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DESY

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Left-invariant Einstein metrics on $S^3 \times S^3$

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Outline

1. Introduction
   - Motivation
   - Some Definitions
   - Previous Results
2. Methods for finding LI Einstein metrics on $S^3 \times S^3 =: G$
   - Reformulation as a Variational Problem
   - Parameter Space
   - System of Polynomial Equations
3. LI Einstein metrics invariant under finite subgroup $\Gamma \subset \mathrm{Ad}(G)$
   - Preliminaries & General Results
   - Specific Results for $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
   - (Partial) Results for $\Gamma \cong \mathbb{Z}_2$
4. Conclusions

Motivation:

Mathematics

- Open problem: **full classification** of homogeneous compact Einstein manifolds in $d = 6$
- Remaining **open case**: LI Einstein metrics on $S^3 \times S^3$

Physics: string theory/supergravity

- Cp. Einstein manifolds play a role in **AdS/CFT corresp.** (string-/M-theory on $\mathrm{AdS}_d \times M$ $\longleftrightarrow$ CFT on $(d-1)$-dim boundary of AdS_d)
- **Flux compactifications** of string theory from 10 to 4 dim:
  - unbroken SUSY $\Longrightarrow$ internal 6d mfd. w/ $\mathrm{SU}(3)$-structure
  - Particular interest: $\mathrm{SU}(3)$-structure nearly Kähler or half-flat
  - (Strict) nearly Kähler $\Longrightarrow$ 6d mfd. is Einstein

Recall:

Definition

A (pseudo-)Riemannian mfd. (M, g) is called **Einstein manifold** if its Ricci tensor Ric_g satisfies

$$Ric_g = \lambda g \ ,$$

for some constant $\lambda \in \mathbb{R}$ called **Einstein constant**.

(Here: only consider the Riemannian case)

Taking the trace yields:

$$S = n\lambda \ ,$$

where S denotes the **scalar curvature** of g and $n := \dim M$.

Later, we need concept of a **symmetry of the metric**.

- For LI Einstein metrics on $S^3 \times S^3 =: G$ [D'Atri, Ziller (1979)]:

  $$L_G \subset \mathrm{Isom}_0(G, g) \subset L_G \cdot R_G \cong (G \times G)/\{(z, z) \mid z \in Z(G)\}$$

  (up to changing metric by an isometric LI metric). Notation:
  - $\mathrm{Isom}_0(G, g)$: conn. isometry grp. of some LI metric g on G
  - L_G (R_G): group of left (right) translations
  - $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$: center of G
- RHS contains group of **inner automorphisms**:

  $$\mathrm{Inn}(G) = C_G := \{C_a \mid a \in G\} \subset L_G \cdot R_G \ ,$$

  where $C_a : G \to G\ ,\ \ x \mapsto axa^{-1}$ (conjugation by a)
- Hence, **isotropy grp.** of neutral el. $e \in G$ in $\mathrm{Isom}_0(G, g)$:

  $$\mathrm{Isom}_0(G, g) \cap C_G =: K_0$$

- K_0 is the **maximal connected subgroup** of the Lie group

  $$\mathrm{Isom}(G, g) \cap C_G =: K$$

[Nikonorov, Rodionov (1999, 2003)]

- **Partial classification**:

Theorem 1 (Nikonorov, Rodionov (1999, 2003))

A simply connected 6*d* homogeneous cp. Einstein mfd. is either

1. a symmetric space, or
2. isometric, up to multiplication of the metric by a constant, to one of the following manifolds:
   a. $\mathbb{CP}^3 = \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\times\mathrm{U}(1)}$ with squashed metric
   b. the Wallach space $\mathrm{SU}(3)/T_{\mathrm{max}}$ with std or Kähler metric
   c. the Lie group $\mathrm{SU}(2)\times\mathrm{SU}(2) = S^3 \times S^3$ with some left-invariant Einstein metric

- Classification of item (2c) is still **open**.
- Progress can be achieved by assuming **additional symmetries** of the metric...

[Nikonorov, Rodionov (2003)]

- **Classification** was achieved for the case that $K = \mathrm{Isom}(G, g) \cap C_G$ contains a $U(1)$ **subgroup**:

Theorem 2 (Nikonorov, Rodionov (2003))

Let g be a LI Einstein metric on $G := S^3 \times S^3$. If K contains a $U(1)$ subgroup, then (G, g) is homothetic to (G, g_{can}) or (G, g_{NK}).

Here: g_{can} = standard metric, g_{NK} = nearly Kähler metric

- These are the **only known** Einstein metrics on $S^3 \times S^3$ (up to isometry and scale)
- These metrics are **rigid** [Kröncke (2015); Moroianu, Semmelmann (2007)]

- Theorem 2 covers the case $\dim K \geq 1$
- Remaining case: $\dim K = 0$, i.e. K is a **finite group**
- Equivalently: require $\mathrm{Isom}_0(G, g) = G$
- i.e. the group of orientation preserving isometries is given by

$$\mathrm{Isom}^+(G, g) = K \ltimes G,$$

where K is a **finite group** of inner automorphisms of G
- $\rightarrow$ **goal of rest of talk** to analyze this case

Section 2

Methods for finding LI Einstein metrics on $S^3 \times S^3 =: G$

[Jensen (1971); Wang, Ziller (1986); Besse (1987); ...]

Finding Einstein metrics reformulated as **variational problem**:

Theorem 3

A Riemannian metric g on a cp. orientable manifold M is **Einstein** iff it is a **critical point** of the Einstein-Hilbert functional

$$S_{\mathrm{EH}}[g] = \int_M S \, \mathrm{vol}_g \ ,$$

subject to the **volume constraint** $\mathcal{V} := \int_M \mathrm{vol}_g = \mathcal{V}_0$, where $\mathcal{V}_0$ is a positive constant.

Here, vol_g is the metric volume form on (M, g).

- Volume constr. incorp. by means of **Lagrange multiplier**
- Instead of $S_{\mathrm{EH}}[g]$, we consider crit. pts. of

$$\tilde{S}_{\mathrm{EH}}[g,\nu] = S_{\mathrm{EH}}[g] - \nu(\mathcal{V} - \mathcal{V}_0) \ ,$$

  where ν is a **Lagrange multiplier**.
- **Simplifications** occur when $(M, g) = (G, g)$ is a **cp. Lie group** G with LI Riemannian metric g
  - Then $S = \mathrm{const.}$ and hence, $S_{\mathrm{EH}}[g] = S\,\mathcal{V}$
  - Crit. pts. of $\tilde{S}_{\mathrm{EH}}[g,\nu]$ (i.e. Einstein metrics) satisfy

$$\mathrm{grad}_g\, S = \frac{2\nu}{2-n}\, \mathrm{grad}_g\, \mathcal{V} \quad \text{and} \quad \mathcal{V} = \mathcal{V}_0$$

  Here, grad_g = variation w.r.t. metric g. Assume $n > 2$.

- LI Riemannian metric g on $G \longleftrightarrow$
  scalar product on Lie algebra $\mathfrak{g}$ of G (also denoted by g)
- Here, $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ w/ scal. prod. $Q(\cdot,\cdot) = -1/2B(\cdot,\cdot)$
  ($B(X, Y) = \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y)) =$ **Killing form** of $\mathfrak{g}$)
- **Any other scalar product** g: $g(\cdot,\cdot) = Q(L\cdot,\cdot)$
  ($L = Q$-symmetric pos. def. endomorphism)
- Thus, **space of LI Riemannian metrics** $\longleftrightarrow$

$$P(\mathfrak{g}) := \{L \in \mathrm{End}(\mathfrak{g}) \mid L \text{ positive definite}\}$$

[Nikonorov, Rodionov (2003)]

- Parameterize $P(\mathfrak{g})$ by considering a **change of basis**:

$$(\mathbf{X},\mathbf{Y}) = (\mathbf{E},\mathbf{F})A^T \ , \qquad A \in \mathrm{GL}(6,\mathbb{R})$$

Notation:

  - Q-orthonormal basis $(\mathbf{E},\mathbf{F})$ of $\mathfrak{g}$
  - $\mathbf{E} := (E_1, E_2, E_3)$, $\mathbf{F} := (F_1, F_2, F_3)$ oriented ONBs of $\mathfrak{su}(2)_{1,2}$
  - g-orthonormal basis $(\mathbf{X},\mathbf{Y})$ of $\mathfrak{g}$

- A satisfies $A^T A = L^{-1}$, i.e. $g(\cdot,\cdot) = Q((A^T A)^{-1}\cdot,\cdot)$
- Can choose $(\mathbf{X},\mathbf{Y})$ s.t. $A = \begin{pmatrix} D & 0 \\ W & \tilde{D} \end{pmatrix}$, where

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \ \tilde{D} = \begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}, \ W = \begin{pmatrix} x & u & v \\ \alpha & y & w \\ \beta & \gamma & z \end{pmatrix}$$

$a, \dots, f$ pos. params, components of W arbitrary real params

[Nikonorov, Rodionov (2003)]

- W.l.o.g. choose $\mathcal{V}_0 = \int_G \mathrm{vol}_Q = 4\pi^4$ (vol. of canonical product metric on $G = S^3 \times S^3$)
- S and $\mathcal{V} = V\,\mathcal{V}_0$ **rational functions in 15 params** $(a, \ldots, f, x, y, z, u, v, w, \alpha, \beta, \gamma)$:
  $V = (\det A)^{-1} = (abcdef)^{-1}$ and

$$
\begin{aligned}
S = {} & a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + x^2 + y^2 + z^2 + u^2 + v^2 + w^2 + \alpha^2 + \beta^2 + \gamma^2 \\
& - \frac{1}{2}\Big\{ a^2b^2c^{-2} + b^2c^2a^{-2} + c^2a^2b^{-2} + d^2e^2f^{-2} + e^2f^2d^{-2} + f^2d^2e^{-2} \\
& + \left(\frac{a^2}{c^2} + \frac{c^2}{a^2}\right)(u^2 + y^2 + \gamma^2) + \left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right)(v^2 + w^2 + z^2) + \left(\frac{b^2}{c^2} + \frac{c^2}{b^2}\right)(x^2 + \alpha^2 + \beta^2) \\
& + a^{-2}\left[\left(uw - vy - \frac{de}{f}\beta\right)^2 + \left(v\gamma - uz - \frac{df}{e}\alpha\right)^2 + \left(yz - w\gamma - \frac{ef}{d}x\right)^2\right] \\
& + b^{-2}\left[\left(v\alpha - xw - \frac{de}{f}\gamma\right)^2 + \left(xz - v\beta - \frac{df}{e}y\right)^2 + \left(w\beta - z\alpha - \frac{ef}{d}u\right)^2\right] \\
& + c^{-2}\left[\left(xy - u\alpha - \frac{de}{f}z\right)^2 + \left(u\beta - x\gamma - \frac{df}{e}w\right)^2 + \left(\alpha\gamma - y\beta - \frac{ef}{d}v\right)^2\right]\Big\}
\end{aligned}
$$

- **Einstein metrics correspond to solutions of**

$$\nabla S = \mu \nabla V \quad \text{and} \quad V = (abcdef)^{-1} = 1$$

Here, ∇ = std. gradient in param. space $(\mathbb{R}_{>0})^6 \times \mathbb{R}^9 \subset \mathbb{R}^{15}$ w/ coords $(a, \ldots, f, x, y, z, u, v, w, \alpha, \beta, \gamma)$ and μ is a **Lagrange multiplier**.

- **Remark:** Lagrange multiplier $\mu \longleftrightarrow$ Einstein constant λ

$$\mu = -2\mathcal{V}_0\lambda$$

Section 3

LI Einstein metrics invariant under finite subgroup $\Gamma \subset \mathrm{Ad}(G)$

[Belgun, Cortés, AH, Lindemann (2017)]

- Consider LI Einstein metrics invariant under **non-trivial finite subgroup** $\Gamma \subset \mathrm{Ad}(G)$
- **Observation:** either all non-trivial elements of Γ are of order 2 or $\exists$ element σ of order $k \geq 3$
- First consider the **latter case:**

Result 1

Let g be a left-invariant and Γ-**invariant** Einstein metric on G, where $\Gamma \subset \mathrm{Ad}(G)$. If Γ contains an element σ of order $k \geq 3$ then K contains a $U(1)$ subgroup and, hence, (G, g) is homothetic to (G, g_{can}) or (G, g_{NK}) by Theorem 2.

- **Proof:** representation-theoretic arguments (quite technical)

[Belgun, Cortés, AH, Lindemann (2017)]

- Remaining case, i.e. all non-triv. elements of Γ are of order 2:

Proposition

If all non-trivial elements of Γ are of order 2, then $\Gamma \cong \mathbb{Z}_2^\ell$, where $1 \leq \ell \leq 4$. If $\ell \geq 3$, then Γ contains an element σ with $\operatorname{tr} \sigma = 2$.

- **Proof:** representation-theoretic arguments & combinatorics
- Cases $\operatorname{tr} \sigma = 2$ and $\operatorname{tr} \sigma = -2$ **treated separately**

Result 2 (case $\operatorname{tr} \sigma = 2$)

Let g be a left-invariant and Γ-invariant Einstein metric on G. If Γ contains an element σ of trace 2, then $g = g_{can}$. (By the previous proposition, this covers the case $\Gamma \cong \mathbb{Z}_2^\ell$, where $\ell \geq 3$.)

Proof:

- Γ-invariance can be used to **simplify basis change matrix** A, namely $y = z = u = v = w = \gamma = 0$
- Need to **solve**: $\nabla S = \mu \nabla V$ and $V = (abcdef)^{-1} = 1$ with

$$\begin{aligned} -2S \quad = \quad & -2(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + x^2 + \alpha^2 + \beta^2) \\ & + a^2 b^2 c^{-2} + b^2 c^2 a^{-2} + c^2 a^2 b^{-2} + d^2 e^2 f^{-2} + e^2 f^2 d^{-2} + f^2 d^2 e^{-2} \\ & + (b^2 c^{-2} + c^2 b^{-2})(x^2 + \alpha^2 + \beta^2) + a^{-2}(d^2 e^2 f^{-2} \beta^2 + d^2 f^2 e^{-2} \alpha^2 + e^2 f^2 d^{-2} x^2) \end{aligned}$$

- 1st consider **gradient in x direction**:

$$\frac{\partial S}{\partial x} = -x \left(\frac{a^2 d^2 \left(b^2 - c^2\right)^2 + b^2 c^2 e^2 f^2}{a^2 b^2 c^2 d^2} \right) \stackrel{!}{=} \frac{\partial V}{\partial x} = 0 \implies x = 0$$

- Same argument: $\alpha = \beta = 0$

Proof continued:

- Remaining eqs: **7 polys in 7 vars** $(a, \ldots, f, \mu)$ w/ $\deg \leq 11$

$$
\begin{aligned}
0 &= abcdef - 1 \,, \\
0 &= abc\mu + a^4b^4def - a^4c^4def - b^4c^4def + 2a^2b^2c^4def \,, \\
0 &= abc\mu - a^4b^4def + a^4c^4def - b^4c^4def + 2a^2b^4c^2def \,, \\
0 &= abc\mu - a^4b^4def - a^4c^4def + b^4c^4def + 2a^4b^2c^2def \,, \\
0 &= def\mu + abcd^4e^4 - abcd^4f^4 - abce^4f^4 + 2abcd^2e^2f^4 \,, \\
0 &= def\mu - abcd^4e^4 + abcd^4f^4 - abce^4f^4 + 2abcd^2e^4f^2 \,, \\
0 &= def\mu - abcd^4e^4 - abcd^4f^4 + abce^4f^4 + 2abcd^4e^2f^2 \,.
\end{aligned}
$$

- Result of simple (< 1 sec) **computer-based Gröbner basis computation** (e.g. using Mathematica):
  $a = b = c = d = e = f = -\mu = 1$
  (only soln w/ $a, \ldots, f \in \mathbb{R}_{>0}$)
- This **proves** that $g = g_{can}$ if Γ contains an element of trace 2.

- **Remaining cases:** $1 \leq \ell \leq 2$ and $\operatorname{tr} \sigma = -2$ for all non-trivial elements $\sigma \in \Gamma$
- **First** consider the case $\ell = 2$, i.e. $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ ...

[Belgun, Cortés, AH, Lindemann (2017)]

Result 3 (case $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$)

Let g be a left-invariant Einstein metric on G that is invariant under a **non-trivial finite subgroup** $\Gamma \subset \mathrm{Ad}(G)$ such that $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then (G, g) is homothetic to (G, g_{can}) or (G, g_{NK}).

Proof:

- Case 1: $\exists \sigma \in \Gamma$ s.t. $\mathrm{tr}\,\sigma = 2$ (**covered by Result 2**) ✓
- Case 2: $\mathrm{tr}\,\sigma = -2$ for all non-trivial elements $\sigma \in \Gamma$
- Γ-invariance $\rightarrow$ **simplify basis change matrix** A, namely $u = v = w = \alpha = \beta = \gamma = 0$
- **Coord. trafo** (diffeo of$(\mathbb{R}_{>0})^6 \times \mathbb{R}^3$) to simplify polys

$$a = \sqrt{BC}\,, \qquad b = \sqrt{AC}\,, \qquad c = \sqrt{AB}\,,$$
$$d = \sqrt{FE}\,, \qquad e = \sqrt{DF}\,, \qquad f = \sqrt{DE}\,,$$
$$x = X\sqrt{BC}\,, \qquad y = Y\sqrt{AC}\,, \qquad z = Z\sqrt{AB}\,.$$

Proof continued:

- **10 polys in 10 vars** $(A, \ldots, F, X, Y, Z, \mu)$ w/ $\deg \leq 6$

$$
\begin{aligned}
0 &= ABCDEF - 1,\\
0 &= BCDEF\mu - AY^2Z^2 + DXYZ - AY^2 - AZ^2 + BZ^2 + CY^2 - A + B + C,\\
0 &= ACDEF\mu - BX^2Z^2 + EXYZ - BZ^2 - BX^2 + CX^2 + AZ^2 + A - B + C,\\
0 &= ABDEF\mu - CX^2Y^2 + FXYZ - CX^2 - CY^2 + AY^2 + BX^2 + A + B - C,\\
0 &= ABCEF\mu + AXYZ - DX^2 - D + E + F,\\
0 &= ABCDF\mu + BXYZ - EY^2 + D - E + F,\\
0 &= ABCDE\mu + CXYZ - FZ^2 + D + E - F,\\
0 &= -B^2XZ^2 - C^2XY^2 - B^2X - C^2X + ADYZ + BEYZ + CFYZ + 2BCX - D^2X,\\
0 &= -C^2X^2Y - A^2YZ^2 - C^2Y - A^2Y + ADXZ + BEXZ + CFXZ + 2ACY - E^2Y,\\
0 &= -A^2Y^2Z - B^2X^2Z - A^2Z - B^2Z + ADXY + BEXY + CFXY + 2ABZ - F^2Z.
\end{aligned}
$$

- **GB computation** using computer algebra system **Magma** (resources: compute-server w/ 24 state-of-the-art Intel Xeon E5-2643 3.40 GHz processors, 512 GB RAM)

Proof continued:

- Running time: 16.5 minutes; consumed 1.8 GB of RAM
- Output: 55 polys w/ ∅ 78.7 terms/poly. Coeffs up to $\mathcal{O}(10^{12})$
- **Real solns** (w/ $a, \ldots, f \in \mathbb{R}_{>0}$):

counter	a	b	c	d	e	f	x	y	z	μ	S
(1)	1	1	1	1	1	1	0	0	0	-1	3
(2)	1	1	1	1	1	1	± 1	± 1	1	-1	3
(3)	1	1	1	1	1	1	± 1	∓ 1	-1	-1	3
(4)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	3
(5)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\pm\frac{1}{\sqrt{2}}$	$\mp\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	3
(6)	$\frac{\sqrt[4]{3}}{\sqrt{2}}$	$\frac{\sqrt[4]{3}}{\sqrt{2}}$	$\frac{\sqrt[4]{3}}{\sqrt{2}}$	$\frac{\sqrt{2}}{\sqrt[4]{3}}$	$\frac{\sqrt{2}}{\sqrt[4]{3}}$	$\frac{\sqrt{2}}{\sqrt[4]{3}}$	$\pm\frac{1}{\sqrt{2}\sqrt[4]{3}}$	$\pm\frac{1}{\sqrt{2}\sqrt[4]{3}}$	$\frac{1}{\sqrt{2}\sqrt[4]{3}}$	$-\frac{5}{3\sqrt{3}}$	$\frac{5}{\sqrt{3}}$
(7)	$\frac{\sqrt[4]{3}}{\sqrt{2}}$	$\frac{\sqrt[4]{3}}{\sqrt{2}}$	$\frac{\sqrt[4]{3}}{\sqrt{2}}$	$\frac{\sqrt{2}}{\sqrt[4]{3}}$	$\frac{\sqrt{2}}{\sqrt[4]{3}}$	$\frac{\sqrt{2}}{\sqrt[4]{3}}$	$\pm\frac{1}{\sqrt{2}\sqrt[4]{3}}$	$\mp\frac{1}{\sqrt{2}\sqrt[4]{3}}$	$-\frac{1}{\sqrt{2}\sqrt[4]{3}}$	$-\frac{5}{3\sqrt{3}}$	$\frac{5}{\sqrt{3}}$

- Sign choices can be absorbed in initial choice of basis $(\mathbf{E}, \mathbf{F})$
- **4 cases remain:** (1), (2), (4), and (6) (w/ $x, y, z \geq 0$)
- By comparing to [Nikonorov, Rodionov (2003)]: (1) = std. metric, (6) = NK metric, (2) and (4) = isometric to std. metric

- **Last** remaining case: $\Gamma \cong \mathbb{Z}_2$, w/ non-triv. $\sigma \in \Gamma$: $\operatorname{tr}\sigma = -2$
- **12 polys in 12 vars** $(A, \ldots, F, X, Y, Z, W, \mathcal{C}, \mu)$ w/ $\deg \leq 6$

$$
\begin{aligned}
0 &= ABCDEF - 1, \\
0 &= -D + E + F + ABCEF\mu - DX^2 + AXYZ - AWX\mathcal{C}, \\
0 &= D - E + F + ABCDF\mu - EW^2 - EY^2 + BXYZ - CWX\mathcal{C}, \\
0 &= A - B + C + ACDEF\mu + AW^2 - BW^2 - BX^2 + CX^2 - BW^2X^2 + EXYZ + AZ^2 - BZ^2 - BX^2Z^2 - FWX\mathcal{C}, \\
0 &= D + E - F + ABCDE\mu + CXYZ - FZ^2 - BWX\mathcal{C} - F\mathcal{C}^2, \\
0 &= -A + B + C + BCDEF\mu - AW^2 + BW^2 - AY^2 + CY^2 + DXYZ - AZ^2 + BZ^2 \\
&\quad - AY^2Z^2 - DWX\mathcal{C} + 2AWYZ\mathcal{C} - A\mathcal{C}^2 + C\mathcal{C}^2 - AW^2\mathcal{C}^2, \\
0 &= A + B - C + ABDEF\mu + BX^2 - CX^2 + AY^2 - CY^2 - CX^2Y^2 + FXYZ - EWX\mathcal{C} + A\mathcal{C}^2 - C\mathcal{C}^2 - CX^2\mathcal{C}^2 \\
0 &= -ADWX - CEWX - BFWX + A^2WYZ - A^2\mathcal{C} + 2AC\mathcal{C} - C^2\mathcal{C} - F^2\mathcal{C} - A^2W^2\mathcal{C} - C^2X^2\mathcal{C}, \\
0 &= ADXY + BEXY + CFXY - A^2Z + 2ABZ - B^2Z - F^2Z - B^2X^2Z - A^2Y^2Z + A^2WY\mathcal{C}, \\
0 &= -A^2Y + 2ACY - C^2Y - E^2Y - C^2X^2Y + ADXZ + BEXZ + CFXZ - A^2YZ^2 + A^2WZ\mathcal{C}, \\
0 &= -A^2W + 2ABW - B^2W - E^2W - B^2WX^2 - ADX\mathcal{C} - CEX\mathcal{C} - BFX\mathcal{C} + A^2YZ\mathcal{C} - A^2W\mathcal{C}^2, \\
0 &= -B^2X + 2BCX - C^2X - D^2X - B^2W^2X - C^2XY^2 + ADYZ + BEYZ + CFYZ \\
&\quad - B^2XZ^2 - ADW\mathcal{C} - CEW\mathcal{C} - BFW\mathcal{C} - C^2X\mathcal{C}^2 .
\end{aligned}
$$

- Solving (using GB techniques) apparently out of reach w/ current technology (tried, but requires > 1 **TB of RAM!**)
- Instead: managed to compute **grevlex GB** (running time **29 days!**, 78 GB of RAM)
- Output: **106 GB (!); 50472 polys w/ $\varnothing$ 593 terms/poly. Coeffs up to $\mathcal{O}(10^{10})$**
- No good for solving eq-sys, but:

Result 4

The system of polynomial equations that describes left-invariant Einstein metrics on G invariant under a subgroup $\mathbb{Z}_2 \subset \mathrm{Ad}(G)$ has a **continuous families of complex solutions**.

- Also: solved eq-sys for **fixed value of** μ ($\sim \lambda \sim S$)
- $\mu = -1$: all real solns isometric to std. metric g_{can}
- $\mu = -5/(3\sqrt{3})$: all real solns isometric to NK metric g_{NK}

Summary

- **Missing link** in classification of homogeneous compact Einstein manifolds in $d = 6$: LI Einstein metrics on $S^3 \times S^3$
- **Previous result** [Nikonorov, Rodionov (2003)]: If $U(1) \subset K$, then (G, g) is homothetic to (G, g_{can}) or (G, g_{NK}).
- Using repn-theoretic arguments & advanced GB techniques, we **extended** this to the case where the metric is, in addition to being LI, inv. under non-triv. finite subgroup $\Gamma \subset \mathrm{Ad}(G)$:
    1. For $\Gamma \ncong \mathbb{Z}_2$: (G, g) is homothetic to (G, g_{can}) or (G, g_{NK})
    2. For $\Gamma \cong \mathbb{Z}_2$: partial results (i.p. for fixed μ; **no new metrics**)

Open Problems

- **Fully solve** $\Gamma \cong \mathbb{Z}_2$-case or gen. case w/out symmetry?
- Is there a **new** metric elsewhere in param. space? Properties?
- Consider **other (related) scenarios**, e.g. (partial) classific. of LI Einst. metrics on (non-cp) $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ (w.i.p.)

Thank you for your attention.