Dynamical symmetry enhancement near black hole horizons

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Outline

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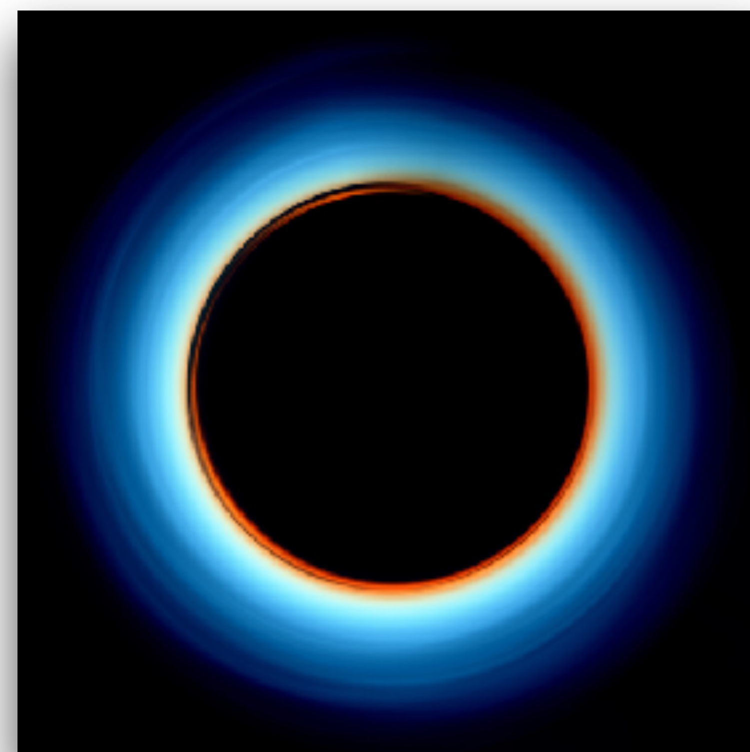


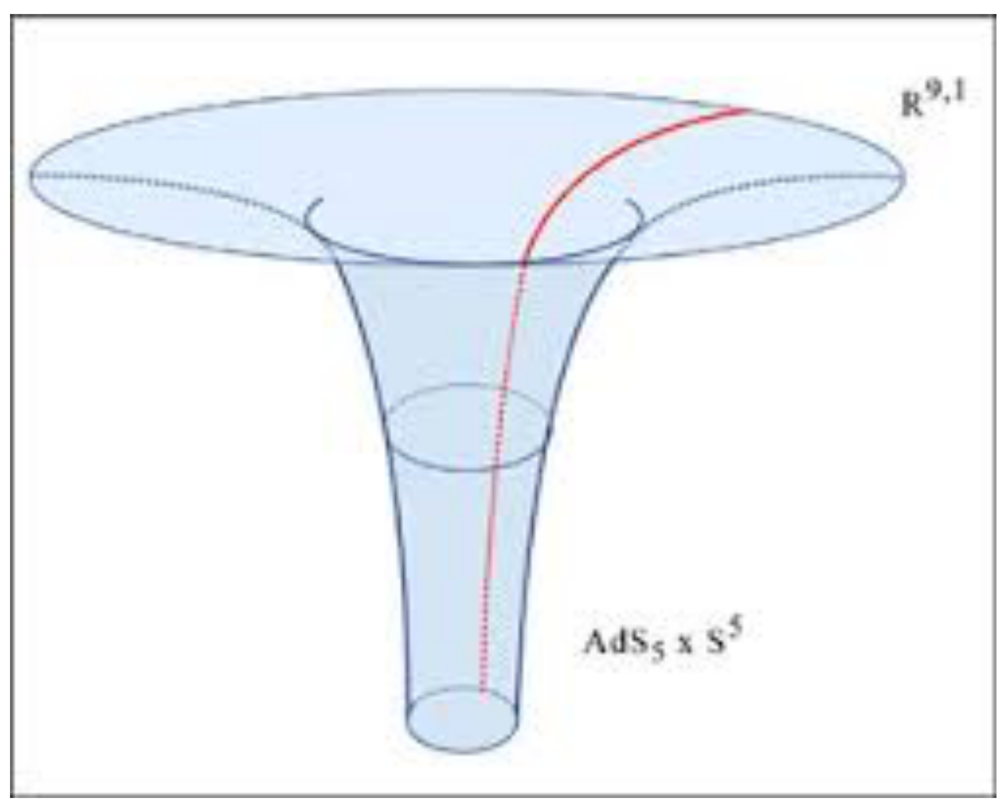
Introduction & Motivation

- The physics near black hole/brane horizons has many interesting features and puzzles (information paradox, BMS transformations, firewalls etc).
- And interesting applications, most notably in the context of the AdS/CFT correspondence.
- One generic feature, observed on a case-by-case basis, is the (super)symmetry enhancement occurring in the near horizon region of supersymmetric black holes and branes.

A familiar example of symmetry enhancement: D3-branes in IIB supergravity

- Near the horizon the symmetry enhances to SO(2,4), i.e. the conformal group in D=4

- In addition the number of preserved SUSY doubles (no SUSY broken)



Questions

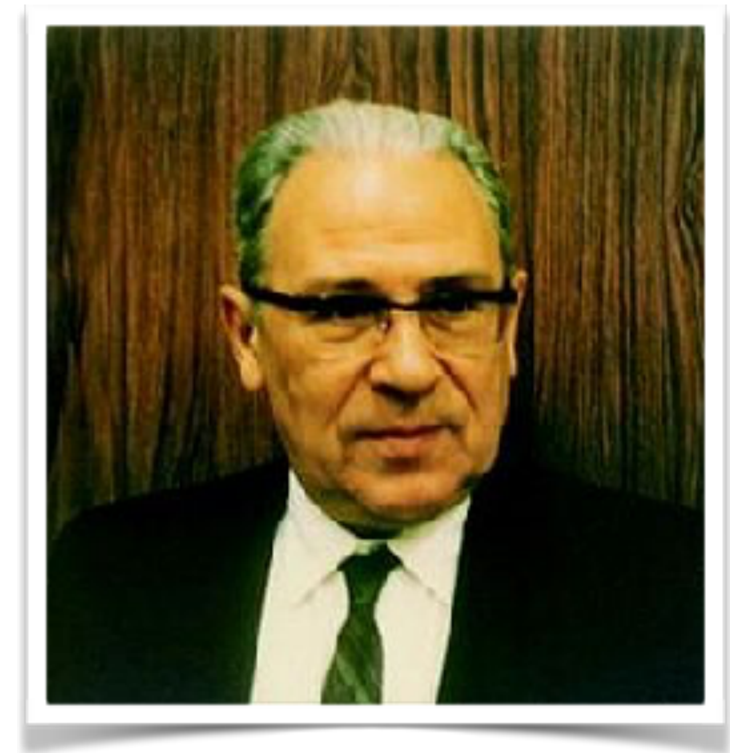
- General picture?
- What is required for (super)symmetry enhancement near the black hole horizon?
- **Aim: General proof of (super)symmetry enhancement for supersymmetric horizons.**
- Talk mainly based on:
  - U. Gran, J. Gutowski, U. Kayani & G. Papadopoulos [arXiv: 1411.5286 & 1409.6303]
  - U. Gran, J. Gutowski & G. Papadopoulos [arXiv:1306.5765]
- Related work include
  - J. Gutowski & G. Papadopoulos [arXiv:1303.0869]
  - J. Grover, J. Gutowski, G. Papadopoulos, & W. Sabra [arXiv: 1303.0853]

Tools

- The Hopf Maximum Principle
- Lichnerowicz Type of Theorems
- Index Theorem
- GAMMA

The Hopf Maximum Principle

- "Classic and bedrock result" in the theory of second order elliptic PDEs.
- The Hopf maximum principle (1927) states that if a function achieves its maximum in the interior of the domain, the function is a constant.
- Requires that the PDE satisfies a certain positive definiteness property.
- Example: Harmonic functions (Gauss 1839).
- We assume *compact* horizons.
- Strategy: Rewrite the equations in such a form that the maximum principle can be applied.



Eberhard Hopf
1902-1983

Lichnerowicz Type of Theorems

- Fundamental equation in the analysis of spinors on pseudo-Riemannian manifolds

$$\mathscr{D}^2\epsilon = \nabla^2\epsilon + \frac{1}{4}R\epsilon$$

- Relates the Dirac and Laplace operator acting on a spinor, and the scalar curvature

- Since the Laplace operator is an *elliptic* operator we apply the Hopf maximum principle.

- In D=4 important in Seiberg-Witten theory

André Lichnerowicz
1915-1998

Index Theorem

- We use the Atiyah–Singer index theorem (1963).
- Index theorems relate *analytic* quantities to *topological* invariants.
- Applicable for *elliptic* differential operators on *compact* manifolds.
- Since we have Dirac-like operators (modified by fluxes) we can use index theorems to count their *zero-modes.*
- The zero-modes will be identified with the Killing spinors.
- Through (modified) Lichnerowicz type of theorems we can combine analytical and topological constraints from spinor theory with constraints on elliptic PDEs for which the Hopf maximum principle applies.

GAMMA

- Mathematica package for performing Γ-matrix algebra.

- Very lengthy calculations…
- Download from www.gran.name
- Compatible with Mathematica 11.1

Equations to Analyze

- The Killing spinor equations (KSE) encode the requirement of preserving SUSY: linear
- Field equations encode the interplay between fluxes and geometry: non-linear
- Integrability conditions relate the KSEs and the field equations.
- Usual approach: Remove the redundant field equations.
- We will do the opposite! (Since some of the KSE are very complicated)

Assumptions

1. The horizons admit at least *one supersymmetry.*

2. The horizons are *compact.*

3. The near horizon geometries are *smooth.*

Incorporate the Horizon

- Coordinate independent definition of an event horizon for an extremal black hole: *Killing horizon*

- Introduce Gaussian null coordinates and use adapted metric and fields [Isenberg & Moncrief '83, Friedrich, Racz & Wald '98]

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j \ , \quad G = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge X + r\mathbf{e}^+ \wedge Y + \tilde{G} \ ,$$
$$H = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge L + r\mathbf{e}^+ \wedge M + \tilde{H} \ , \quad F = \mathbf{e}^+ \wedge \mathbf{e}^- S + r\mathbf{e}^+ \wedge T + \tilde{F} \ ,$$

where

$$\mathbf{e}^+ = du, \qquad \mathbf{e}^- = dr + rh - \frac{1}{2}r^2\Delta du, \qquad \mathbf{e}^i = e^i{}_I dy^I \ .$$

- The forms in the right hand sides above are forms on the spatial horizon section ***S*** given by $u = r = 0$.

- NB: All u and r dependence is explicit.

- Smoothness required for the field equations and Bianchis to be valid.

Killing Spinor Equations

- The Killing spinor equations (KSE) are the gravitino and dilatino supersymmetry variations evaluated at the locus where all fermions vanish.

$$
\begin{aligned}
\mathcal{D}_\mu \epsilon \equiv \nabla_\mu \epsilon + \frac{1}{8} H_{\mu\nu_1\nu_2}\Gamma^{\nu_1\nu_2}\Gamma_{11}\epsilon + \frac{1}{16} e^\Phi F_{\nu_1\nu_2}\Gamma^{\nu_1\nu_2}\Gamma_\mu\Gamma_{11}\epsilon \\
+ \frac{1}{8\cdot 4!} e^\Phi G_{\nu_1\nu_2\nu_3\nu_4}\Gamma^{\nu_1\nu_2\nu_3\nu_4}\Gamma_\mu\epsilon + \frac{1}{8} e^\Phi m\Gamma_\mu\epsilon = 0 \ , \\
\mathcal{A}\epsilon \equiv \partial_\mu\Phi\,\Gamma^\mu\epsilon + \frac{1}{12} H_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3}\Gamma_{11}\epsilon + \frac{3}{8} e^\Phi F_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2}\Gamma_{11}\epsilon \\
+ \frac{1}{4\cdot 4!} e^\Phi\, G_{\mu_1\mu_2\mu_3\mu_4}\Gamma^{\mu_1\mu_2\mu_3\mu_4}\epsilon + \frac{5}{4} e^\Phi m\epsilon = 0 \ ,
\end{aligned}
$$

- Use the Ansatz for the metric and the fields to re-express the field equation, Bianchi identities and the KSEs.

Integrate along the light cone

- Since all *u* and *r* dependence is explicit it is possible to integrate the KSE along the light-cone directions.

- This leads to conditions in terms of the KSEs restricted to the horizon

$$\nabla_i^{(\pm)}\eta_\pm = 0 \ , \quad \mathcal{A}^{(\pm)}\eta_\pm = 0 \ ,$$

where

$$\nabla_i^{(\pm)} = \tilde{\nabla}_i + \Psi_i^{(\pm)} \ ,$$

$$\Psi_i^{(\pm)} = \left(\mp \frac{1}{4}h_i \mp \frac{1}{16}e^\Phi X_{l_1 l_2}\Gamma^{l_1 l_2}\Gamma_i + \frac{1}{8\cdot 4!}e^\Phi \tilde{G}_{l_1 l_2 l_3 l_4}\Gamma^{l_1 l_2 l_3 l_4}\Gamma_i + \frac{1}{8}e^\Phi m\Gamma_i \right)$$
$$+ \Gamma_{11}\left(\mp \frac{1}{4}L_i + \frac{1}{8}\tilde{H}_{i l_1 l_2}\Gamma^{l_1 l_2} \pm \frac{1}{8}e^\Phi S\Gamma_i - \frac{1}{16}e^\Phi \tilde{F}_{l_1 l_2}\Gamma^{l_1 l_2}\Gamma_i \right) \ ,$$

$$\mathcal{A}^{(\pm)} = \partial_i\Phi\Gamma^i + \left(\mp \frac{1}{8}e^\Phi X_{l_1 l_2}\Gamma^{l_1 l_2} + \frac{1}{4\cdot 4!}e^\Phi \tilde{G}_{l_1 l_2 l_3 l_4}\Gamma^{l_1 l_2 l_3 l_4} + \frac{5}{4}e^\Phi m \right)$$
$$+ \Gamma_{11}\left(\pm \frac{1}{2}L_i\Gamma^i - \frac{1}{12}\tilde{H}_{ijk}\Gamma^{ijk} \mp \frac{3}{4}e^\Phi S + \frac{3}{8}e^\Phi \tilde{F}_{ij}\Gamma^{ij} \right) \ .$$

Modified Dirac operators

- Goal: Identify the Killing spinors with the zero-modes of modified Dirac operators allowing us to use index theorems to count the Killing spinors.

- NB: There are two Killing spinor equations!

- Dirac operators: $$\mathscr{D}^{(\pm)} = \mathcal{D}^{(\pm)} - \mathcal{A}^{(\pm)} ,$$

  where
  $$\mathcal{D}^{(\pm)} \equiv \Gamma^i \nabla_i^{(\pm)} = \Gamma^i \tilde{\nabla}_i + \Psi^{(\pm)} ,$$

  and
  $$\Psi^{(\pm)} \equiv \Gamma^i \Psi_i^{(\pm)} = \mp \frac{1}{4} h_i \Gamma^i \mp \frac{1}{4} e^\Phi X_{ij} \Gamma^{ij} + e^\Phi m + \Gamma_{11} \left(\pm \frac{1}{4} L_i \Gamma^i - \frac{1}{8} \tilde{H}_{ijk} \Gamma^{ijk} \mp e^\Phi S + \frac{1}{4} e^\Phi \tilde{F}_{ij} \Gamma^{ij} \right) .$$

Lichnerowicz Type of Theorems

- Needed to show

$$\nabla_i^{(+)}\eta_+ = 0 \ , \quad \mathcal{A}^{(+)}\eta_+ = 0 \iff \mathscr{D}^{(+)}\eta_+ = 0 \ .$$

i.e. that there is a 1-1 correspondence between Killing spinors and zero modes.

$$\mathscr{D}^{(+)}: \qquad \tilde{\nabla}^i\tilde{\nabla}_i \parallel \eta_+ \parallel^2 - (2\tilde{\nabla}^i\Phi + h^i)\tilde{\nabla}_i \parallel \eta_+ \parallel^2 = 2 \parallel \hat{\nabla}^{(+)}\eta_+ \parallel^2 + (-4\kappa - 16\kappa^2) \parallel \mathcal{A}^{(+)}\eta_+ \parallel^2 \ ,$$

$$\mathscr{D}^{(-)}: \qquad \tilde{\nabla}^i\left(e^{-2\Phi}V_i\right) = -2e^{-2\Phi} \parallel \hat{\nabla}^{(-)}\eta_- \parallel^2 + e^{-2\Phi}(4\kappa + 16\kappa^2) \parallel \mathcal{A}^{(-)}\eta_- \parallel^2 \ ,$$

where

$$\hat{\nabla}_i^{(\pm)} = \nabla_i^{(\pm)} + \kappa\Gamma_i\mathcal{A}^{(\pm)}$$

$$V = -d \parallel \eta_- \parallel^2 - \parallel \eta_- \parallel^2 h$$

Supersymmetry enhancement

- Now, use the index theorem to arrive at $N_+ = N_-$

$$N = N_+ + N_- = 2N_-$$

- Then, it is possible to show that if η_- is a Killing spinor, then so is

  where $$\eta_+ = \Gamma_+ \Theta_- \eta_-$$

$$\Theta_- = \frac{1}{4} h_i \Gamma^i + \frac{1}{4} \Gamma_{11} L_i \Gamma^i - \frac{1}{16} e^\Phi \Gamma_{11} (-2S + \tilde{F}_{ij} \Gamma^{ij}) - \frac{1}{8 \cdot 4!} e^\Phi (-12 X_{ij} \Gamma^{ij} + \tilde{G}_{ijkl} \Gamma^{ijkl}) - \frac{1}{8} e^\Phi m$$

- Using the Hopf maximum principle one can show that

$$\mathrm{Ker}\, \Theta_- = \{0\}$$

  (otherwise a contradiction) unless all fields vanish, so the extra Killing spinor is non-zero.

Symmetry enhancement

- We just showed that there is always at least two Killing spinors.
- The three Killing vectors that can be constructed from them form an sl(2,R) algebra

$$
\begin{aligned}
K_1(\epsilon_1, \epsilon_2) &= -2u \parallel \eta_+ \parallel^2 \partial_u + 2r \parallel \eta_+ \parallel^2 \partial_r + \tilde{V} \ , \\
K_2(\epsilon_2, \epsilon_2) &= -2 \parallel \eta_+ \parallel^2 \partial_u \ , \\
K_3(\epsilon_1, \epsilon_1) &= -2u^2 \parallel \eta_+ \parallel^2 \partial_u + (2 \parallel \eta_- \parallel^2 + 4ru \parallel \eta_+ \parallel^2)\partial_r + 2u\tilde{V} \ ,
\end{aligned}
$$

where

$$\tilde{V} = \langle \Gamma_+ \eta_-, \Gamma^i \eta_+ \rangle \, \tilde{\partial}_i \ ,$$

and

$$[K_1, K_2] = 2 \parallel \eta_+ \parallel^2 K_2 \ , \quad [K_2, K_3] = -4 \parallel \eta_+ \parallel^2 K_1 \ , \quad [K_3, K_1] = 2 \parallel \eta_+ \parallel^2 K_3 \ .$$

Results

- There is always supersymmetry enhancement

$$N = 2N_{-} + \mathrm{Index}(D_E)$$

- Killing horizons in IIA SUGRA always preserve and **even** number of supersymmetries.

- Horizons with $N_{-} \neq 0$ and non-trivial fluxes have an sl(2,R) symmetry subalgebra.

- Conjecture that this holds in general for all supergravity theories.

- The sl(2,R) symmetry provides evidence that all such horizons have an AdS/CFT dual.

- Reduction from D=11 breaks an even number of supersymmetries.

Outlook

- All information obtained solving neither the (horizon) KSEs nor the field equations!

- Solving the Killing spinor equations is feasible and could give more information on the structure of generic black holes.

- Can we exploit the sl(2,R) symmetry to get information about the dual CFT?