Symplectic duality bundles and four-dimensional Supergravity

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Context

Supergavity theories are supersymmetric theories of gravity that some of us love, some of us hate, but sooner or later we all have to face when working in String Theory. There is no doubt that supergravity theories are interesting by themselves as well as because of their relation to String Theory. But... how well do *we* understand supergravity?

- The local structure of the theory has been extensively studied in the literature since the 70's.
- In particular, the local structure of all ungauged supergravities is relatively well known and has been classified by dimension and amount supersymmetry preserved.
- The local gaugings of supergravity have also been extensively studied, but there is still a lot of work to be done before arriving to a complete classification which clarifies their possible String/M-theory origin.
- Various formalisms, such as generalized geometry and double field theory, have have been developed to further explore the local structure of supergravity, uncovering interesting mathematical structures.
- Local supersymmetric (and non-supersymmetric to a lesser extent) solutions of supergravity have been systematically studied in the literature.

It looks like there is a well-established program to study the **local!** structure of supergravity and its zoo of solutions.

What about the **GLOBAL** structure of supergravity?

But wait...what does *global structure of supergravity* even mean?

Any proper theory of gravity should be formulated on a Lorentzian manifold whose physical Lorentzian metric determines the causality of the space-time (and a corresponding topology). Supergravity contains many other fields, aside from the metric, so what about them?

- Obtaining the global structure of supergravity on a differentiable manifold consists of two steps:
  1. Determining its complete matter content in terms of connections and global sections of the appropriate fiber bundles (more generally, submersions and other geometric structures such as gerbes may be needed) equipped with the appropriate geometric structures.
  2. Determining the global differential operators acting on the corresponding spaces of sections (generically infinite-dimensional Fréchet manifolds, important for application to moduli problems) which are required to formulate the equations of motion of the theory.

It turns out that the problem of studying the mathematical global formulation of supergravity has not been systematically addressed in the literature.

The goal is to develop the **mathematical theory of supergravity**: necessary step in order to perform a systematic mathematical study of supergravity. Some motivation:

1. Supergravity theories contain many alluring moduli problems involving several mathematical structures of great interest in a delicate interplay dictated by supersymmetry. Most of the differential-geometric moduli problems considered in Mathematics are particular cases of supersymmetry equations.
2. Possible applications to the study the differential topology of manifolds, in the spirit of Donaldson's theory.
3. Supergravity theories on globally hyperbolic manifolds give rise to potentially interesting *supersymmetric* flow equations and reduced systems.
4. Maximal analytic extension of supergravity solutions, of interest in Lorentzian geometry.
5. Requires a better understanding of spin geometry beyond standard spinorial structures: Lipschitz structures are required.

OK, so this is mathematically relevant. However, is it also physically relevant? Unsurprisingly, understanding the global structure of supergravity clarifies a number physical aspects of this theory and its solutions.

In the process of understanding the global mathematical formulation of supergravity we will:

1. find new supergravity theories, namely new non-trivial (and non-equivalent) extensions of local supergravity. Hence, **we are not just rewritting local supergravity using fancy mathematical tools. We are obtaining new physical supersymmetric theories**.
2. obtain a **global and manifiestly U-duality invariant formulation of supergravity**.
3. be able to give a precise definition of the global U-duality group of a supergravity theory as the automorphism group of the appropriate geometric structures (polarized symplectic flat vector bundles).
4. find that generic supergravity solutions are in fact locally geometric U-folds. In addition, we will be able to give a precise definition of locally geometric U-fold.
5. provide a precise geometric model to locally geometric of four-dimensional U-fold backgrounds, characterizing some of their global topological properties.

Strategy

In order to obtain the global structure of supergravity we need to split the problem in two parts:

1. We first consider the bosonic sector of supergravity, obtaining its global mathematical structure and formulation. This leads to a *generalized Einstein-Section-Maxwell theory*.
2. We then supersymmetrize the previous bosonic theoory, obtaining a *generalized Supergravity*. For this we need to use the appropriate mathematical framework to work with spinors in supergravity. Such framework is yet not available: it requires a complete understanding of real and complex bundles of Clifford modules over the bundle of Clifford algebras (and its even part) of the underlying space-time manifold.

Point (2) is very subtle and delicate. Progress has been recently made by C. Lazaroiu and CSS by exploring the theory of real Lipschitz structures and bundles of irreducible real Clifford modules building on the seminal work of T. Friedrich and A. Trautman on the topological classification of bundles of faithful complex Clifford-modules. In this talk we will exclusively consider point (1) in the particular case of four dimensions.

Understanding the global mathematical formulation of the bosonic sector of four-dimensional supergravity boils down to answer the following two fundamental questions:

1. What is a *scalar field* in supergravity?
2. What is a *vector field* and a *field strength* (or rather a bunch of them) in supergravity?

These look like very elementary questions. However, properly answering them in the context of supergravity requires some subtle mathematical technology. Standard local four-dimensional supergravity action:

$$\mathcal{L} = \frac{1}{2\kappa}\mathrm{R}(g) - \frac{1}{2}\mathcal{G}_{AB}\partial_\mu\varphi^A\partial^\mu\varphi^B - \gamma_{ij}(\varphi)\, F^i_{\mu\nu}F^{j\mu\nu} - \theta_{ij}(\varphi)\, F^i_{\mu\nu} *_g F^{j\,\mu\nu} + \Phi(\varphi)\,.$$

We have:

- Scalar fields $\varphi^A(x^\mu)$.
- Field strengths $F^i_{\mu\nu}$.
- Scalar-dependent couplings $\mathcal{G}_{AB}(\varphi^A)$, $\gamma_{ij}(\varphi^A)$ and $\theta_{ij}(\varphi^A)$.

All these are local objects! They make sense on a contractible open set of a four-dimensional manifold M.

The answer to the previous two fundamental questions we requires to consider the following three structures:

- A *flat Kaluza Klein space* $\pi\colon (E, h) \to (M, g)$. This is a particular type of surjective pseudo-Riemannian submersion π defined over space-time M, whose total space carries a geodesically complete Lorentzian metric h making the fibers into totally-geodesic Riemannian submanifolds. In particular, π is a fiber bundle endowed with a complete Ehresmann connection whose transport acts through isometries between the fibers.
- A *duality bundle*. This is a flat symplectic vector bundle $\mathbf{\Delta} = (\boldsymbol{\mathcal{S}}, \boldsymbol{\omega}, \mathbf{D})$ defined over the total space E of the previously introduced Kaluza-Klein space.
- A *vertical taming* J on $\mathbf{\Delta}$, which defines a complex polarization of the complexified bundle $(\boldsymbol{\mathcal{S}}_{\mathbb{C}}, \boldsymbol{\omega}_{\mathbb{C}})$, inducing a splitting $\boldsymbol{\mathcal{S}}_{\mathbb{C}} = L \oplus \bar{L}$.

The previous three structures are the basic structures required to formulate bosonic generalized supergravity: if we fix them we completely fix the bosonic sector of the theory. The previous observation suggests a way of *classifying* GESM theories, as a preliminary step to classify generalized supergravities.

Vertical tamings

Definition

The *extended horizontal transport* along a path $\gamma \in \mathcal{P}(M)$ is the unbased isomorphism of vector bundles $\mathbf{T}_\gamma : \mathcal{S}_{\gamma(0)} \to \mathcal{S}_{\gamma(1)}$ defined through:

$$\mathbf{T}_\gamma(e) \stackrel{\text{def.}}{=} U_{\gamma_e} : \mathcal{S}_e \to \mathcal{S}_{T_\gamma(e)}\,, \quad \forall\, e \in E_{\gamma(0)}\,,$$

which linearizes the Ehresmann transport $T_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ along γ.

Definition

A taming $\mathbf{J}$ of $(\boldsymbol{\mathcal{S}}, \boldsymbol{\omega}, \mathbf{D})$ is called *vertical* if it is $\mathbf{T}$-invariant, which means that it satisfies:

$$\mathbf{T}_\gamma \circ J_{\gamma(0)} = J_{\gamma(1)} \circ \mathbf{T}_\gamma\,, \qquad \forall\, \gamma \in \mathcal{P}(M)\,.$$

It is clear that $\mathbf{J}$ is vertical if and only if it satisfies:

$$\mathbf{D}_X \circ \mathbf{J} = \mathbf{J} \circ \mathbf{D}_X\,, \qquad \forall\, X \in \Gamma(E, H)\,.$$

In this case, $\mathbf{T}_\gamma$ is an isomorphism of tamed flat symplectic vector bundles:

$$\mathbf{T}_\gamma \colon (\mathcal{S}_{\gamma(0)}, D_{\gamma(0)}, \omega_{\gamma(0)}, J_{\gamma(0)}) \xrightarrow{\sim} (\mathcal{S}_{\gamma(1)}, D_{\gamma(1)}, \omega_{\gamma(1)}, J_{\gamma(1)}) \, ,$$

which covers the isometry $T_\gamma : (E_{\gamma(0)}, h_{\gamma(0)}) \to (E_{\gamma(1)}, h_{\gamma(1)})$, i.e., the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{S}_{\gamma(0)}, D_{\gamma(0)}, \omega_{\gamma(0)}, J_{\gamma(0)}) \arrow[r, "\mathbf{T}_\gamma"] \arrow[d, "\pi_{\gamma(0)}"'] & (\mathcal{S}_{\gamma(1)}, D_{\gamma(1)}, \omega_{\gamma(1)}, J_{\gamma(1)}) \arrow[d, "\pi_{\gamma(1)}"] \\
(E_{\gamma(0)}, h_{\gamma(0)}) \arrow[r, "T_\gamma"] & (E_{\gamma(1)}, h_{\gamma(1)})
\end{tikzcd}$$

Definition

Let $\pi \colon (E, h) \to (M, g)$ be a Lorentzian submersion. An *electromagnetic bundle* $\mathbf{\Xi} \stackrel{\text{def.}}{=} (\boldsymbol{\mathcal{S}}, \boldsymbol{\omega}, \mathbf{D}, \mathbf{J})$ is a duality bundle $\mathbf{\Delta}$ over E equipped with a vertical taming $\mathbf{J}$.

An electromagnetic bundle $\mathbf{\Xi} \stackrel{\text{def.}}{=} (\boldsymbol{\mathcal{S}}, \boldsymbol{\omega}, \mathbf{D}, \mathbf{J})$ uniquely specifies the global couplings of the *scalars* and the *abelian gauge fields* of the theory.

Definition

A *scalar-electromagnetic bundle* defined over (M, g) is a triple:

$$\mathcal{D} = (\pi : (E, h) \to (M, g), \bar{\Phi}, \Xi)$$

consisting of a Kaluza-Klein space $\pi : (E, h) \to (M, g)$, a vertical potential $\bar{\Phi}$ and an electromagnetic bundle Ξ defined over the total space E of π. The scalar-electromagnetic bundle $\mathcal{D}$ is called *integrable* if $\pi : (E, h) \to (M, g)$ is an integrable Kaluza-Klein space.

A choice of scalar-electromagnetic bundle uniquely defines a GESM theory, namely the bosonic sector of generalized supergravity on a four-manifold. This observation can be used to initiate a classification of GESM and Supergravity theories. For example, the character variety:

$$\mathcal{M} = \mathrm{Hom}(\pi_1(E), \mathrm{Sp}(2n, \mathbb{R}))/\mathrm{Sp}(2n, \mathbb{R}) \,,$$

classifies duality bundles modulo natural isomorphism. This shows that there exits an *uncountable infinity* of inequivalent choices of duality bundles, yielding inequivalent theories which are however locally equivalent: **this infinite amount of choices is missed in the standard local formulation of the theory**.

Definition

Let $\mathcal{D} = (\pi, \bar{\Phi}, \Xi)$ be a scalar-electromagnetic bundle with associated Lorentzian submersion $\pi\colon (E,h) \to (M,g)$ and let $s \in \Gamma(\pi)$ be a section of π. An *electromagnetic field strength* is a two-form $\boldsymbol{\mathcal{V}} \in \Omega^2(M, \boldsymbol{\mathcal{S}}^s)$ having the following properties:

1. $\boldsymbol{\mathcal{V}}$ is positively-polarized with respect to $\mathbf{J}^s$, i.e. the following relation is satisfied:

$$*_g \boldsymbol{\mathcal{V}} = -\mathbf{J}^s \boldsymbol{\mathcal{V}}\,, \quad \Leftrightarrow \quad * \boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}\,.$$

2. $\boldsymbol{\mathcal{V}}$ satisfies the *electromagnetic equation* with respect to s:

$$\mathrm{d}_{\mathbf{D}^s} \boldsymbol{\mathcal{V}} = 0 \quad .$$

Once a choice of scalar-electromagnetic bundle has been made, the matter content of a generalized Einstein-Section-Maxwell theory is given by:

- A Lorentzian metric g on M.
- A section $s \in \Gamma(\pi)$ of the flat Kaluza-Klein space $\pi\colon (E,h) \to (M,g)$.
- A positively-polarized electromagnetic field strenght $\boldsymbol{\mathcal{V}} \in \Omega^2(M, \boldsymbol{\mathcal{S}}^s)$.

This gives a global definition of the configuration space of the theory: crucial for understanding moduli spaces!

Let $\mathcal{D} = (\pi, \bar{\Phi}, \Xi)$ be a scalar-electromagnetic bundle with associated electromagnetic bundle $\Xi = (\mathcal{S}, \omega, \mathbf{D}, \mathbf{J})$ and Lorentzian submersion $\pi\colon (E, h) \to (M, g)$. Let:

$$\mathbf{D}^{\mathrm{ad}} \colon \Omega^0(E, End(\mathcal{S})) \to \Omega^1(E, End(\mathcal{S}))\,,$$

be the connection induced by $\mathbf{D}$ on the endomorphism bundle $End(\mathcal{S})$ of $\mathcal{S}$.

Definition

The *fundamental bundle form* Θ associated to $\mathcal{D}$ is the $End(\mathcal{S})$-valued one-form defined on E as follows:

$$\Theta \stackrel{\mathrm{def.}}{=} \mathbf{D}^{\mathrm{ad}}\mathbf{J} \in \Omega^0(E, V^* \otimes End(\mathcal{S}))\,.$$

Key fact: we have $\Theta^s = 0$ if and only if there always exists a local duality frame such that $\mathbf{J}^s$ is constant, namely the local couplings of scalars and vectors are constant.

Definition

The *fundamental bundle field* Ψ associated to $\mathcal{D}$ is the $End(\mathcal{S})$-valued vector field defined on E as follows:

$$\Psi \stackrel{\mathrm{def.}}{=} \left(\sharp_h \otimes \mathrm{Id}_{End(\mathcal{S})}\right) \circ D^{\mathrm{ad}}\mathbf{J} \in \Omega^0(E, V \otimes End(\mathcal{S}))\,.$$

Local Object	Global Object
Local metric g on $U \subset M$	Global metric g on M
Scalars $\phi^i \in C^\infty(U)$	Section $s\colon M \to E$
Field strengths $F^\Lambda, G_\Lambda \in \Omega^2_{cl}(U)$	Positively polarized two-form $\mathcal{V} \in \Omega^2(M, \boldsymbol{\mathcal{S}}^s)$
Local couplings $\gamma_{ij}(\varphi^A), \theta_{ij}(\varphi^A)$	Symplectic taming J on $\boldsymbol{\mathcal{S}}$

Definition

The *vertical Lagrange density* of π is the functional $e^v_\Phi \colon \Gamma(\pi) \to \mathcal{C}^\infty(M, \mathbb{R})$ defined, for every $s \in \Gamma(\pi)$, as follows:

$$S_{\rm sc}[g,h,s] = -\int_U \nu_M(g) e^v_\Phi(g,h,s)\,, \qquad e^v_\Phi(g,h,s) \stackrel{\rm def.}{=} \frac{1}{2}{\rm Tr}_g s^*(h_V) + \bar{\Phi}^s \quad .$$

Here $\bar{\Phi}^s = \bar{\Phi} \circ s \in \mathcal{C}^\infty(M, \mathbb{R})$.

Let $s \in \Gamma(\pi)$ be a section. The differential ${\rm d}s \colon TM \to TE$ of s is an unbased morphism of vector bundles, which is equivalent to a section ${\rm d}s \in \Omega^1(M, TE^s)$ which for simplicity we denote by the same symbol. We define $d^v s \stackrel{\rm def.}{=} P_V \circ {\rm d}s \in \Omega^1(M, V^s)$. The Levi-Civita connection on (M, g) together with the s-pull-back of the connection ∇^v on V induce a connection on $T^*M \otimes V^s$, which we denote again by ∇^v.

Definition

The *vertical tension field* of $s \in \Gamma(\pi)$ is defined through:

$$\tau^v(g,h,s) \stackrel{\rm def.}{=} {\rm Tr}_g \nabla^v {\rm d}^v s \in \Gamma(M, V^s) \quad .$$

Definition

Let $\mathcal{D} = (\pi, \bar{\Phi}, \Xi)$ be a scalar-electromagnetic bundle with associated Lorentzian submersion $\pi\colon (E, h) \to (M, g)$. A GESM-theory associated to $\mathcal{D}$ is defined by the following set of partial differential equations on (M, g):

- The Einstein equations:

$$\mathcal{E}_E(g, s, \mathcal{V}) \stackrel{\text{def.}}{=} \mathrm{G}(g) - \kappa\, \mathrm{T}(g, s, \mathcal{V}) = 0\,, \tag{1}$$

  where $\mathrm{T}(g, s, \mathcal{V}) \in \Omega^0(S^2 T^* M)$ is the energy-momentum tensor of the theory, which is given by:

$$\mathrm{T}(g, s, \mathcal{V}) = g\, e_0^v(g, , h, s) - h^s + \frac{g}{2}\bar{\Phi}^{\varphi} + 2\, \mathcal{V} \oslash \mathcal{V}\,.$$

- The scalar equations:

$$\mathcal{E}_S(g, s, \mathcal{V}) \stackrel{\text{def.}}{=} \tau^v(g, h, s) + (\mathrm{grad}_h \bar{\Phi})^s - \frac{1}{2}(*\mathcal{V}, \Psi^s \mathcal{V}) = 0\,. \tag{2}$$

- The electromagnetic (or Maxwell) equations:

$$\mathcal{E}_K(g, s, \mathcal{V}) \stackrel{\text{def.}}{=} \mathrm{d}_{\mathbf{D}^s} \mathcal{V} = 0\,. \tag{3}$$

with unknowns given by triples $(g, s, \mathcal{V}) \in \mathrm{Conf}_{\mathcal{D}}(M)$.

Theorem (C. Lazaroiu + CSS)

Let $\boldsymbol{\mathcal{D}} = (\pi, \bar{\Phi}, \Xi)$ be a scalar-electromagnetic bundle of type $\mathcal{D}$ with integrable horizontal distribution $H \subset TE$ and let $(U_\alpha, \mathbf{q}_\alpha)_{\alpha\in I}$ be a special trivializing atlas for $\boldsymbol{\mathcal{D}}$. Let (U_α, x) and (W_α, u, y) be a π-adapted system of coordinates. Then, associated to the previous data, there exists a canonical bijection of sets:

$$C_{U_\alpha} : \mathbf{Conf}_{\boldsymbol{\mathcal{D}}}(U_\alpha) \xrightarrow{\simeq} \mathrm{Conf}_{\mathcal{D}}(U_\alpha)\,,$$

for every $U_\alpha \in (U_\alpha)_{\alpha\in I}$. Furthermore, C_{U_α} induces a bijection onto $\mathrm{Sol}_{\mathcal{D}}(U_\alpha)$ when restricted to $\mathbf{Sol}_{\boldsymbol{\mathcal{D}}}(U_\alpha)$:

$$C_{U_\alpha} : \mathbf{Sol}_{\boldsymbol{\mathcal{D}}}(U_\alpha) \xrightarrow{\simeq} \mathrm{Sol}_{\mathcal{D}}(U_\alpha)\,,$$

which for simplicity we denote by the same symbol. In particular, if $(g, s, \boldsymbol{\mathcal{V}}) \in \mathbf{Sol}_{\boldsymbol{\mathcal{D}}}(M)$, then:

$$(g, s, \boldsymbol{\mathcal{V}})|_{U_\alpha} \in \mathrm{Sol}_{\mathcal{D}}(U_\alpha)\,.$$

Here $\mathbf{Conf}_{\boldsymbol{\mathcal{D}}}$ and $\mathbf{Sol}_{\boldsymbol{\mathcal{D}}}$ respectively denote the configurations and solutions sheaves of a GESM theory and $\mathrm{Conf}_{\boldsymbol{\mathcal{D}}}$ and $\mathrm{Sol}_{\boldsymbol{\mathcal{D}}}$ respectively denote the configurations and solutions sheaves of a local ESM theory.

- Based on a fiber sub-bundle Λ of full lattices inside $\mathcal{S}$, which is preserved by the parallel transport of the flat connection D and which has the property that ω is integer-valued when restricted to Λ. Pairs $\boldsymbol{\Delta} = (\Delta, \Lambda)$ are called *integral duality structures*.

- Integral duality structures correspond to local systems T_Δ valued in the groupoid of Symp_0 of *integral symplectic spaces*.

- A taming J of $(\mathcal{S}, \omega)$ defines an *integral electromagnetic structure* $\boldsymbol{\Xi} = (\Xi, \Lambda)$, where $\Xi = (\mathcal{S}, D, J, \omega)$ is the electromagnetic structure defined by J and Δ. The fibers of an integral electromagnetic structure are *integral tamed symplectic spaces*, being objects of the category $\mathrm{TamedSymp}_0$ of *integral tamed symplectic spaces*.

- We can encode the data of an integral electromagnetic structure using a (smooth) bundle $\mathcal{X}_s(\boldsymbol{\Xi})$ of polarized Abelian varieties, endowed with a flat Ehresmann connection whose transport preserves the symplectic structure of the torus fibers but need not preserve their complex structure.

The Dirac quantization condition requires that $[\mathcal{V}] \in H^2_{d_{D^\varphi}}(M, \mathcal{S}^\varphi)$ be *integral* in the sense that it belongs to the image through the universal coefficient map of the second cohomology of M with coefficients in the Symp_0-valued local system T_{Δ^φ}. This condition shows that a geometric model of semiclassical Abelian field configurations can be constructed using a certain version of twisted differential cohomology.

Let $H^\bullet(M, \mathbf{\Delta}^\varphi)$ be the twisted singular cohomology of M with coefficients in the local system $T_{\mathbf{\Delta}^\varphi}$. Since $\mathcal{S}^\varphi = \Lambda^\varphi \otimes_{\mathbb{Z}} \mathbb{R}$, the coefficient sequence gives a map $j_* : H^\bullet(M, \mathbf{\Delta}^\varphi) \to H^\bullet_{d_{D^\varphi}}(M, \mathcal{S}^\varphi)$, whose image is a lattice $H^\bullet_{\Lambda^\varphi}(M, \Delta^\varphi) \subset H^\bullet(M, \Delta^\varphi)$:

$$H^\bullet_{\Lambda^\varphi}(M, \Delta^\varphi) \stackrel{\text{def.}}{=} H^\bullet(M, \mathbf{\Delta}^\varphi)/\mathrm{Tors}(H^\bullet(M, \mathbf{\Delta}^\varphi))$$

Definition

An electromagnetic field $\mathcal{V} \in \Omega^2(M, \mathcal{S}^\varphi)$ is called Λ^φ*-integral* if its D^φ-twisted cohomology class $[\mathcal{V}] \in H^2_{d_{D^\varphi}}(M, \mathcal{S}^\varphi)$ belongs to $H^2_{\Lambda^\varphi}(M, \Delta^\varphi)$:

$$[\mathcal{V}] \in H^2_{\Lambda^\varphi}(M, \Delta^\varphi) = j_*(H^2(M, \mathbf{\Delta}^\varphi)) .$$

Since $H^2_{\Lambda^\varphi}(M, \mathcal{S}^\varphi)$ does not capture $\mathrm{Tor}\, H^2(M, \Lambda^\varphi)$, the integrality condition is weaker than the expected condition. In this case, the natural model for semiclassical fields is provided by a twisted version of differential cohomology.

Further directions

1. Characterize the U-duality group of a general GESM theory and compute it in explicit important cases: spectral sequence expected.
2. Implement Dirac quantization on a general GESM theory, developing the appropriate model in differential cohomology.
3. Explore more general submersions as possible models of locally non-geometric U-folds.
4. Supersymmetrize GESM theories $\Rightarrow$ Generalized supergravities.
5. Study the higher-dimensional origin of GESM theories by using the appropriate notion of reduction.
6. Explore a geometric model for Freudenthal duality in terms of the taming J.
7. Gauged GESM theories.
8. Characterize all the supersymmetric solutions of generalized supergravity.
9. GESM on globally hyperbolic Lorentzian four-manifolds and associated flow equations.
10. Moduli spaces and applications.

Thanks!

Definition

The *twisted exterior pairing* $(\ ,\ ) := (\ ,\ )_{g,\boldsymbol{Q}^s}$ is the unique pseudo-Euclidean scalar product on the twisted exterior bundle $\wedge_M(\boldsymbol{\mathcal{S}}^s)$ which satisfies:

$$(\rho_1 \otimes \xi_1, \rho_2 \otimes \xi_2)_{g,\boldsymbol{Q}^s} = (\rho_1, \rho_2)_g \boldsymbol{Q}^s(\xi_1, \xi_2)\,,$$

for any $\rho_1, \rho_2 \in \Omega(M)$ and any $\xi_1, \xi_2 \in \Omega^0(M, \boldsymbol{\mathcal{S}}^s)$. Here $\boldsymbol{Q}(\xi_1, \xi_2) = \boldsymbol{\omega}(\mathbf{J}\xi_1, \xi_2)$ and the superscript denotes pull-back by s.

For any vector bundle W, we trivially extend the twisted exterior pairing to a W-valued pairing, which for simplicity we denote by the same symbol, between the bundles $W \otimes (\wedge_M(\mathcal{S}^\varphi))$ and $\wedge_M(\mathcal{S}^\varphi)$. Thus:

$$(e \otimes \eta_1, \eta_2)_{g,\boldsymbol{Q}^s} \stackrel{\text{def.}}{=} e \otimes (\eta_1, \eta_2)_{g,\boldsymbol{Q}^s}\,, \quad \forall\, e \in \Omega^0(M, W)\,, \quad \forall\, \eta_1, \eta_2 \in \wedge_M(\boldsymbol{\mathcal{S}}^s)\,.$$

The *inner g-contraction of two-tensors* is the bundle morphism $\oslash_g : (\otimes^2 T^* M)^{\otimes 2} \to \otimes^2 T^* M$ uniquely determined by the condition:

$$(\alpha_1 \otimes \alpha_2) \oslash_g (\alpha_3 \otimes \alpha_4) = (\alpha_2, \alpha_3)_g \alpha_1 \otimes \alpha_4 \,, \quad \forall\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Omega^1(M) \,.$$

We define the *inner g-contraction of two-forms* to be the restriction of $\oslash_g$ to $\wedge^2 T^* M \otimes \wedge^2 T^* M \overset{\oslash_g}{\to} \otimes^2 T^* M$.

Definition

We define the *twisted inner contraction* of $\boldsymbol{\mathcal{S}}^s$-valued two-forms to be the unique morphism of vector bundles $\oslash\colon\ \wedge^2_M (\boldsymbol{\mathcal{S}}^s) \times_M \wedge^2_M(\boldsymbol{\mathcal{S}}^s) \to \otimes^2 (T^* M)$ which satisfies:

$$(\rho_1 \otimes \xi_1) \oslash (\rho_2 \otimes \xi_2) = \boldsymbol{Q}^s(\xi_1, \xi_2) \rho_1 \oslash_g \rho_2 \,,$$

for all $\rho_1, \rho_2 \in \Omega^2(M)$ and all $\xi_2, \xi_2 \in \Omega^0(M, \boldsymbol{\mathcal{S}}^s)$.

- The twisted inner contraction is necessary in order to globally write the equations of motion of supergravity!

Let $(\pi : (E, h) \to (M, g), \bar{\Phi})$ be an *integrable* bundle of scalar data of type $(\mathcal{M}, \mathcal{G}, \Phi)$. We consider a special trivializing atlas of π defined by the geodesically convex open sets $(U_\alpha)_{\alpha\in I}$ (which cover M). Since π is integrable, the appropriate trivializing maps give isometries $q_\alpha : (E_\alpha, h_\alpha) \xrightarrow{\sim} (U_\alpha \times \mathcal{M}, g_\alpha \times \mathcal{G})$. For any pair of indices $\alpha, \beta \in I$ such that $U_{\alpha\beta} \stackrel{\text{def.}}{=} U_\alpha \cap U_\beta$ is non-empty, the composition:

$$q_{\alpha\beta} \stackrel{\text{def.}}{=} q_\beta \circ q_\alpha^{-1} : U_{\alpha\beta} \times \mathcal{M} \to U_{\alpha\beta} \times \mathcal{M} \,,$$

has the form $q_{\alpha\beta}(m, p) = (m, \mathbf{g}_{\alpha\beta}(p))$, where:

$$\mathbf{g}_{\alpha\beta} \in \mathrm{Iso}(\mathcal{M}, \mathcal{G}, \Phi) \,.$$

Setting $\mathbf{g}_{\alpha\beta} = \mathrm{id}_{\mathcal{M}}$ for $U_{\alpha\beta} = \emptyset$, the collection $(\mathbf{g}_{\alpha\beta})_{\alpha,\beta\in I}$ satisfies the cocycle condition:

$$\mathbf{g}_{\beta\delta}\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\delta} \,, \quad \forall\, \alpha, \beta, \delta \in I \,. \tag{4}$$

For any section $s \in \Gamma(\pi)$, the restriction $s_\alpha \stackrel{\text{def.}}{=} s|_{U_\alpha}$ corresponds through q_α to the graph $\mathrm{graph}(\varphi_\alpha) \in \Gamma(\pi^0_\alpha)$ of a uniquely-defined smooth map $\varphi_\alpha \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})$:

$$s_\alpha = q_\alpha^{-1} \circ \mathrm{graph}(\varphi_\alpha) \;\; \text{i.e.} \;\; q_\alpha(s_\alpha(m)) = (m, \varphi_\alpha(m)) \,, \quad \forall m \in U_\alpha \,. \tag{5}$$

Composing the first relation from the left with $p^0_\alpha \colon E^0_\alpha \simeq U_\alpha \times \mathcal{M} \to \mathcal{M}$ gives:

$$\varphi^\alpha = \hat{q}_\alpha \circ s_\alpha \quad .$$

Which in turn implies:

$$\varphi^\beta(m) = \mathbf{g}_{\alpha\beta}\varphi^\alpha(m) \quad \forall m \in U_{\alpha\beta} \quad , \tag{6}$$

where juxtaposition in the right hand side denotes the tautological action of the group $\mathrm{Isom}(\mathcal{M}, \mathcal{G}, \Phi)$ on $\mathcal{M}$. Conversely, any family of smooth maps $\{\varphi^\alpha \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})\}_{\alpha \in I}$ satisfying (6) defines a smooth section $s \in \Gamma(\pi)$ whose restrictions to U_α are given by (5). The equation of motion for s is equivalent with the condition that each φ^α satisfies the equation of motion of the ordinary sigma model defined by the scalar data $(\mathcal{M}, \mathcal{G}, \Phi)$ on the space-time (U_α, g_α):

$$\tau^v(h, s) = -(\mathrm{grad}\bar{\Phi})^s \Leftrightarrow \tau(g_\alpha, \varphi^\alpha) = -(\mathrm{grad}\Phi)^{\varphi^\alpha} \quad \forall \alpha \in I \quad . \tag{7}$$

Thus global solutions s of the equations of motion are *glued* from local solutions $\varphi^\alpha \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})$ of the equations of motion of the ordinary sigma model using the $\mathrm{Iso}(\mathcal{M}, \mathcal{G}, \Phi)$-valued constant transition functions $\mathbf{g}_{\alpha\beta}$ which satisfy the cocycle condition (4).

Let $\boldsymbol{\mathcal{D}} = (\pi, \bar{\Phi}, \Xi)$ be an *integrable* scalar-electromagnetic bundle and let $(U_\alpha, \mathbf{q}_\alpha)_{\alpha\in I}$ be a special trivializing atlas. Let $s \in \Gamma(\pi)$ and and $\mathcal{V} \in \Omega^2(M, \boldsymbol{\mathcal{S}}^s)$. Let $s_\alpha \stackrel{\text{def.}}{=} s|_{U_\alpha}$ and $\mathcal{V}_\alpha \stackrel{\text{def.}}{=} \mathcal{V}|_{U_\alpha} \in \Omega^2(U_\alpha, \boldsymbol{\mathcal{S}}^s)$. The diffeomorphisms $q_\alpha : E_\alpha \to E^{\mathbf{0}}_\alpha = U_\alpha \times \mathcal{M}$ and their linearizations $\mathbf{q}_\alpha : \boldsymbol{\mathcal{S}}_\alpha \to \boldsymbol{\mathcal{S}}^{\mathbf{0}}_\alpha \stackrel{\text{def.}}{=} \mathcal{S}^{p^{\mathbf{0}}_\alpha}$ identify s_α with maps $\varphi_\alpha \stackrel{\text{def.}}{=} p^{\mathbf{0}}_\alpha \circ q_\alpha(s_\alpha) \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})$. The isomorphism of electromagnetic structures $\mathbf{q}_\alpha : (\boldsymbol{\mathcal{S}}_\alpha, \boldsymbol{\omega}_\alpha, \mathbf{D}_\alpha, \mathbf{J}_\alpha) \xrightarrow{\sim} (\boldsymbol{\mathcal{S}}^{\mathbf{0}}_\alpha, \boldsymbol{\omega}^{\mathbf{0}}_\alpha, \mathbf{D}^{\mathbf{0}}_\alpha, \mathbf{J}^{0}_\alpha)$ pulls back to an isomorphism:

$$\mathbf{q}^s_\alpha : (\boldsymbol{\mathcal{S}}^s_\alpha, \boldsymbol{\omega}^s_\alpha, \mathbf{D}^s_\alpha, \mathbf{J}^s_\alpha) \xrightarrow{\sim} (\mathcal{S}^{\varphi^\alpha}, \omega^{\varphi^\alpha}, D^{\varphi^\alpha}, J^{\varphi^\alpha})\,,$$

This isomorphism identifies $\boldsymbol{\mathcal{V}}_\alpha$ with a $\mathcal{S}^{\varphi^\alpha}$-valued two-form defined on U_α through:

$$\mathcal{V}^\alpha \stackrel{\text{def.}}{=} \mathbf{q}^s_\alpha \circ \boldsymbol{\mathcal{V}}_\alpha \in \Omega^2(U_\alpha, \mathcal{S}^{\varphi^\alpha})\,, \tag{8}$$

and we have:

$$\begin{aligned} \boldsymbol{\mathcal{V}}_\alpha \in \Omega^{2+,s}_{g_\alpha, \boldsymbol{\mathcal{S}}^s_\alpha, \mathbf{J}^s_\alpha}(U_\alpha) \quad &\text{iff} \quad \mathcal{V}^\alpha \in \Omega^{2+,\varphi^\alpha}_{g_\alpha, \mathcal{S}^{\varphi^\alpha}, J^{\varphi^\alpha}}(U_\alpha) \\ \mathrm{d}_{\boldsymbol{\mathcal{D}}^s} \boldsymbol{\mathcal{V}}_\alpha = 0 \quad &\text{iff} \quad \mathrm{d}_{D^{\varphi^\alpha}} \mathcal{V}^\alpha = 0\,. \end{aligned}$$

$$(s, \boldsymbol{\mathcal{V}}) \in \mathrm{Sol}^g_{\boldsymbol{\mathcal{D}}}(M) \;\; \text{iff} \;\; (\varphi^\alpha, \mathcal{V}^\alpha) \in \mathrm{Sol}^{g_\alpha}_{\mathcal{D}}(U_\alpha) \;\; \forall \alpha \in I \;\; .$$

Relations (8) imply the gluing conditions:

$$\mathcal{V}^{\beta}|_{U_{\alpha\beta}} = \mathbf{f}^{s}_{\alpha\beta}\mathcal{V}^{\alpha}|_{U_{\alpha\beta}} \quad , \tag{9}$$

which accompany the gluing conditions (6) for φ^{α}.
Conversely, any pair of families $(\varphi_{\alpha})_{\alpha\in I}$ and $(\mathcal{V}^{\alpha})_{\alpha\in I}$ of solutions of the equations of motion of the ordinary scalar sigma model with Abelian gauge fields of type $\mathcal{D} = (\mathcal{M}, \mathcal{G}, \Phi, \mathcal{S}, \omega, D, J)$ defined on the open sets $U_{\alpha} \subset M$ of a special open cover for (M, g) which satisfy conditions (6) and (9) corresponds to a global solution $(s, \mathcal{V})$ of the equations of motion of the section sigma model coupled to Abelian gauge fields defined by the scalar-electromagnetic bundle $\boldsymbol{\mathcal{D}}$. Hence such global solutions are obtained by gluing local solutions of the ordinary sigma model coupled to Abelian gauge fields using scalar-electromagnetic symmetries of the latter.

Definition

Let $\boldsymbol{\mathcal{D}}$ be a scalar-electromagnetic structure. A classical locally-geometric U-fold of type $\boldsymbol{\mathcal{D}}$ is a global solution $(g, s, \boldsymbol{\mathcal{V}}) \in \mathrm{Sol}_{\boldsymbol{\mathcal{D}}}(M)$ of the equations of motion a GESM theory associated to an scalar-electromagnetic bundle $\boldsymbol{\mathcal{D}}$ of type $\mathcal{D}$ with integrable associated Lorentzian submersion.

Consider an integral symplectic space $(\mathcal{S}_0, \omega_0, \Lambda_0)$ defined on M. One can define a version of differential cohomology valued in such objects; an explicit construction can be given for example using Cheeger-Simons characters valued in the (affine) symplectic torus $\mathcal{S}_0/\Lambda_0$ defined by $(\mathcal{S}_0, \omega_0, \Lambda_0)$. This can be promoted to a twisted version $\check{H}^k(N, T)$, whereby the coefficient object is replaced by a local system $T : \Pi_1(N) \to \mathrm{Symp}_0^\times$. Given an integral electromagnetic structure Ξ whose underlying integral duality structure $\boldsymbol{\Delta}$ corresponds to the local system T, the correct model for the set of semiclassical Abelian gauge fields is provided by a certain subset of $\check{H}^2(N, T)$; the curvature $\mathcal{V} = \mathrm{curv}(\alpha)$ of any element α of this subset is a polarized closed 2-form which satisfies the integrality condition. When (M, g) is globally hyperbolic, the initial value problem for the twisted electromagnetic field is well-posed (because the twisted d'Alembert operator is normally hyperbolic) and restriction to a Cauchy hypersurface allows one to describe explicitly the space of semiclassical fields. In that case, one can to quantize the electromagnetic theory in a manner which reproduces this model for the space of semiclassical fields. It turns out that elements $\alpha \in \check{H}^2(N, T)$ classify affine symplectic T^{2n}-bundles[1] with connection. This allows one to represent $\mathcal{V}$ as the curvature of a connection on such a bundle, which gives the geometric interpretation of semiclassical Abelian gauge fields.

[1] Non-principal fiber bundles with fiber a symplectic $2n$-torus and whose structure group reduces to the affine symplectic group of such a torus.

Let $\mathcal{D} \stackrel{\text{def.}}{=} (\mathcal{M}, \mathcal{G}, \Phi, \mathcal{S}, D, \omega, J)$.

Definition

The *scalar-electromagnetic symmetry group* of $\mathcal{D}$ is the subgroup $\mathrm{Aut}(\mathcal{D})$ of $\mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, D, J, \omega)$ defined through:

$$\mathrm{Aut}(\mathcal{D}) \stackrel{\text{def.}}{=} \{f \in \mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, D, J, \omega) \mid f_0 \in \mathrm{Aut}(\mathcal{M}, \mathcal{G}, \Phi)\}\,.$$

An element of this group is called a *scalar-electromagnetic symmetry*.

Theorem

For all $f \in \mathrm{Aut}(\mathcal{D})$, we have:

$$f \diamond \mathrm{Sol}^{g}_{\mathcal{D}}(M) = \mathrm{Sol}^{g}_{\mathcal{D}}(M)\quad .$$

Thus $\mathrm{Aut}(\mathcal{D})$ consists of symmetries of the equations of motion of a GESM theory.

It is natural then to define the U-duality group of a GESM theory as its scalar-electromagnetic symmetry group!

Theorem

Let $\Sigma \stackrel{\text{def.}}{=} (\mathcal{M}, \mathcal{G}, \Phi)$, $\Delta \stackrel{\text{def.}}{=} (\mathcal{S}, D, \omega)$, $\Xi \stackrel{\text{def.}}{=} (\mathcal{S}, D, J, \omega)$ *and* $\mathcal{D}_0 \stackrel{\text{def.}}{=} (\mathcal{M}, \mathcal{G}, \Phi, \mathcal{S}, D, \omega)$. *We have short exact sequences:*

$$1 \to \mathrm{Aut}(\Delta) \hookrightarrow \mathrm{Aut}(\mathcal{D}_0) \longrightarrow \mathrm{Aut}^{\Delta}(\Sigma) \to 1$$
$$1 \to \mathrm{Aut}(\Xi) \hookrightarrow \mathrm{Aut}(\mathcal{D}) \longrightarrow \mathrm{Aut}^{\Xi}(\Sigma) \to 1 \quad ,$$

where $\mathrm{Aut}(\Delta)$ *and* $\mathrm{Aut}(\Xi)$ *are the groups of based symmetries of* Δ *and* Ξ. *The groups appearing in the right hand side consist of those automorphisms of the scalar structure* Σ *which respectively admit lifts to scalar-electromagnetic dualities of* $\mathcal{D}_0 = (\Sigma, \Delta)$ *and scalar-electromagnetic symmetries of* $\mathcal{D} = (\Sigma, \Xi)$. *Fixing a point* $p \in \mathcal{M}$, *we can identify* $\mathrm{Aut}(\Delta)$ *with the commutant of* Hol_D^p *inside the group* $\mathrm{Aut}(\mathcal{S}_p, \omega_p) \simeq \mathrm{Sp}(2n, \mathbb{R})$. *In particular, the exact sequences above show that* $\mathrm{Aut}(\mathcal{D}_0)$ *and* $\mathrm{Aut}(\mathcal{D})$ *are Lie groups.*

Definition

The *fundamental matrix* of (V, J) with respect to $\mathcal{E}$ and $\mathcal{W}$ is the matrix $\Pi := \Pi^{\mathcal{W}}_{\mathcal{E}} \in \mathrm{Mat}(n, 2n, \mathbb{C})$ defined through:

$$e_\alpha = \sum_{k=1}^{n} \Pi_{k,\alpha} w_k = \sum_{k=1}^{n} [(\mathrm{Re}\Pi_{k,\alpha}) w_k + (\mathrm{Im}\Pi_{k,\alpha}) J w_k] \quad (\alpha = 1 \dots 2n) \quad . \tag{10}$$

Siegel upper half space $\mathbb{SH}_n$:

$$\mathbb{SH}_n \overset{\text{def.}}{=} \{\tau \in \mathrm{Mat}_s(n, \mathbb{C}) | \mathrm{Im}\tau \text{ is strictly positive definite}\} \quad ,$$

When endowed with the natural complex structure induced from the affine space $\mathrm{Mat}_s(n, \mathbb{C})$, the space $\mathbb{SH}_n$ is a complex manifold of complex dimension $\frac{n(n+1)}{2}$ which is biholomorphic with the simply-connected bounded complex domain:

$$\{Z \in \mathrm{Mat}_s(n, \mathbb{C}) | I - \bar{Z} Z^T > 0\}$$

A symplectic basis of (V, ω) is a basis of the form $\mathcal{E} = (e_1 \dots e_n, f_1 \dots f_n)$, whose elements satisfy the conditions:

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad , \quad \omega(e_i, f_j) = -\omega(f_i, e_j) = \delta_{ij}$$

Theorem

Let $\mathcal{E} = (e_1 \dots e_n, f_1 \dots f_n)$ be a symplectic basis of (V, ω). Then the vectors $\mathcal{E}_2 \stackrel{\text{def.}}{=} (f_1, \dots, f_n)$ form a basis of the complex vector space (V, J) over $\mathbb{C}$ and the fundamental matrix of (V, J) with respect to $\mathcal{E}$ and $\mathcal{E}_2$ has the form:

$$\Pi_{\mathcal{E}}^{\mathcal{E}_2} = [\boldsymbol{\tau}^{\mathcal{E}}, I_n]^T \,,$$

where $\boldsymbol{\tau}^{\mathcal{E}} \in \mathrm{Mat}(n, n, \mathbb{C})$ is a complex-valued square matrix of size n. Moreover, J is a taming of (V, ω) iff $\boldsymbol{\tau}_{\mathcal{E}}$ belongs to $\mathbb{SH}_n$. In this case, the matrix of the Hermitian form h with respect to the basis $\mathcal{E}_2$ equals $(\mathrm{Im}\boldsymbol{\tau}^{\mathcal{E}})^{-1}$.

Theorem

Let $\boldsymbol{\tau} = \boldsymbol{\tau}_R + i\boldsymbol{\tau}_I \in \mathbb{SH}_n$ be a point of the Siegel upper half space, where $\boldsymbol{\tau}_R, \boldsymbol{\tau}_I \in \mathrm{Mat}_s(n, n, \mathbb{R})$ are the real and imaginary parts of $\boldsymbol{\tau}$:

$$\boldsymbol{\tau}_R \stackrel{\text{def.}}{=} \mathrm{Re}\boldsymbol{\tau} \quad , \quad \boldsymbol{\tau}_I \stackrel{\text{def.}}{=} \mathrm{Im}\boldsymbol{\tau} \quad .$$

Then the matrix of the taming $J(\boldsymbol{\tau}) \stackrel{\text{def.}}{=} \boldsymbol{\tau}_{\mathcal{E}}^{-1}(\boldsymbol{\tau})$ in the symplectic basis $\mathcal{E}$ is given by:

$$\hat{J}(\boldsymbol{\tau}) = \left[\begin{array}{cc} \boldsymbol{\tau}_I^{-1}\boldsymbol{\tau}_R & \boldsymbol{\tau}_I^{-1} \\ -\boldsymbol{\tau}_I - \boldsymbol{\tau}_R\boldsymbol{\tau}_I^{-1}\boldsymbol{\tau}_R & -\boldsymbol{\tau}_R\boldsymbol{\tau}_I^{-1} \end{array} \right] \quad .$$