

# Quaternionic Heisenberg group and Solutions to Strominger system with non-constant dilaton

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Workshop Geometry, Gravity and Supersymmetry, Mainz, April 2017

based on joint works with Marisa Fernandez, Luis Ugarte & Dimiter Vassilev-JHEP-2014,Comm.Math.Phys.-2015

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<sup>1</sup>The author is partially supported by Contract DFNI I02/4/2014 and Contract 195/2016 with the University of Sofia "St. Kliment Ohridski"

The bosonic fields of the ten-dimensional supergravity which arises as low energy effective theory of the heterotic string are:

- i) the spacetime metric  $g$ ;
- ii) the NS three-form field strength (flux)  $H$ ;
- iii) the dilaton  $\phi$ ;
- iv) the gauge connection  $A$  with curvature 2-form  $F^A$ .

- The bosonic geometry is of the form  $\mathbb{R}^{1,9-d} \times M^d$ .
- The bosonic fields are non-trivial only on  $M^d$ ,  $d \leq 8$ .
- The two torsion connections  $\nabla^\pm = \nabla^g \pm \frac{1}{2}H$ ,  $\nabla^g$  is the Levi-Civita connection of the Riemannian metric  $g$ .
- Both connections preserve the metric,  $\nabla^\pm g = 0$  and have totally skew-symmetric torsion  $\pm H$ , respectively.
- $R^g, R^\pm$  - the corresponding curvatures.

# The action and equations of motion

The bosonic part of the ten-dimensional supergravity action in the string frame is ( $R = R^-$ )

$$S = \frac{1}{2k^2} \int d^{10}x \sqrt{-g} e^{-2\phi} \left[ \text{Scal}^g + 4(\nabla^g \phi)^2 - \frac{1}{2}|H|^2 - \frac{\alpha'}{4} \left( \text{Tr}|F^A|^2 \right) - \text{Tr}|R|^2 \right].$$

- The string frame field equations (the equations of motion) are;

$$\begin{aligned} \text{Ric}_{ij}^g - \frac{1}{4} H_{imn} H_j^{mn} + 2\nabla_i^g \nabla_j^g \phi - \frac{\alpha'}{4} \left[ (F^A)_{imab} (F^A)_j^{mab} - R_{imnq} R_j^{mnq} \right] &= 0, \\ \nabla_i^g (e^{-2\phi} H_{jk}^i) &= 0, \quad \nabla_i^+ (e^{-2\phi} (F^A)_j^i) = 0. \end{aligned} \quad (1)$$

The field equation of the dilaton  $\phi$  is implied from the first two equations above.

- The Green-Schwarz anomaly cancellation mechanism requires that the three-form Bianchi identity receives an  $\alpha'$  correction of the form

$$dH = \frac{\alpha'}{4} 8\pi^2 (p_1(M^d) - p_1(E)) = \frac{\alpha'}{4} \left( \text{Tr}(R \wedge R) - \text{Tr}(F^A \wedge F^A) \right), \quad (2)$$

where  $p_1(M^d)$  and  $p_1(E)$  are the first Pontrjagin forms of  $M^d$  with respect to a connection  $\nabla$  with curvature  $R$  and the vector bundle  $E$  with connection  $A$ ;

# Heterotic supersymmetry and the Strominger system

A heterotic geometry preserves supersymmetry iff there exists at least one Majorana-Weyl spinor  $\epsilon$  such that the following Killing-spinor equations hold

•

$$\begin{aligned}\delta_\lambda &= \nabla_m \epsilon = \left( \nabla_m^g + \frac{1}{4} H_{mnp} \Gamma^{np} \right) \epsilon = \nabla^+ \epsilon = 0, \\ \delta_\psi &= \left( \Gamma^m \partial_m \phi - \frac{1}{12} H_{mnp} \Gamma^{mnp} \right) \epsilon = (d\phi - \frac{1}{2} H) \cdot \epsilon = 0, \\ \delta_\xi &= F_{mn}^A \Gamma^{mn} \epsilon = F^A \cdot \epsilon = 0,\end{aligned}\tag{3}$$

- $\lambda, \psi, \xi$  are the gravitino, the dilatino and the gaugino fields,
- $\Gamma_i$  generate the Clifford algebra  $\{\Gamma_i, \Gamma_j\} = 2g_{ij}$
- $\cdot$  means Clifford action of forms on spinors.

**THE STROMINGER SYSTEM** The system of Killing spinor equations (3) together with the anomaly cancellation condition (2) is known as the Strominger system.

- The last equation in (3) is the instanton condition which means that the curvature  $F^A$  is contained in a Lie algebra of a Lie group which is a stabilizer of a non-trivial spinor.

- In dimension 7 this group is  $G_2$ . Denoting the  $G_2$  three-form by  $\Theta$ , the  $G_2$ -instanton condition has the form

$$\sum_{k,l=1}^7 (F^A)^j_i(E_k, E_l) \Theta(E_k, E_l, E_m) = 0.\tag{4}$$

# Heterotic supersymmetry and equations of motion

In the presence of a curvature term  $\text{Tr}(R \wedge R)$  the solutions of the Strominger system (3), (2) obey the second and the third equations of motion (the second and the third equations in (1)) but do not always satisfy the Einstein equations of motion.

## Theorem (Iv, Phys.Lett. 2010)

*The solutions of the Strominger system ((3) and (2)) also solve the heterotic supersymmetric equations of motion (1) if and only if the curvature  $R$  of the connection on the tangent bundle is an instanton in dimensions 5,6,7,8. In dimension 7,  $R$  is required to be an  $G_2$ -instanton.*

The result was re-proved by Martelli, Sparks '11, de la Ossa et al '14 with different proofs which holds for all dimensions.

The physically relevant connection on the tangent bundle to be considered in (2), (1) is the  $(-)$ -connection, Bergshoeff, de Roo' 89, Hull' 86.

- Reason: the curvature  $R^-$  is an instanton up to the first order of  $\alpha'$ .
- a consequence of the first equation in (3), (2) and the well known identity

$$R^+(X, Y, Z, U) - R^-(Z, U, X, Y) = \frac{1}{2}dH(X, Y, Z, U). \quad (5)$$

Indeed, (2) together with (5) imply

$$R^+(X, Y, Z, U) - R^-(Z, U, X, Y) = O(\alpha').$$

The first equation in (3) yields the holonomy group of  $\nabla^+$  is contained in  $G_2$ , i.e.  $R^+(X, Y) \subset \mathfrak{g}_2$ . Therefore  $R^-$  satisfies the instanton condition (4) up to the first order of  $\alpha'$ .

# The dimension 7, $G_2$ structures

In dimension seven  $Hol(\nabla^+)$  has to be contained in the exceptional group  $G_2$ .

- The precise conditions to have a solution to the gravitino Killing spinor equation in D=7 are:

- There exists a parallel spinor with respect to a  $G_2$ -connection with torsion 3-form  $T$  iff there exists an integrable  $G_2$ -structure  $\Theta$ ,

$$d * \Theta = \theta^7 \wedge * \Theta, \quad 3\theta^7 = - * (*d\Theta \wedge \Theta) = *(*d * \Theta \wedge * \Theta), \quad \theta^7 \quad \text{-- the Lee form}$$

The connection  $\nabla^+$  is unique and the torsion 3-form  $T$  is given by the formula (Fridrich-Iv'01)

$$H = T = \frac{1}{6}(d\Theta, * \Theta) \Theta - *d\Theta + *(\theta^7 \wedge \Theta).$$

- The necessary conditions to have a solution to the dilatino and gravitino Killing spinor equations in D=7 were derived in Gauntlett et al'01, and the sufficiency was proved in Fridrich- Iv,01.

- There exists a solution to both dilatino and gravitino Killing spinor equations in D=7 iff there exists a  $G_2$ -structure  $(\Theta, g)$  satisfying the equations

$$d * \Theta = \theta^7 \wedge * \Theta, \quad d\Theta \wedge \Theta = 0, \quad \theta^7 = -2d\phi.$$

The torsion 3-form (the flux  $H$ ) is given by

$$H = T = - * d\Theta - 2 * (d\phi \wedge \Theta)$$

the Riemannian scalar curvature is given by  $s^g = 8\|d\phi\|^2 - \frac{1}{12}\|T\|^2 - 6\delta d\phi$

- The solution exists exactly when the  $G_2$ -structure  $(\bar{\Theta} = e^{-\frac{3}{2}\phi} \Theta, \bar{g} = e^{-\phi} g)$  obeys the equations  $d\bar{*}\bar{\Theta} = d\bar{\Theta} \wedge \bar{\Theta} = 0$ , i.e., **a SOLUTION WITH CONSTANT DILATON,  $\theta^7 = 0 = d\phi$ .**

A geometric model which fits the above structures - a certain  $\mathbb{T}^3$ -bundle over a Calabi-Yau surface.

- Let  $\Gamma_i$ ,  $1 \leq i \leq 3$ , be three closed ANTI-SELF-DUAL 2-forms on a Calabi-Yau surface  $M^4$ , which represent integral cohomology classes.

- Let  $\omega_1$  and  $\omega_2 + \sqrt{-1}\omega_3$  be the (closed) Kähler form and the holomorphic volume form on  $M^4$ .

- There is a compact 7-dimensional manifold  $M^{1,1,1}$  which is the total space of a  $\mathbb{T}^3$ -bundle over  $M^4$  and has a  $G_2$ -structure

$$\Theta = \omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 - \omega_3 \wedge \eta_3 + \eta_1 \wedge \eta_2 \wedge \eta_3,$$

solving the first two Killing spinor equations in (3) with CONSTANT DILATON in dimension 7, where  $\eta_i$ ,  $1 \leq i \leq 3$ , is a 1-form on  $M^{1,1,1}$  such that  $d\eta_i = \Gamma_i$ ,  $1 \leq i \leq 3$ .

- For any smooth function  $f$  on  $M^4$ , the  $G_2$ -structure on  $M^{1,1,1}$  given by

$$\Theta_f = e^{2f} \left[ \omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 - \omega_3 \wedge \eta_3 \right] + \eta_1 \wedge \eta_2 \wedge \eta_3$$

solves the first two Killing spinor equations in (3) with non-constant dilaton  $\phi = -2f$ .

- The metric has the form  $g_f = e^{2f} g_{cy} + \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3$ .

- The ansatz guaranties solution to first two killing spinor equations. To achieve a smooth solution to the Strominger system we still have to determine an auxiliary vector bundle with an instanton and a linear connection on  $M^{1,1,1}$  in order to satisfy the anomaly cancellation condition (2).

# The quaternionic Heisenberg group

- The seven dimensional quaternionic Heisenberg group  $G(\mathbb{H})$  is the connected simply connected Lie group determined by the Lie algebra  $\mathfrak{g}(\mathbb{H})$  with structure equations

$$de^1 = de^2 = de^3 = de^4 = 0, \\ d\gamma^5 = \sigma_1 = e^{12} - e^{34}, \quad d\gamma^6 = \sigma_2 = e^{13} + e^{24}, \quad d\gamma^7 = \sigma_3 = e^{14} - e^{23}.$$

- Note  $\sigma_1 = e^{12} - e^{34}$ ,  $\sigma_2 = e^{13} + e^{24}$ ,  $\sigma_3 = e^{14} - e^{23}$  are the **three anti-self-dual 2-forms on  $\mathbb{R}^4$** .

- To obtain results in dimensions less than seven through contractions of  $\mathfrak{g}(\mathbb{H})$  consider the orbit of  $G(\mathbb{H})$  under the natural action of  $GL(3, \mathbb{R})$  on the *span*  $\{\gamma^5, \gamma^6, \gamma^7\}$ .

- The group  $K_A$  is determined by the Lie algebra  $\mathfrak{k}_A$  with structure equations:

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^{4+i} = \sum_{j=1}^3 a_{ij} \sigma_j, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

- Denote the norm of  $A$  by  $|A|$ ,  $|A|^2 = \sum_{i,j=1}^3 a_{ij}^2$ .

$\mathfrak{k}_A$  is isomorphic to  $\mathfrak{g}(\mathbb{H})$  then  $K_A$  is isomorphic to  $G(\mathbb{H})$ .

Any lattice  $\Gamma_A$  gives rise to a (compact) nilmanifold  $M_A = K_A/\Gamma_A$ , which is a  $\mathbb{T}^3$ -bundle over a  $\mathbb{T}^4$  whose connection 1-form is the anti-self-dual curvature on the four torus.



# The $G_2$ structure on $G(\mathbb{H})$ -constant dilaton

Consider the  $G_2$ -structure on the Lie group  $K_A$  defined by the 3-form

$$\Theta = \omega_1 \wedge e^7 + \omega_2 \wedge e^5 - \omega_3 \wedge e^6 + e^{567},$$
$$\omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} - e^{24}, \quad \omega_3 = e^{14} + e^{23}$$

are the three closed self-dual 2-forms on  $\mathbb{R}^4$ .

- The corresponding Hodge dual 4-form  $*\Theta$  is given by

$$*\Theta = \omega_1 \wedge e^{56} + \omega_2 \wedge e^{67} + \omega_3 \wedge e^{57} + \frac{1}{2}\omega_1 \wedge \omega_1.$$

- The property  $\sigma_i \wedge \omega_j = 0$  for  $1 \leq i, j \leq 3$  implies

$$d*\Theta = 0, \quad d\Theta \wedge \Theta = 0,$$

- solves the gravitino and dilatino equations with constant dilaton.

# The $G_2$ structure on $G(\text{III})$ -non constant dilaton

-  $f$  a smooth function on  $\mathbb{R}^4$ . Consider the  $G_2$  form

$$\bar{\Theta} = e^{2f} [\omega_1 \wedge e^7 + \omega_2 \wedge e^5 - \omega_3 \wedge e^6] + e^{567}.$$

- The corresponding metric  $\bar{g}$  on  $K_A$  has an orthonormal basis of 1-forms

$$\bar{e}^1 = e^f e^1, \quad \bar{e}^2 = e^f e^2, \quad \bar{e}^3 = e^f e^3, \quad \bar{e}^4 = e^f e^4, \quad \bar{e}^5 = e^5, \quad \bar{e}^6 = e^6, \quad \bar{e}^7 = e^7.$$

With respect to  $\bar{g}$ , the self-dual forms  $\bar{\omega}_i$  and anti-self-dual forms  $\bar{\sigma}_i$  are

$$\bar{\omega}_i = e^{2f} \omega_i, \quad \bar{\sigma}_i = e^{2f} \sigma_i, \quad i = 1, 2, 3.$$

- The Hodge dual 4-form  $\bar{*}\bar{\Theta}$ :

$$\bar{*}\bar{\Theta} = e^{2f} \left[ \omega_1 \wedge e^{56} + \omega_2 \wedge e^{67} + \omega_3 \wedge e^{57} + \frac{e^{2f}}{2} \omega_1 \wedge \omega_1 \right].$$

$$d\bar{*}\bar{\Theta} = 2df \wedge \bar{*}\bar{\Theta}, \quad d\bar{\Theta} \wedge \bar{\Theta} = 0.$$

The Lie form  $\bar{\theta}$ :

$$\bar{\theta} = 2df$$

-  $\bar{\Theta}$  solves the gravitino and dilatino equations with non-constant dilaton  $\phi = -2f$

Recall, the torsion of the (+)-connection  $\nabla^+$  (the flux  $H$ ) is the 3-form

$$T = - * d\Theta + *( \theta \wedge \Theta ).$$

- Calculate

$$d\bar{\Theta} = 2df \wedge \bar{\Theta} - 2df \wedge e^{567} + de^{567}.$$

- The torsion  $\bar{T}$  :

$$\begin{aligned} \bar{T} = \bar{*}(2df \wedge e^{567} - de^{567}) &= e^{-f} \left[ -2f_1 \bar{e}^{234} + 2f_2 \bar{e}^{134} - 2f_3 \bar{e}^{124} + 2f_4 \bar{e}^{123} \right] \\ &+ e^{-2f} \left[ (a_{11} \bar{\sigma}_1 + a_{12} \bar{\sigma}_2 + a_{13} \bar{\sigma}_3) \wedge \bar{e}^5 + (a_{21} \bar{\sigma}_1 + a_{22} \bar{\sigma}_2 + a_{23} \bar{\sigma}_3) \wedge \bar{e}^6 \right] \\ &+ e^{-2f} \left[ (a_{31} \bar{\sigma}_1 + a_{32} \bar{\sigma}_2 + a_{33} \bar{\sigma}_3) \wedge \bar{e}^7 \right], \end{aligned}$$

-  $f_i = \frac{\partial f}{\partial x_i}$ ,  $1 \leq i \leq 4$ , and  $f_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ ,  $1 \leq i, j \leq 4$ .

- Calculate

$$d\bar{T} = -e^{-4f} \left[ \Delta e^{2f} + 2|A|^2 \right] \bar{e}^{1234} = - \left[ \Delta e^{2f} + 2|A|^2 \right] e^{1234},$$

$\Delta e^{2f} = (e^{2f})_{11} + (e^{2f})_{22} + (e^{2f})_{33} + (e^{2f})_{44}$  - the standard Laplacian on  $\mathbb{R}^4$ .

# The first Pontrjagin form of (-)-connection

Recall:

The connection 1-forms  $\omega_{ji}$  of a metric connection  $\nabla$ ,  $\nabla g = 0$  with respect to a local orthonormal basis  $\{E_1, \dots, E_d\}$  are:

$$\omega_{ji}(E_k) = g(\nabla_{E_k} E_j, E_i), \quad \nabla_X E_j = \omega_j^s(X) E_s$$

The curvature 2-forms  $\Omega_j^i$  of  $\nabla$ :

$$\Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k, \quad \Omega_{ji} = d\omega_{ji} + \omega_{ki} \wedge \omega_{jk}, \quad R_{ijk}^l = \Omega_k^l(E_i, E_j), \quad R_{ijkl} = R_{ijk}^s g_{ls}$$

The first Pontrjagin 4-form:

$$8\pi^2 p_1(\nabla) = \sum_{1 \leq i < j \leq d} \Omega_j^i \wedge \Omega_j^i$$

- **OUR CASE:**

The Koszul's formula for the Levi-Civita connection 1-forms  $(\omega^{\bar{g}})^{\bar{j}}$  of the metric  $\bar{g}$ :

$$(\omega^{\bar{g}})^{\bar{j}}(\bar{e}_k) = \frac{1}{2} \left( d\bar{e}^i(\bar{e}_j, \bar{e}_k) - d\bar{e}^k(\bar{e}_i, \bar{e}_j) + d\bar{e}^i(\bar{e}_k, \bar{e}_i) \right), \quad \bar{g}(\bar{e}_i, [\bar{e}_j, \bar{e}_k]) = -d\bar{e}^i(\bar{e}_j, \bar{e}_k)$$

- The connection 1-forms  $(\omega^-)^{\bar{j}}$  of the connection  $\nabla^-$ ,

$$(\omega^-)^{\bar{j}} = (\omega^{\bar{g}})^{\bar{j}} - \frac{1}{2}(\bar{T})^{\bar{j}}, \quad (\bar{T})^{\bar{j}}(\bar{e}_k) = \bar{T}(\bar{e}_i, \bar{e}_j, \bar{e}_k).$$

# The first Pontrjagin form of $\nabla^-$

- Calculate the curvature 2-forms  $(\Omega^-)_{\bar{j}}^{\bar{i}}$ . Then calculate the trace:

$$\begin{aligned} \sum_{1 \leq \bar{i} < \bar{j} \leq 7} (\Omega^-)_{\bar{j}}^{\bar{i}} \wedge (\Omega^-)_{\bar{j}}^{\bar{i}} = & \left( 6|A|^2 e^{-2f} (f_{11} + f_{22} + f_{33} + f_{44} - 2f_1^2 - 2f_2^2 - 2f_3^2 - 2f_4^2) \right. \\ & + 24 f_1^2 f_{11} - 8 f_{12}^2 - 8 f_{13}^2 - 8 f_{14}^2 + 32 f_1 f_{12} f_2 + 8 f_{11} f_2^2 + 8 f_1^2 f_{22} + 8 f_{11} f_{22} + 24 f_2^2 f_{22} - 8 f_{23}^2 - 8 f_{24}^2 + \\ & 32 f_1 f_{13} f_3 + 32 f_2 f_{23} f_3 + 8 f_{11} f_3^2 + 8 f_{22} f_3^2 + 8 f_1^2 f_{33} + 8 f_{11} f_{33} + 8 f_2^2 f_{33} + 8 f_{22} f_{33} + 24 f_3^2 f_{33} - 8 f_{34}^2 + \\ & 32 f_1 f_{14} f_4 + 32 f_2 f_{24} f_4 + 32 f_3 f_{34} f_4 + 8 f_{11} f_4^2 + 8 f_{22} f_4^2 + 8 f_{33} f_4^2 + 8 f_1^2 f_{44} + 8 f_{11} f_{44} + 8 f_2^2 f_{44} + \\ & \left. 8 f_{22} f_{44} + 8 f_3^2 f_{44} + 8 f_{33} f_{44} + 24 f_4^2 f_{44} \right) e^{1234}. \end{aligned}$$

The first Pontrjagin form of  $\nabla^-$  is a scalar multiple of  $e^{1234}$  given by

$$\pi^2 p_1(\nabla^-) = \left[ \mathcal{F}_2[f] + \Delta_4 f - \frac{3}{8} |A|^2 \Delta e^{-2f} \right] e^{1234},$$

- $\mathcal{F}_2[f]$  is the 2-Hessian of  $f$ , i.e., the sum of all principle  $2 \times 2$ -minors of the Hessian,
- $\Delta_4 f = \text{div}(|\nabla f|^2 \nabla f)$  is the 4-Laplacian of  $f$ .

Even though the curvature 2-forms of  $\nabla^-$  are quadratic in the gradient of the dilaton, the Pontrjagin form of  $\nabla^-$  is also quadratic in these terms. Fourth order in general!

- If  $f$  depends on two of the variables then  $\mathcal{F}_2[f] = \det(\text{Hess}f)$ ;
- If  $f$  is a function of one variable  $\mathcal{F}_2[f]$  vanishes.

## Proposition

Let  $D_\Lambda$ ,  $\Lambda = (\lambda_{ij}) \in \mathfrak{gl}_3(\mathbb{R})$ , be the linear connection on the Lie group  $K_A$  whose possibly non-zero 1-forms are given as follows

$$(\omega^{D_\Lambda})_{\bar{2}}^{\bar{1}} = -(\omega^{D_\Lambda})_{\bar{1}}^{\bar{2}} = -(\omega^{D_\Lambda})_{\bar{4}}^{\bar{3}} = (\omega^{D_\Lambda})_{\bar{3}}^{\bar{4}} = \lambda_{11} \bar{e}^5 + \lambda_{12} \bar{e}^6 + \lambda_{13} \bar{e}^7,$$

$$(\omega^{D_\Lambda})_{\bar{3}}^{\bar{1}} = -(\omega^{D_\Lambda})_{\bar{1}}^{\bar{3}} = (\omega^{D_\Lambda})_{\bar{4}}^{\bar{2}} = -(\omega^{D_\Lambda})_{\bar{2}}^{\bar{4}} = \lambda_{21} \bar{e}^5 + \lambda_{22} \bar{e}^6 + \lambda_{23} \bar{e}^7,$$

$$(\omega^{D_\Lambda})_{\bar{4}}^{\bar{1}} = -(\omega^{D_\Lambda})_{\bar{1}}^{\bar{4}} = -(\omega^{D_\Lambda})_{\bar{3}}^{\bar{2}} = (\omega^{D_\Lambda})_{\bar{2}}^{\bar{3}} = \lambda_{31} \bar{e}^5 + \lambda_{32} \bar{e}^6 + \lambda_{33} \bar{e}^7.$$

$D_\Lambda$  is a  $G_2$ -instanton with respect to the  $G_2$  structure  $\bar{\Theta}$  which preserves the metric iff  $\text{rank}(\Lambda) \leq 1$ .

In this case, the first Pontrjagin form  $p_1(D_\Lambda)$  of the  $G_2$ -instanton  $D_\Lambda$  is given by

$$8\pi^2 p_1(D_\Lambda) = -4\lambda^2 e^{1234},$$

$\lambda = |\Lambda A|$  is the norm of the product matrix  $\Lambda A$ .

## Theorem

The conformally compact manifold  $M^7 = (\Gamma \backslash K_A, \bar{\Theta}, \nabla^-, D_\Lambda, f)$  is a  $G_2$ -manifold which solves the Strominger system with non-constant dilaton  $f$ , non-trivial flux  $H = \bar{T}$ , non-flat instanton  $D_\Lambda$  using the first Pontrjagin form of  $\nabla^-$  and negative  $\alpha'$ . The dilaton  $f$  depends on one variable and is determined as a real slice of the Weierstrass' elliptic function.

The conformally compact manifold  $M^7 = (\Gamma \backslash K_A, \bar{\Theta}, \nabla^-, D_\Lambda, f)$  satisfies the heterotic equations of motion (1) up to first order of  $\alpha'$ .

**Sketch of the proof:** We are left with solving the anomaly cancellation condition

$d\bar{T} = \frac{\alpha'}{4} 8\pi^2 (p_1(\nabla^-) - p_1(D_\Lambda))$  in our case- **the single non-linear equation**

$$\Delta e^{2f} + 2|A|^2 + \frac{\alpha'}{4} [8\mathcal{F}_2[f] + 8\Delta_4 f - 3|A|^2 \Delta e^{-2f} + 4\lambda^2] = 0.$$

Assume the function  $f$  depends on one variable,  $f = f(x^1)$ . For **a negative  $\alpha'$  we choose  $2|A|^2 + \alpha'\lambda^2 = 0$** . Let  $\alpha' = -\alpha^2$  so that  $2|A|^2 = \alpha^2\lambda^2$ . The PDE goes to the ODE

$$(e^{2f})' + \frac{3}{4}\alpha^2|A|^2 (e^{-2f})' - 2\alpha^2 f'^3 = C_0 = \text{const.}$$

The substitution  $u = \alpha^{-2} e^{2f}$  allows us to write

$$(e^{2f})' + \frac{3}{4}\alpha^2|A|^2 (e^{-2f})' - 2\alpha^2 f'^3 = \frac{\alpha^2 u'}{4u^3} \left( 4u^3 - 3\frac{|A|^2}{\alpha^2} u - u'^2 \right).$$

For  $C_0 = 0$  we solve the following ODE for the function  $u = u(x^1) > 0$

$$u'^2 = 4u^3 - 3 \frac{|A|^2}{\alpha^2} u = 4u(u-d)(u+d), \quad d = \sqrt{3|A|^2/\alpha}.$$

Replacing the real derivative with the complex derivative leads to the Weierstrass' equation

$$\left(\frac{d\mathcal{P}}{dz}\right)^2 = 4\mathcal{P}(\mathcal{P}-d)(\mathcal{P}+d)$$

for the doubly periodic Weierstrass  $\mathcal{P}$  function with a pole at the origin.

- Letting  $\tau_+$  be the basic half-period such that  $\tau_+$  is real we have that  $\mathcal{P}$  is real valued on the line  $\Re z = m\tau_+, m \in \mathbb{Z}$ .
- Thus,  $u(x^1) = \mathcal{P}(x^1)$  defines a non-negative  $2\tau_+$ -periodic function with zeros at the points  $2n\tau_+, n \in \mathbb{Z}$ , which solves the real ODE. By construction,  $f = \frac{1}{2} \ln(\alpha^2 u)$  is a periodic function with singularities on the real line which is a solution to the ODE.
- The  $G_2$  structure  $\bar{\Theta}$  descends to the 7-dimensional nilmanifold  $M^7 = \Gamma \backslash K_A$  with singularity, determined by the singularity of  $u$ , where  $\Gamma$  is a lattice with the same period as  $f$ , i.e.,  $2\tau_+$ .
- $M^7$  is the total space of a  $\mathbb{T}^3$  bundle over the asymptotically hyperbolic manifold  $M^4$  with metric

$$\bar{g}_H = u(x^1) \left( (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 \right),$$

which is a conformally compact 4-torus with conformal boundary at infinity a flat 3-torus.



# A complete solution with positive $\alpha'$ - the instanton

## Lemma

The  $(-)$ -connection of the  $G_2$  structure  $\bar{\Theta}$  is a  $G_2$  instanton with respect to  $\bar{\Theta}$  if and only if the torsion 3-form is closed,  $d\bar{T} = 0$ , i.e. the dilaton function  $f$  satisfies the equality

$$\Delta e^{2f} + 2|A|^2 = 0.$$

**Proof:** Using (5) we investigate the  $G_2$ -instanton condition (4) for  $R^-$  as follows

$$\begin{aligned} 0 &= \sum_{i,j=1}^7 R^-(\bar{e}_i, \bar{e}_j, \bar{e}_l, \bar{e}_m) \bar{\Theta}(\bar{e}_i, \bar{e}_j, \bar{e}_k) = \sum_{i,j=1}^7 [R^+ - d\bar{T}](\bar{e}_i, \bar{e}_j, \bar{e}_l, \bar{e}_m) \bar{\Theta}(\bar{e}_i, \bar{e}_j, \bar{e}_k) \\ &= - \sum_{i,j=1}^7 d\bar{T}(\bar{e}_i, \bar{e}_j, \bar{e}_l, \bar{e}_m) \bar{\Theta}(\bar{e}_i, \bar{e}_j, \bar{e}_k), \end{aligned}$$

Use the fact that the holonomy of  $\nabla^+$  is contained in  $G_2$  and the expression of  $d\bar{T}$

$$\sum_{i,j=1}^7 R^+(\bar{e}_i, \bar{e}_j, \bar{e}_l, \bar{e}_m) \bar{\Theta}(\bar{e}_i, \bar{e}_j, \bar{e}_k) = 0, \quad d\bar{T} = -e^{-4f} [\Delta e^{2f} + 2|A|^2] \bar{e}^{1234}.$$

Let  $D_B$  be the  $\nabla^-$  connection obtained by replacing  $A$  with matrix  $B \in \mathfrak{gl}_3(\mathbb{R})$ .

From the Lemma, the connection  $D_B$  is a  $G_2$ -instanton iff the dilaton function satisfies

$$\Delta e^{2f} = -2|B|^2.$$

## A complete solution with positive $\alpha'$

The difference between the first Pontrjagin forms of  $\nabla^-$  and  $D_B$  is given by the formula

$$8\pi^2 \left( p_1(\nabla^-) - p_1(D_B) \right) = -3 \left( |A|^2 - |B|^2 \right) \left( \Delta e^{-2f} \right) e^{1234}.$$

The anomaly cancellation condition is

$$d\bar{T} - \frac{\alpha'}{4} 8\pi^2 \left( p_1(\nabla^-) - p_1(D_B) \right) = - \left[ \Delta e^{2f} + 2|A|^2 - \frac{3}{4} \alpha' \left( |A|^2 - |B|^2 \right) \left( \Delta e^{-2f} \right) \right] e^{1234} = 0.$$

Coupled with  $\Delta e^{2f} = -2|B|^2$  yields

$$d\bar{T} - \frac{\alpha'}{4} 8\pi^2 \left( p_1(\nabla^-) - p_1(D_B) \right) = - \frac{|A|^2 - |B|^2}{4} \left[ 8 - 3\alpha' \Delta e^{-2f} \right] e^{1234} = 0.$$

I.  $|A|^2 - |B|^2 = 0$  and  $\Delta e^{2f} + 2|A|^2 = 0$ . The torsion is closed, the anomaly condition is trivially satisfied, both  $\nabla^-$  and  $D_B$  are  $G_2$ -instantons. A particular solution defined in the unit ball is:

$$e^{2f} = \frac{|A|^2}{4} (1 - |x|^2)$$

II.  $|A|^2 - |B|^2 \neq 0$  - the anomaly condition is non-trivial.

Take  $B = 0$ , the zero matrix, we arrive to the next two equations for the dilaton  $f$ :

$$\Delta e^{2f} = 0, \quad \Delta e^{-2f} = 8/(3\alpha').$$

The solution with a singularity is given by

$$e^{2f} = \frac{3\alpha'}{|x-b|^2}, \quad x, b \in \mathbb{R}^4, b = \text{const.}$$

Logarithmic radial coordinates near the singularity imply the metric induced on  $\mathbb{R}^4$  is complete.

Taking the singularity at the origin, in the coordinate

$$q = \sqrt{3\alpha'}/2 \ln(|x|^2/3\alpha') = -\sqrt{3\alpha'} f$$

the dilaton and the 4 –  $D$  metric can be expressed as follows

$$\bar{g}_H = \sum_{i=1}^4 e^{2f} (e^i)^2 = dq^2 + 3\alpha' ds_3^2, \quad f = -q\sqrt{3\alpha'},$$

where  $ds_3^2$  is the metric on the unit 3 sphere in the 4- dimensional Euclidean space.

## Theorem

*The non-compact complete simply connected manifold  $(K_A, \bar{\Theta}, \nabla^-, D_O, f)$  described above is a complete  $G_2$  manifold which solves the Strominger system with non-constant dilaton  $f$  determined by the fundamental solution of the laplacian on  $R^4$ , non-zero flux  $H = \bar{T}$  and non-flat instanton  $D_O$  using the first Pontrjagin form of  $\nabla^-$  and positive  $\alpha'$  solving the heterotic equations of motion (1) up to the first order of  $\alpha'$ .*

# The geometric model in D=6, Goldstein-Prokushkin'04

$(M, J, g)$  - Hermitian 6-manifold with Kaehler form -  $F(\cdot, \cdot) = g(\cdot, J\cdot)$  and Lee form  $\theta(\cdot) = \delta F(J\cdot)$ .  
The flux  $H$ , i.e. the torsion of the connection  $\nabla^+$  preserving the hermitian structure  $(J, g)$ :

$$H = T = d^c F, \quad \text{where} \quad d^c F(X, Y, Z) = -dF(JX, JY, JZ).$$

An  $SU(3)$ -structure - an additional non-degenerate  $(3,0)$ -form  $\Psi = \Psi^+ + \sqrt{-1}\Psi^-$ , satisfying the compatibility conditions  $F \wedge \Psi^\pm = 0$ ,  $\Psi^+ \wedge \Psi^- = \frac{2}{3}F \wedge F$ .

The necessary and sufficient condition for the existence of solutions to the first two equations in (3) derived by Strominger imply that the 6-manifold should be a complex conformally balanced manifold (the Lee form  $\theta = 2d\phi$ ) with non-vanishing holomorphic volume form  $\Psi$  satisfying

$$2F \lrcorner dF + \Psi^+ \lrcorner d\Psi^+ = 0,$$

The geometric model in dimension six - the total space of a  $\mathbb{T}^2$ -bundle over  $M^4$ : It is non-Kaehler and it has an  $SU(3)$ -structure

$$g = g_{CY} + \eta_1^2 + \eta_2^2, \quad F = \omega_1 + \eta_1 \wedge \eta_2, \quad \Psi^+ = \omega_2 \wedge \eta_1 - \omega_3 \wedge \eta_2, \quad \Psi^- = \omega_2 \wedge \eta_2 + \omega_3 \wedge \eta_1,$$

The  $SU(3)$  structure solves the first two Killing spinor equations in (3) with constant dilaton. For any smooth function  $f$  on  $M^4$ , the  $SU(3)$ -structure on  $M^6$  given by

$$F = e^{2f}\omega_1 + \eta_1 \wedge \eta_2, \quad \Psi^+ = e^{2f}[\omega_2 \wedge \eta_1 - \omega_3 \wedge \eta_2], \quad \Psi^- = e^{2f}[\omega_2 \wedge \eta_2 + \omega_3 \wedge \eta_1]$$

solves the first two Killing spinor equations in (3) with non-constant dilaton  $\phi = 2f$ .

The existence of  $\nabla^+$ -parallel spinor in dimension 5 determines an almost contact metric structure and, equivalently, a reduction of the structure group  $SO(5)$  to  $SU(2)$ .

**Almost contact metric structure** - an odd dimensional manifold  $M^{2k+1}$ , a Riemannian metric  $g$ , a unit vector field  $\xi$ , its dual 1-form  $\eta$  and an endomorphism  $\psi$  of the tangent bundle such that

$$\psi(\xi) = 0, \quad \psi^2 = -id + \eta \otimes \xi, \quad g(\psi \cdot, \psi \cdot) = g(\cdot, \cdot) - \eta \otimes \eta.$$

The Reeb vector field  $\xi$  is determined by the equations  $\eta(\xi) = 1, \quad \xi \lrcorner d\eta = 0$ . The Nijenhuis tensor  $N$ , the fundamental form  $F$  and the Lee form  $\theta$  are defined by

$$N = [\psi \cdot, \psi \cdot] + \psi^2[\cdot, \cdot] - \psi[\psi \cdot, \cdot] - \psi[\cdot, \psi \cdot] + d\eta \otimes \xi, \quad F(\cdot, \cdot) = g(\cdot, \psi \cdot), \quad \theta = \frac{1}{2} F \lrcorner dF.$$

- Friedrich-Ivanov'03: The gravitino and the dilatino equation admit a solution in dimension five iff the Nijenhuis tensor is totally skew-symmetric, the Reeb vector field  $\xi$  is a Killing and the next equalities hold

$$2d\phi = \theta, \quad *_{\mathbb{H}} d\eta = -d\eta$$

$*_{\mathbb{H}}$  denotes the Hodge operator acting on the 4-dimensional space  $\mathbb{H} = \text{Ker } \eta$ .

There is NO solution on Sasakian manifold

## Reduction of the structure group $SO(5)$ to $SU(2)$ :

An  $SU(2)$ -structure on a 5-dimensional manifold  $M$  is  $(\eta, F = \omega_1, \omega_2, \omega_3)$ , where  $\eta$  is a 1-form dual to  $\xi$  via the metric and  $\omega_s, s = 1, 2, 3$ , are 2-forms on  $M$  satisfying

$$\omega_s \wedge \omega_t = \delta_{st} v, \quad v \wedge \eta \neq 0, \quad X \lrcorner \omega_1 = Y \lrcorner \omega_2 \Rightarrow \omega_3(X, Y) \geq 0.$$

The 2-forms  $\omega_s, s = 1, 2, 3$ , can be chosen to form a basis of the  $\mathbb{H}$ -self-dual 2-forms.

Fernandez-I-Ugarte-Villacampa'11 - The first two equations in (3) admit a solution in dimension five exactly when there exists a five dimensional manifold  $M$  endowed with an  $SU(2)$ -structure  $(\eta, F = \omega_1, \omega_2, \omega_3)$  satisfying the structure equations:

$$d\omega_s = 2df \wedge \omega_s, \quad *_\mathbb{H}d\eta = -d\eta, \quad df(\xi) = 0.$$

The flux  $H$  is

$$H = T = \eta \wedge d\eta + 2d^\psi f \wedge F, \quad \text{where} \quad d^\psi f(X) = -df(\psi X).$$

The dilaton  $\phi$  is equal to  $\phi = 2f$ , the metric has the form  $g_f = e^{2f} g_{|\mathbb{H}} + \eta \otimes \eta$ .

$S^1$  bundles over a conformally hyper-Kähler manifold - guaranties solution to the first two equations in (3).

The  $SU(2)$ -instanton condition in  $D=5$  reads:

$$(F^A)_j^i(\psi E_k, \psi E_l) = (F^A)_j^i(E_k, E_l), \quad \sum_{k=1}^5 (F^A)_j^i(E_k, \psi E_k) = 0$$

Recall: The quaternionic Heisenberg group  $K_A$  is determined by the Lie algebra  $\mathfrak{k}_A$ :

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^{4+i} = \sum_{j=1}^3 a_{ij}\sigma_j, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Consider the matrix

$$A_\varepsilon = \begin{pmatrix} 0 & b & 0 \\ a & 0 & -b \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

We have  $de^{\bar{7}} = de^7_\varepsilon = \varepsilon\sigma_3 \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . In the limit,  $\mathfrak{k}_A = \mathfrak{h}_5 \times \mathbb{R}$ .

Write the  $G_2$ -form in the usual way as

$$\bar{\Theta}_\varepsilon = \bar{F} \wedge e^7_\varepsilon + \bar{\Psi}^+, \quad \bar{F} = e^{2f}\omega_1 + e^{56}, \quad \bar{\Psi}^+ = e^{2f}(\omega_2 \wedge e^5 - \omega_3 \wedge e^6)$$

and  $\bar{\Psi}^- = e^{2f}(\omega_2 \wedge e^6 + \omega_3 \wedge e^5)$ .

In the limit  $\varepsilon \rightarrow 0$ , the forms  $\bar{F}$ ,  $\bar{\Psi}^\pm$  define an  $SU(3)$  structure  $(\bar{F}, \bar{\Psi}^\pm)$  on a six dimensional Lie group determined by the Lie algebra  $\mathfrak{h}_5$  which is a  $\mathbb{T}^2$  bundle over  $\mathbb{T}^4$  (corresponding to  $f = 0$ ).

This  $SU(3)$  structure solves the first two Killing spinor equations.

The curvature 2-forms  $((\Omega_\varepsilon^-)^i_{\bar{j}} \rightarrow 0$  for all  $i$ ) converge to those of the  $\nabla^-$  of the  $SU(3)$  case. The  $G_2$  instantons converge to the  $SU(3)$ -instantons.

The seven dimensional anomaly cancellation conditions turn into the anomaly cancellation conditions for the corresponding six dimensional structures. As a consequence we obtain the six-dimensional solutions with non-constant dilaton.



## Contraction-Five dimensional solutions

Consider the matrix:

$$A_\varepsilon \stackrel{\text{def}}{=} \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$$

Letting  $\varepsilon \rightarrow 0$  we have

$$d\bar{e}^i = e_\varepsilon^j = d\varepsilon \sigma^i \rightarrow 0, \quad i = 6, 7.$$

The limit is the complex heisenberg group in  $D=5$ ,  $\mathfrak{h}(2, 1)$  with structure equations

$$de^j = 0, \quad j = 1, 2, 3, 4, \quad de^5 = \sum_{i=1}^3 a_i \sigma_i, \quad a_i \in \mathbb{R}, \quad (a_1, a_2, a_3) \neq (0, 0, 0).$$

We consider  $\nabla^-$  as a contraction limit of the  $G_2$ -solutions.

This results in five dimensional solutions with non-constant dilaton. Applying the limit one gets precisely the expression for the torsion in dimension five

The Pontrjagin form of the  $\nabla^-$  connection in  $D=5$  is obtained as a limit of that for  $D=7$ .

The connection forms and the corresponding curvature 2-forms (notice that  $(\Omega_\varepsilon^-)_{\bar{6}}^{\bar{i}} \rightarrow 0$  and  $(\Omega_\varepsilon^-)_{\bar{7}}^{\bar{i}} \rightarrow 0$  for all  $i$ ) converge to those of the  $\nabla^-$  connection of the  $SU(2)$  structure in  $D=5$ .

The seven dimensional anomaly cancellation conditions turn into the anomaly cancellation conditions for the corresponding five dimensional structures. As a consequence we turn to five dimensional solutions with non-constant dilaton.

The  $SU(2)$  instanton  $D_\Lambda$  below corresponds to the instanton obtained from the one in  $D=7$  setting the last two columns equal to zero and letting  $\varepsilon \rightarrow 0$ :

## Lemma

Let  $D_\Lambda, \Lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ , be the linear connection on the Lie group  $H(2, 1)$  whose possibly non-zero 1-forms are given as follows

$$(\omega^{D_\Lambda})_{\bar{2}}^{\bar{1}} = -(\omega^{D_\Lambda})_{\bar{1}}^{\bar{2}} = -(\omega^{D_\Lambda})_{\bar{4}}^{\bar{3}} = (\omega^{D_\Lambda})_{\bar{3}}^{\bar{4}} = \lambda_1 \bar{e}^5,$$

$$(\omega^{D_\Lambda})_{\bar{3}}^{\bar{1}} = -(\omega^{D_\Lambda})_{\bar{1}}^{\bar{3}} = (\omega^{D_\Lambda})_{\bar{4}}^{\bar{2}} = -(\omega^{D_\Lambda})_{\bar{2}}^{\bar{4}} = \lambda_2 \bar{e}^5,$$

$$(\omega^{D_\Lambda})_{\bar{4}}^{\bar{1}} = -(\omega^{D_\Lambda})_{\bar{1}}^{\bar{4}} = -(\omega^{D_\Lambda})_{\bar{3}}^{\bar{2}} = (\omega^{D_\Lambda})_{\bar{2}}^{\bar{3}} = \lambda_3 \bar{e}^5.$$

Then,  $D_\Lambda$  is an  $SU(2)$ -instanton with respect to the  $SU(2)$  structure defined by  $(e^5, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)$ .

## Theorem

Let  $(e^5, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)$  be the  $SU(2)$  structure on the Lie group  $H(2, 1)$ .

The conformally compact five manifold  $M^5 = (\Gamma \backslash H(2, 1), \eta^5, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \nabla^-, D_\Lambda, f)$  is a conformally quasi-Sasakian five manifold which solves the Strominger system with non-constant dilaton  $f$ , non-trivial flux  $H = \bar{T}$  and non-flat instanton  $D_\Lambda$  using the first Pontrjagin form of  $\nabla^-$  and negative  $\alpha'$ . The dilaton  $f$  depends on one variable and is determined as a real slice of the Weierstrass' elliptic function. In addition,  $M^5$  satisfies the heterotic equations of motion (1) up to first order of  $\alpha'$ .

## Theorem

The non-compact simply connected five manifold  $(H(2, 1), e^5, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \nabla^-, D_O, f)$  is a complete conformally quasi-Sasakian five manifold which solves the Strominger system with non-constant dilaton  $f$  determined by the fundamental solution of the laplacian in  $R^4$ , non-trivial flux  $H = \bar{T}$ , non-flat instanton  $D_O$  using the first Pontrjagin form of  $\nabla^-$  and positive  $\alpha'$ .

The complete five manifold  $(H(2, 1), e^5, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \nabla^-, D_O, f)$  satisfies the heterotic equations of motion (1) up to first order of  $\alpha'$ .