-metrics.