A Riemann-Hilbert approach to rotating attractors and extremal black holes

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Outline

- Stationary-axisymmetric solutions and 2d sigma models
- The Breitenlohner-Maison linear system
- Riemann-Hilbert problems
- Explicit factorization

Based on: M.C. Câmara, G.L. Cardoso, T.M. and S. Nampuri, arXiv:1703.10366.

Motivation and Objectives

Systematic construction of solutions with non-trivial dependence on > 1 coordinates (i.e. solving PDEs rather than ODEs). Specifically: rotating black holes and their near horizon asymptotic attractor geometries.

Integrability and hidden symmetries in gravitational theories. Specifically: Breitenlohner-Maison linear system, Geroch group.

Establish a factorization method which allows higher order poles of the monodromy matrix in the spectral parameter, and space-time geometries which are not asymptotically flat: attractors and extremal black holes. Proof of concept.

Obtain explicit matrix factorisation using 'vectorial' auxiliary problem $\rightarrow$ explicit solution of both space-time equations of motion and of linear system.

Use hidden, infinite-dimensional symmetries to relate and generate solutions.

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Selected earlier work

- P. Breitenlohner and D. Maison, *On the Geroch Group*, Ann. Inst. H. Poincaré Phys. Theor. **46** (1987) 215.
- H. Nicolai, *Two-dimensional gravities and supergravities as integrable systems*, in: Lect. Notes Phys. **396** (1991) 231.
- D. Katsimpouri, A. Kleinschmidt, and A. Virmani, JHEP **02** (2013) 011, JHEP **03** (2014) 101, JHEP **12** (2014) 070.
- B. Chakrabarty and A. Virmani, JHEP **11** (2014) 068.

See paper for a more complete list of references.

Assumptions

n-dimensional gravity coupled to matter.

- $n-2$ commuting Killing vectors. (We'll take one to be time-like.)
- Resulting two-dimensional (Euclidean) theory is a generalized ('Ernst') sigma model with symmetric target space G/H.

Specifically

1. Pure 4d gravity: $G/H = SL(2,\mathbb{R})/SO(2)$.
2. 4d Einstein-Maxwell: $G/H = SU(2,1)/(SL(2,\mathbb{R}) \times SO(2))$.
3. Pure 5d gravity: $G/H = SL(3,\mathbb{R})/SO(2,1)$.

Symmetric spaces

G simple real Lie group with Lie algebra $\mathfrak{g}$. $H \subset G$ maximal Lie subgroup (not necessarily comact) with Lie algebra $\mathfrak{h}$. Symmetric decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$.

- Involutive automorphism:

$$\theta(Z) = Z, \forall Z \in \mathfrak{h} \ , \quad \theta Z = -Z, \forall Z \in \mathfrak{p} \ .$$

- Generalized transposition: $Z^\natural = -\theta(Z)$

$$Z^\natural = -Z, \forall Z \in \mathfrak{h} \ , \quad Z^\natural = Z, \forall Z \in \mathfrak{p} \ ,$$

At group level: $h^\sharp = h^{-1}, \forall h \in H$, $g^\sharp = g, \forall g \in \exp \mathfrak{p}$.
Note: $\natural$ acts anti-homorphically $(g_1 g_2)^\natural = g_2^\natural g_1^\natural$.

G/H sigma models

We take G to be given as a matrix group. $V(x)$ is G-(matrix)-valued function on space-time (N, g_N).
'Gauge' $H \subset G$: $V(x) \simeq h(x)V(x)$.
G acts as a 'rigid' symmetry:

$$H \times G \; : V(x) \mapsto h(x)V(x)g^{-1} \; .$$

Metric on G/H induced by (right- or left-invariant) MC form on G, is pulled back to space-time to define the action:

$$S[g_N, V] = \int d^p x \sqrt{|g_N|} \left(\frac{1}{2} R[g_N] - g_N^{mn} \mathrm{Tr}(D_m V V^{-1} D_n V V^{-1}) \right) \; .$$

D is LC connection wrt g_N and H-covariant connection wrt target space. Trace Tr might include a numerical factor.

Explicit parametrizations:

- Impose gauge condition on $V(x)$, for example 'Borel gauge' using the Iwasawa decomposition, $G = HL$, where L triangular. (If H not compact, use triangular subgroup acting with open orbit on G/H).
- Use $\natural$-symmetric, H-gauge invariant representative:

$$M(x) = V^{\natural}(x)V(x)$$

$$H \times G \ : M(x) \mapsto g^{\natural,-1}M(x)g^{-1}$$

Equation of motion = current conservation for rigid G symmetry:

$$D^m A_m = D^m(M^{-1}\partial_m M) = 0 \Leftrightarrow d \star A = 0 \ , \quad A = M^{-1}dM \ .$$

Symmetry enhancement in $d = 2$.

Two dimensions: one-forms $\star$-dual to one-forms. Hidden symmetries ('Twist potentials.')

$$d \star A = 0 \Rightarrow A = \star dX_1$$

Observe:

Define: $J_1 := (d + A)X_1$ Find: $d \star J_1 = 0 \Rightarrow J_1 = \star dX_2 \ldots$

Define: $J_k = (d + A)X_k$ Find: $d \star J_k = 0 \Rightarrow J_k = \star dX_{k+1} \ldots$

To derive this hierarchy of conserved currents, the only equation needed apart from previously obtained relations, is the flatness of $A = M^{-1}dM$, $dA + A \wedge A = 0$.

Natural to introduce generating functions

$$
\begin{aligned}
J(\tau, x) &= A\tau + \sum_{k=1}^{\infty} J_k \tau^{k+1} \\
X(\tau, x) &= \mathbb{I} + \sum_{k=1}^{\infty} X_k \tau^{k}
\end{aligned}
$$

The relations between currents and potentials combine into the linear system

$$J = \tau(d + A)X = \star dX$$

which has the combined conservation equations $d \star J = 0$ as integrability condition. This includes the sigma model equation of motion $d \star A = 0$, which is the conservation equation for rigid G-symmetry.

Enhanced, infinite-dimensional 'rigid' symmetry group is identified as the loop group $\tilde{G}$ associated to G, with Lie algebra $\tilde{\mathfrak{g}}$.

Note i.p. that $J(x) \in \tilde{\mathfrak{g}}$, and $X(x) \in \tilde{G}$.

Generalized ('Ernst') sigma models

Reduction from four to two dimensions:

$$
\begin{aligned}
ds_4^2 &= -\Delta(dy + B_m dx^m) + \Delta^{-1} ds_3^2 \\
ds_3^2 &= e^{\psi} ds_2^2 + \rho^2 d\phi^2
\end{aligned}
$$

where Δ, ψ, ρ, B_m are functions on (N_2, ds_2^2).

Remaining two-dimensional field equations:

$$
d(\rho \star A) = 0 \ , \quad d \star d\rho = 0 \ .
$$

Field equation for A modified by ρ.

NB: Δ, B_m have been absorbed into $A = M^{-1} dM$.
NB: Equation for ψ is integrable, given A, ρ.

The spectral curve

The modified field equation

$$d(\rho \star A) = 0$$

is still the integrability of the linear system

$$J = \tau(d + A)X = \star dX$$

provided that the ‘spectral parameter’ τ becomes a function on (N_2, ds_2^2), and satisfies

$$\omega = i(f - \bar{f}) + \frac{f + \bar{f}}{2\tau}(1 - \tau^2) \, ,$$

where $\omega \in \mathbb{C}$ is called the ‘constant’ (space-time independent) spectral parameter, and where $\rho = f + \bar{f}$ is a solution of $d \star d\rho = 0$.

The relation between ω and τ defines the 'spectral curve.' Assume that we can choose Weyl coordinates $x = (\rho, v)$, where v is chosen such that $f = \frac{1}{2}(\rho - iv)$ (i.e. $\star d\rho = -dv$) and consequently $ds_2^2 = d\rho^2 + dv^2$:

$$\tau(\omega, x) = \frac{1}{\rho}\left(v - \omega \pm \sqrt{\rho^2 + (v - \omega)^2}\right) \ , \quad \text{for} \quad \rho \neq 0 \ .$$

The monodromy matrix

Given a solution $M(x) = V^\natural(x)V(x)$ of the field equations, and a solution $X(\tau, x)$ of the linear system, one can define 'spectral deformations' of $V(x)$ and $M(x)$:

$$\mathcal{P}(\tau, x) = V(x)X(\tau, x) \;, \quad \mathcal{M}(\tau, x) = \mathcal{P}^\natural(-1/\tau, x)\mathcal{P}(\tau, x) \;.$$

The 'monodromy matrix' only depends on the constant spectral parameter:

$$\mathcal{M}(\tau(\omega, x), x) = \mathcal{M}(\omega)$$

If the solution $X(\tau, x)$ satisfies $X(0, x) = \mathbb{I}$, then the monodromy matrix $\mathcal{M}(\omega)$ satisfies a canonical Riemann-Hilbert factorisation problem.

Conversely, given a matrix function $\mathcal{M}(\omega) = \mathcal{M}^\natural(\omega)$ which admits a canonical factorization, we obtain a solution to the linear system, and to the field equations.

Riemann-Hilbert problems

Ingredients:

- Γ: closed simple contour in $\mathbb{C}$, taken to be the unit circle. $D_\pm$ interior/exterior of Γ.
- $n \times n$ matrix function $\mathcal{M}$, defined and continuous with continuous inverse $\mathcal{M}^{-1}$ on Γ.

We seek a *canonical* Birkhoff decomposition (aka Wiener Hopf factorization) on Γ,

$$\mathcal{M} = M_- M_+$$

where M_+, M_+^{-1} are bounded and analytic on D_+ and where M_-, M_-^{-1} are bounded and analytic on D_-.

A general ('non-cananonical') factorization takes the form

$$\mathcal{M} = M_- D M_+ \ , \quad D = \mathrm{diag}(\tau^{k_j})_{j=1,\ldots,n}$$

with 'partial indices' $k_j \in \mathbb{Z}$. For the existence of a non-canonical factorization it is sufficient that $\mathcal{M}, \mathcal{M}^{-1}$ have Hölder continuous components.

But we must require a *canonical* factorization, where all partial indices are zero.

Theorem: $\mathcal{M} \in C_\alpha^{n\times n}$ admits a canonical factorization if and only if

1. $\det \mathcal{M}$ admits a (scalar) canonical factorization,

$$\det \mathcal{M} = \gamma_- \gamma_+ \ , \quad \gamma_-^{\pm 1} \in C_\alpha^- \ , \quad \gamma_+^{\pm 1} \in C_\alpha^+ \ .$$

2. The vectorial Riemann-Hilbert problem

$$\mathcal{M}\phi_+ = \phi_- \ , \quad \phi_\pm \in (C_\alpha^\pm)^{n\times 1}, \quad \phi_-(\infty) = 0$$

   has only the trivial solution.

Remark: While implied by known results (and probably 'well known' to experts in Riemann-Hilbert problems), this theorem has to our knowledge not been formulated (and proved) before and in the above form.

Notation:

- $C_\alpha = C_\alpha(\Gamma)$ Hölder continuous functions.
- $C_\alpha^\pm$: functions in C_α admitting a bounded analytic extension to $D_\pm$

Canonical factorizations are unique up to a constant matrix factor. Fix solution by imposing:

$$M_+(0) = \mathbb{I} \,.$$

Notation $M_+ = X$ for normalized canonical factorization.

The map $\tau \to -1/\tau$ exchanges the interior and exterior of the unit circle, and the two Riemann sheets of $\tau(\omega, x)$. The constant spectral parameter ω is invariant.

Recovering solutions from monodromy matrices:

$$\mathcal{M}(\omega) = M_-(\tau, x) M_+(\tau, x) = X^\natural(-1/\tau, x) M(x) X(\tau, x) \,.$$

Note that

$$M_+(\tau, x) = X(\tau, x) \,, \quad M_-(\infty, x) = M(x)$$

are solutions to the linear system and to the field equations, respectively.

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In fact, any $\natural$-symmetric matrix function $\mathcal{M}(\omega)$ which admits a canonical factorization with ω and (τ, x) related by the spectral curve provides a solution to the linear system and to the field equations.

Proof: see Section 6 of our paper. Important stepping stone is the equivalent form

$$dXX^{-1} = -\frac{\tau}{1+\tau^2} \star A - \frac{\tau^2}{1+\tau^2} A$$

of the linear system $\tau(d+A)X = \star dX$.

N.B. Breitenlohner and Maison use the corresponding expression involving $\mathcal{P}(\tau, x) = V(x)X(\tau, x)$:

$$d\mathcal{P}\mathcal{P}^{-1} = Q + \frac{1-\tau^2}{1+\tau^2} P - \frac{2\tau}{1+\tau^2} \star P$$

where $A = Q + P \in \mathfrak{h} \oplus \mathfrak{p}$.

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Group theoretical interpretation

Here I follow Breitenlohner-Maison and Nicolai.

Extend involution θ to $\tilde{\mathfrak{g}} \ni \delta g(\tau)$ by

$$\tilde{\theta}(\delta g(\tau)) = \theta(\delta g)(-1/\tau)$$

Invariant subalgebra $\tilde{\mathfrak{h}}$ defined by $\tilde{\theta}$-invariance,
$\tilde{\theta}((\delta h)(\tau)) = \theta(\delta h)(-1/\tau) = h(\tau)$:

$$\delta h(\tau) = \delta h_0 + \sum_{n=1}^{\infty} \delta h_n \left(\tau^n + (-1)^n \frac{1}{\tau^n}\right) + \sum_{n=1}^{\infty} \delta p_n \left(\tau^n - (-1)^n \frac{1}{\tau^n}\right) \ ,$$

$\delta h_n \in \mathfrak{h}\ , \delta p_n \in \mathfrak{p}$.

Define infinitesimal action of infinite-dimensional rigid symmetry group $\tilde{G}$ and of infinite-dimensional gauge group $\tilde{H}$ on $\mathcal{P}(\tau, x) = V(x)X(\tau, x)$ by

$$\mathcal{P}(\tau, x) \mapsto \delta h(\tau, x)\mathcal{P}(\tau, x) - \mathcal{P}(\tau, x)\delta g^{-1}(\omega)$$

NB: both spectral parameters occur in this construction.

Define 'infinite-dimensional coset space' $\tilde{G}/\tilde{H}$ by

$$\mathcal{P}(\tau, x) \simeq h(\tau, x)\mathcal{P}(\tau, x) \ , \ \ h(\tau, x) \in H \ .$$

Breitenlohner-Maison: $\tilde{G}/\tilde{H} =$ space of solutions of the linear system. Further explanation (Nicolai): $d\mathcal{P}\mathcal{P}^{-1} \in \tilde{\mathfrak{h}}$ 'suggests' that $\mathcal{P} = h(\tau, x)g^{-1}(\omega)$.

Comment: to be able to extract the space-time solution, we need to choose $h(\tau, x)$ sucht that $\mathcal{P}(\tau, x)$ is regular at $\tau = 0$. ('Generalized Borel gauge'). This property is not preserved under $\tilde{H}$ gauge transformations.

The Riemann-Hilbert problem picks the unique representative out of an $\tilde{H}$-orbit which is regular (and normalized) at $\tau = 0$

$$X(0, x) = M_+(0, x) = \mathbb{I}$$

and thus allows to obtain a corresponding space-time solution as

$$M(x) = M_-(\infty, x) \ .$$

Our Method of explicit factorization

Lemma: Let $\phi_1 = r_1\phi_+$, $\phi_2 = r_2\phi_-$ where r_1 , r_2 are rational functions, bounded on Γ, and ϕ_+ , $\phi_- \in C_\alpha^\pm$ do not vanish at any of the poles of r_1 in D_+, r_2 in D_-, respectively. If $\phi_1 = \phi_2$ on Γ, then both ϕ_1 and ϕ_2 are equal to a rational function whose poles are the poles of r_1 in D_+ and r_2 in D_- (including ∞), counting their multiplicity.

Explicit canonical factorization example 1

Coset: $G/H = SU(2,1)/SL(2,\mathbb{R}) \times U(1)$,
Theory: Einstein-Maxwell.

Monodromy matrix:

$$\mathcal{M}(\omega) = \frac{1}{\omega^2} \begin{pmatrix} \frac{1}{2}|a|^2 & a\,\omega & \omega^2 \\ -\bar{a}\,\omega & -\omega^2 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix} \;,\; \det \mathcal{M} = 1 \;,\; \mathcal{M}^\natural(\omega) = \mathcal{M}(\omega) \;,$$

where $a \in \mathbb{C}$.
Observe: double pole at $\omega = 0$, bounded for $\omega \to \infty$, triangular structure.

Write factorization problem in the form

$$\mathcal{M}(\omega)\, M_+^{-1} = M_- \quad , \quad \tau \in \Gamma \; .$$

Start with vectorial factorization problem for the first column:

$$\frac{1}{\omega^2}\begin{pmatrix} \frac{1}{2}|a|^2 & a\,\omega & \omega^2 \\ -\bar{a}\,\omega & -\omega^2 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_{1+} \\ \phi_{2+} \\ \phi_{3+} \end{pmatrix} = \begin{pmatrix} \phi_{1-} \\ \phi_{2-} \\ \phi_{3-} \end{pmatrix} ,$$

Normalization conditions:

$$M_+^{-1}(\tau = 0) = \mathbb{I} \Rightarrow \phi_{1+}(0) = 1 \ , \quad \phi_{2+}(0) = \phi_{3+}(0) = 0 \ .$$

Compare third component: $\phi_{1+} = \phi_{3-}$ for $\tau \in \Gamma$, with $\phi_{1+} \in C_+^\alpha$, $\phi_{3-} \in C_-^\alpha$.
Liouville's theorem plus normalization condition:

$$\phi_{1+} = \phi_{3-} = K = 1 \ .$$

Compare second component (using previous result):

$$-\frac{\bar{a}}{\omega} - \phi_{2+} = \phi_{2-} \quad , \quad \tau \in \Gamma \; .$$

Decompose ω using the spectral curve:

$$\omega = -\frac{\rho}{2}\frac{(\tau - \tau_0^+)(\tau - \tau_0^-)}{\tau} \; ,$$

where $\tau_0^\pm \in D_\pm$. Use decomposition into partial fractions

$$\frac{\tau}{(\tau - \tau_0^+)(\tau - \tau_0^-)} = \frac{A}{(\tau - \tau_0^+)} + \frac{B}{(\tau - \tau_0^-)}$$

$$A = \frac{\tau_0^+}{\tau_0^+ - \tau_0^-} \quad , \quad B = -\frac{\tau_0^-}{\tau_0^+ - \tau_0^-} \; .$$

and re-arrange terms such that C_{+}^{α}-functions and C_{-}^{α}-functions are on different sides of the equation. Then use Liouville and normalization conditions:

$$\frac{2\bar{a}}{\rho}\,\frac{B}{(\tau-\tau_0^{-})} - \phi_{2+} = \phi_{2-} - \frac{2\bar{a}}{\rho}\,\frac{A}{(\tau-\tau_0^{+})} = k = -\frac{2\bar{a}}{\rho}\,\frac{B}{\tau_0^{-}}\ .$$

From this we obtain ϕ_{2+} and ϕ_{2-}. Continue until M_{+}^{-1} and M_{-} have been determined.

In this particular example we find

$$M_+^{-1}(\tau,x) \quad = \quad \begin{pmatrix} 1 & 0 & 0 \\ \frac{-2\bar{a}}{\rho\,(\tau_0^+ - \tau_0^-)}\;\frac{\tau}{\tau-\tau_0^-} & 1 & 0 \\ \frac{2|a|^2}{\rho^2\,(\tau_0^+ - \tau_0^-)^2}\;\frac{\tau^2}{(\tau-\tau_0^-)^2} & \frac{-2a}{\rho\,(\tau_0^+ - \tau_0^-)}\;\frac{\tau}{\tau-\tau_0^-} & 1 \end{pmatrix}$$

as well as

$$M_-(\tau,x) \quad = \quad \begin{pmatrix} \frac{2|a|^2}{\rho^2\,(\tau_0^+ - \tau_0^-)^2}\;\frac{\tau^2}{(\tau-\tau_0^+)^2} & \frac{-2a}{\rho\,(\tau_0^+ - \tau_0^-)}\;\frac{\tau}{\tau-\tau_0^+} & 1 \\ \frac{2\bar{a}}{\rho\,(\tau_0^+ - \tau_0^-)}\;\frac{\tau}{\tau-\tau_0^+} & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Extract space-time solution:

$$M(x) = M_-(\tau = \infty, x) = \begin{pmatrix} \frac{|a|^2}{2(\rho^2+v^2)} & \frac{a}{\sqrt{\rho^2+v^2}} & 1 \\ \frac{-\bar{a}}{\sqrt{\rho^2+v^2}} & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

Expression in four-dimensional coordinates (t, r, θ, ϕ), where $\rho = r \sin\theta,\ v = r\cos\theta$:

$$ds_4^2 = -e^{-\varphi} dt^2 + e^{\varphi} \left(dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right) , \quad e^{-\varphi} = \frac{r^2}{|Q|^2} ,$$

where $Q = q + ip = a/\sqrt{2}$ combines electric and magnetic charge. Potentials for electromagnetic field:

$$\chi_e(r) = \frac{q}{|Q|^2} r , \quad \chi_m(r) = \frac{p}{|Q|^2} r .$$

Static attractor solution $AdS_2 \times S^2$ with covariantly constant electromagnetic field (Bertotti-Robinson).

Explicit canonical factorization example 2

Coset: $G/H = SL(3,\mathbb{R})/SO(2,1)$.
Theory: 5d gravity $\to$ 4d Einstein-Maxwell-Scalar.

Mondromy matrix:

$$\mathcal{M}(\omega) = \frac{1}{\omega^2}\begin{pmatrix} A & B\omega & C\omega^2 \\ -B\omega & D\omega^2 & 0 \\ C\omega^2 & 0 & 0 \end{pmatrix}$$

$A, B, C, D \in \mathbb{R}$, where $-C^2 D = 1$. Again: second order pole at $\omega = 0$, bounded at $\omega \to \infty$, triangular structure.

Factorization:

$$M_{-}(\tau,x)=\begin{pmatrix} m_{11} & -\frac{2B}{\rho(\tau_0^{+}-\tau_0^{-})}\,\frac{\tau}{(\tau-\tau_0^{+})} & C \\ \frac{2B}{\rho(\tau_0^{+}-\tau_0^{-})}\,\frac{\tau}{(\tau-\tau_0^{+})} & D & 0 \\ C & 0 & 0 \end{pmatrix}\;,$$

$$M_{+}^{-1}(\tau,x)=\begin{pmatrix} 1 & 0 & 0 \\ \frac{2B}{D\,\rho(\tau_0^{+}-\tau_0^{-})}\,\frac{\tau}{(\tau-\tau_0^{-})} & 1 & 0 \\ m_{31} & -\frac{2B}{C\,\rho(\tau_0^{+}-\tau_0^{-})}\,\frac{\tau}{\tau-\tau_0^{-})} & 1 \end{pmatrix}\;.$$

where m_{11}, m_{31} are given in our paper.

Space-time solution:

$$M(\rho, v) = \frac{1}{\rho^2 + v^2} \begin{pmatrix} A + \frac{(2AD+B^2)}{2D} \left(\frac{v}{\sqrt{\rho^2+v^2}} - 1 \right) & B\sqrt{\rho^2 + v^2} & C(\rho^2 + v^2) \\ -B\sqrt{\rho^2 + v^2} & D(\rho^2 + v^2) & 0 \\ C(\rho^2 + v^2) & 0 & 0 \end{pmatrix} .$$

Four-dimensional coordinates (t, r, θ, ϕ):

$$ds_4^2 = -\frac{r^2}{v_1(\theta)} \left(dt - \frac{J \sin^2\theta}{8\pi\, r}\, d\phi \right)^2 + v_1(\theta) \left(\frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta\, d\phi^2 \right)$$

$$v_1(\theta) = \frac{1}{8\pi} \sqrt{P^2 Q^2 - J^2 \cos^2\theta} \, .$$

$(A, B, C, D) \leftrightarrow (Q, P, J)$, electric/magnetic charges and angular momentum.

Scalar field:

$$e^{2\phi_1/\sqrt{3}} = \left(\frac{P}{Q} \right)^{2/3} \frac{PQ - J\cos\theta}{PQ + J\cos\theta} \, .$$

Underrotating attractor solution.

Underrotating attractor = near horizon geometry of an extremal, underrotating black hole.

Underrotating: $P^2 Q^2 > J^2$, static limit $J = 0$, no ergoregion, scalar takes attractor values determined by Q, P, J at horizon, irrespective of its value at infinity (universal θ-dependence).

Overrotating: $J^2 > P^2 Q^2$, have extremal Kerr limit $PQ = 0$, ergoregion, scalars do not necessarily attain fixed point values at the horizon, but entropy is still independent of values of scalars at infinity.

Rotating attractors (both under- and overrotating ones) have reduced isometry $SL(2,\mathbb{R}) \times SO(2) \subset SL(2,\mathbb{R}) \times SO(3)$ compared to static attractors.

Reference for 'rotating attractors': D. Astefanasi, K. Goldstein, R.P. Jena, A. Sen and S.P. Trivedi, JHEP **10** (2006) 058.

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Remark: the static attractor considered previously (though for a different G/H) is contained as the special case where $2AD + B^2 = 0 \Leftrightarrow J = 0$. As far as factorization is concerned, finding underrotating attractors is as involved as finding static attractors.

Consistent with static attractors being a limiting case of the underrotating branch of extremal rotating solutions.

Generating solutions 1. G-action

G-action is compatible with factorization. Given a factorization

$$\mathcal{M}_{\mathrm{seed}} = M_-^{\mathrm{seed}} M_+^{\mathrm{seed}}$$

and $g \in G$, we obtain

$$\mathcal{M}(\omega) = M_- M_+ = (g^\natural M_-^{\mathrm{seed}} g)(g^{-1} M_+^{\mathrm{seed}} g)$$

and

$$M(x) = g^\sharp M^{\mathrm{seed}}(x) g \ .$$

Some $g \in G$ just generate gauge transformations, i.p. constant shifts of scalar fields which correspond to electric or magnetic gauge potentials. NB: such gauge transformations do not preserve the 'triangular shape' we found useful in our examples.

Other $g \in G$ correspond to Harrison-like transformations, which map near-horizon solutions to the corresponding full black holes solutions. We show this in particular for static and underrotating solutions.

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Generating solutions 2. Full Geroch group

Action of $g(\omega) \in \tilde{G}$ requires to solve a new factorization problem. We have studied i.p. the following deformation of an underrotating attractor

$$\mathcal{M}(\omega) = g^{\natural}(\omega)\, \mathcal{M}_{\rm seed}(\omega)\, g(\omega) = \frac{1}{\omega^2} \begin{pmatrix} A & B\,\omega + \alpha & C\,\omega^2 \\ -B\,\omega - \alpha & D\,\omega^2 & 0 \\ C\,\omega^2 & 0 & 0 \end{pmatrix}$$

where

$$g(\omega) = e^{N/\omega^2}\ , \quad N = \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ \mu & \gamma & 0 \end{pmatrix}\ .$$

and we have chosen $\beta = \mu = 0$, and $\alpha = \gamma C$.

This can be factorized (see paper) and yields a new space-time solution asymptotic to $AdS_2 \times S^2$. For a static seed solution, the new solution is regular, except at $\rho = v = 0$ (where canonical factorization does not apply). Ricci scalar blows up, scalar runs away.

How to find monodromy matrices to factorize?

How to choose $\mathcal{M}(\omega)$ to start with? We have mostly relied on known solutions and the 'substitution rule' of Chakrabarty/Virmani,

$$\mathcal{M}(\omega = v) = M(\rho = 0, v) \, .$$

Comment: the limit $\rho \to 0^+$ is delicate, and the validity of the substitution rule requires that the limits

$$\lim_{\rho \to 0^+} \mathcal{M}(\omega(\tau, x)) = \left(\lim_{\rho \to 0^+} X^\natural(-1/\tau, x) \right) \left(\lim_{\rho \to 0^+} M(x) \right) \left(\lim_{\rho \to 0^+} X(\tau, x) \right)$$

exist individually. While it is not clear to us why this should hold in general, we have verified it to be true in all examples where $M(x)$ itself has a limit (one example with singular limit of $M(x)$, where substitution rule cannot be applied, while a regular solution corresponding to the monodromy matrix does exist.)

Remark: It is known from Breitenlohner-Maison that the solution on the axis of rotation determines the solution uniquely.

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Open Questions

- How to choose monodromy matrices that correspond to interesting space-time solutions? How are space-time properties encoded in the monodromy matrix?
- What can still be done in presence of a cosmological constant or scalar potential (gauged supergravity)?
- What can one learn about the 'stringy' Geroch group and hidden symmetries in supergravity and string theory?

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