Compact pseudo-Riemannian Einstein manifolds in low dimensions

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I Properties of compact pseudo-Riemannian homogeneous spaces

The objects of interest

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- $M = G/H$ for a connected Lie group G and a closed subgroup H,
- G acts transitively and by isometries (in particular volume-preserving),
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Question:
When is a space of this type an Einstein manifold ($\mathrm{Ric} = \lambda \mathrm{g}$)?

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- $\langle \cdot, \cdot \rangle$ is $\mathrm{Ad}_{\mathfrak{g}}(H)$-invariant (that is, $\langle \mathrm{Ad}_{\mathfrak{g}}(h)x, \mathrm{Ad}_{\mathfrak{g}}(h)y \rangle = \langle x, y \rangle$).
- The kernel $\mathfrak{g}^{\perp} = \{x \in \mathfrak{g} \mid \langle x, \cdot \rangle = 0\}$ is precisely $\mathfrak{h}$.

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Since $\langle \cdot, \cdot \rangle$ is a quadratic function on $\mathfrak{g}$, a density theorem by Mostow (1971) implies:

Corollary

1. $\langle \cdot, \cdot \rangle$ *is* $\overline{\mathrm{Ad}_{\mathfrak{g}}(H)}^{\mathrm{Z}}$*-invariant.*
2. $\langle \cdot, \cdot \rangle$ *is invariant under* $\mathrm{Ad}_{\mathfrak{g}}(S)$ *and* $\mathrm{Ad}_{\mathfrak{g}}(R)_{\mathrm{trig}}$.

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In particular, $\langle\cdot,\cdot\rangle$ is invariant under all nilpotent elements in of $\mathfrak{Lie}(\overline{\mathrm{Ad}_{\mathfrak{g}}(H)}^{\mathrm{Z}})$.
We say $\langle\cdot,\cdot\rangle$ is nil-invariant (generalizes (bi-)invariance).

Properties of nil-invariant metrics (Baues, Globke, Zeghib)

The Lie algebra

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Algebraic Theorem

1. $\langle \cdot, \cdot \rangle$ *is invariant under* $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{s} \ltimes \mathfrak{r})$.
2. *The restriction* $\langle \cdot, \cdot \rangle_{\mathfrak{s} \ltimes \mathfrak{r}}$ *to* $\mathfrak{s} \ltimes \mathfrak{r}$ *is invariant under* $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$.
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Geometric Theorem

Suppose $G = S \ltimes R$, *and* $M = G/H$ *is a pseudo-Riemannian homogeneous space of finite volume on which* G *acts almost effectively and isometrically. Then:*

1. H *is a lattice in* G.
2. *The pseudo-Riemannian metric* g *on* M *pulls back to a bi-invariant metric on* G.
3. M *is a locally symmetric space.*

II Nil-invariant pseudo-Riemannian Einstein metrics

(M, g) is called Einstein manifold if

$$\mathrm{Ric} = \lambda \mathrm{g}$$

for some constant $\lambda \in \mathbb{R}$.

For a bi-invariant metric g on G,

$$\mathrm{Ric} = -\frac{1}{4}\kappa$$

where $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))$ is the Killing form of $\mathfrak{g}$.

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- *either $\mathfrak{g}$ is semisimple and $\langle\cdot,\cdot\rangle$ is a non-zero multiple of the Killing form κ,*
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Remark

- This holds in particular if the Levi subgroup of G has no compact semisimple factors.
- Unclear for nil-invariant $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ with $\mathfrak{k} \neq \mathbf{0}$.

III Semisimple isometry group

Theorem A

Let

- M *be a pseudo-Riemannian homogeneous Einstein manifold of finite volume,*
- $G = K \times S$ *a semisimple, connected and simply connected Lie group acting transitively and almost effectively by isometries on* M*,*
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Then M *is a pseudo-Riemannian product of Einstein manifolds*

$$M = M_K \times M_S,$$

where

- $M_K = K/(H \cap K)$ *for a closed subgroup* $H \leq G$, *and* $H^\circ \leq H \cap K$,
- $M_S = S/\Gamma$ *and a lattice* $\Gamma \leq S$, *and the Einstein metric on* M_S *is induced by a multiple of the Killing form on* S.

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- $\mathfrak{k} \perp \mathfrak{s}$, and K, S commute. Hence S-orbits and M_K are orthogonal everywhere.
- $\pi_S(H) = \Gamma$ is a lattice in S, so $M_S = S/\Gamma$.
  The metric g_S is Einstein and bi-invariant, hence $\mathrm{g}_S \sim \kappa_S$. $\square$

IV Einstein solvmanifolds

Let (M, g_M) be a compact pseudo-Riemannian solvmanifold.

Recall:

- $M = G/\Gamma$ for a lattice $\Gamma \leq G$.
- g_M pulls back to bi-invariant g_G.
- $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is non-degenerate and invariant.

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If (M, g_M) is Einstein,

$$\mathrm{Ric} = 0 = \kappa.$$

Solvable, but not nilpotent?

If $\mathfrak{g}$ is nilpotent, then $\kappa = 0$.

Question:
Are there solvable $\mathfrak{g}$, not nilpotent, with

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Yes!
Construct examples of dimension 6 and index 2 using Medina's double extension.

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- $\mathrm{ad}(x)$ for $x \in \mathfrak{n}$ is nilpotent, so $\kappa(x, x) = 0$.
- For $a \in \mathfrak{a}$, the Einstein condition (Ricci-flat, $\kappa = 0$) becomes

$$\lambda_1^2 + \ldots + \lambda_k^2 + 2\alpha_1^2 + \ldots + 2\alpha_m^2 - 2\beta_1^2 - \ldots - 2\beta_m^2 = 0,$$

where $\lambda_i \in \mathbb{R}$ and $\zeta_j = \alpha_j + \mathrm{i}\beta_j \in \mathbb{C} \backslash \mathbb{R}$ are the eigenvalues of $\mathrm{ad}(a)$.

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Theorem B

Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be a solvable Lie algebra with invariant Einstein scalar product. If $\mathfrak{g}$ is not nilpotent, then $\dim \mathfrak{g} \geq 6$ and the index of $\langle\cdot,\cdot\rangle$ is ≥ 2.

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Corollary

Every Lorentzian Einstein Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is abelian.

This follows from the Theorem B and the classification of invariant Lorentzian scalar products (Medina 1985, Hilgert & Hofmann 1985).

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Necessary conditions for the existence of a lattice (Auslander 1973):

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The characteristic polynomial of $\exp(\mathrm{ad}(a))$ has coefficients in $\mathbb{Z}$ for all $a \in \mathfrak{a}$...
which means that all eigenvalues e^{λ_i}, e^{ζ_j} of $\exp(\mathrm{ad}(a))$ are algebraic numbers.

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Schanuel's Conjecture states:
Let $\alpha_1, \ldots, \alpha_d$ be complex numbers, linearly independent over $\mathbb{Q}$. Then

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha_1, \ldots, \alpha_d, e^{\alpha_1}, \ldots, e^{\alpha_d}) \;\geq\; d.$$

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Let X be a matrix in the normal form of $\mathrm{ad}(a)$ *with eigenvalues* $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ *and* $\zeta_1, \ldots, \zeta_m, \overline{\zeta}_1, \ldots, \overline{\zeta}_m \in \mathbb{C} \backslash \mathbb{R}$*. Suppose the eigenvalues satisfy*

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Conjecture

If Schanuel's Conjecture is true, then the Algebraic Lemma holds without "$n \leq 5$".

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Every compact pseudo-Riemannian Einstein solvmanifold is a nilmanifold.

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