The ASK/PSK-correspondence and the r-map

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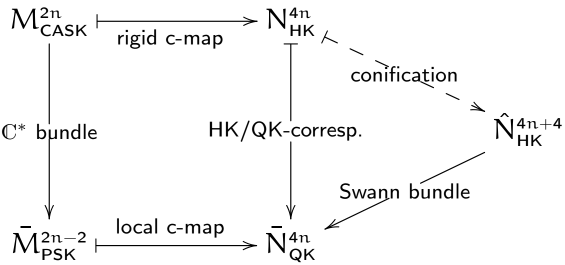
We are interested in dimensional reduction of $\mathcal{N} = 2$ supersymmetric theories, in particular to the scalar geometries of vector multiplets.

	rigid $\mathcal{N} = 2$ SUSY	$\mathcal{N} = 2$ SUGRA
5d vector multiplets	affine special real (ASR)	projective special real (PSR)
	↓ rigid r-map	↓ local r-map
4d vector multiplets	affine special Kähler (ASK)	projective special Kähler (PSK)
	↓ rigid c-map	↓ local c-map
3d hyper-multiplets	hyper Kähler (HK)	quaternionic Kähler (QK)

c-map (Ferrara, Sabharwal ’90 [FS90])



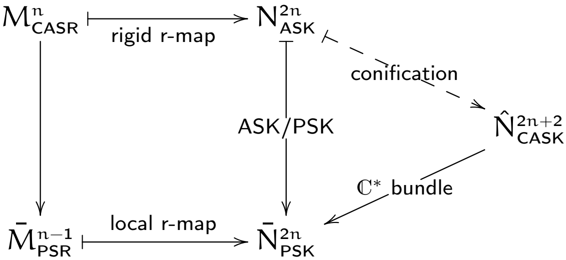
d	rigid	local
5	ASR	PSR
4	ASK	PSK
3	HK	QK

- The HK/QK-correspondence[1] constructs a quaternionic Kähler manifold out of a hyper-Kähler manifold with symmetry of equal dimension. The metric is explicitly given.
- Construction is more general. Gives one-parameter deformation of the supergravity c-map metric (one-loop deformation[2]) preserving completeness.

[1] Hay08; ACM13; ACDM15.

[2] LSV06; APP11.

r-map (de Wit, Van Proeyen '92, [WV92])



d	rigid	local
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- "Naive" approach analogous to HK/QK fails because of no distinguished symmetry and problems with signatures.
- Establish ASK/PSK-correspondence, assigning (locally) to any ASK manifold a PSK manifold. Gives one-parameter family of metrics, interpreted as perturbative α'-correction.

r-map (de Wit, Van Proeyen '92, [WV92])

- (Projective) special real geometry (i.e., $d = 5$) is defined by a cubic homogeneous polynomial $h(x) = a_{ijk}x^i x^j x^k$ defined on a conic domain $U \subset \mathbb{R}^n$.
- Let $M = \mathbb{R}^n + iU$ with $z = y + ix$.
- Rigid r-map assigns to h ASK geometry generated by holomorphic prepotential

$$F(z) = -h(z).$$

- Sugra r-map assigns to h CASK/PSK geometry generated by homogeneous holomorphic prepotential

$$\hat{F}(Z^0, Z) = -(Z^0)^2 h(Z/Z^0)$$

on $\hat{M} = \mathbb{C}^* \times M$ with homogeneous coordinates $(Z^0, Z = Z^0 z)$.

Idea

"Local conification" of special Kähler geometry on M generated by prepotential $F(z)$ by assigning to it the homogeneous prepotential $\hat{F} = (Z^0)^2 F(Z/Z^0)$ on the cone $\hat{M} = \mathbb{C}^* \times M$.

Questions

- How does this depend on the coordinates?
- How does this depend on the choice of prepotential (unique up to constant)? E.g., $F \mapsto F + C$, $C \in \mathbb{C}$, then

  $$\hat{F}' = \hat{F} + (Z^0)^2 C$$

  changes metric.
- Can this construction be globalized?

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Definition

An *affine special Kähler* (ASK) manifold (M, J, g, ∇), $\dim_{\mathbb{C}} M = n$, is a pseudo Kähler manifold (M, J, g), $\omega = g(\cdot, J\cdot)$, such that

1. ∇ is a flat torsion-free connection,
2. $\nabla\omega = 0$,
3. $d^{\nabla}J = 0$, i.e., $(\nabla_X J)Y = (\nabla_Y J)X$.

Definition

An ASK manifold $(\hat{M}, \hat{J}, \hat{g}, \hat{\nabla})$, $\dim_{\mathbb{C}} \hat{M} = n + 1$, is called *conical* (CASK) if there is a vector field ξ such that $\hat{g}(\xi, \xi) \neq 0$ and $\hat{\nabla}\xi = \hat{D}\xi = \mathrm{id}$.

Definition

If ξ induces a principal $\mathbb{C}^*$-action, then the quotient $\overline{M} := \hat{M}/\mathbb{C}^*$ is called *projective special Kähler* (PSK).

Local description [ACD02]

An ASK manifold M can locally be realized as a Kählerian Lagrangian immersion (KLI), i.e., $\phi : U \subset M \to \mathbb{C}^{2n}$ holomorphic immersion such that

1. $\phi^*\Omega = 0$, where $\Omega = dz^i \wedge dw_i$ symplectic form of $\mathbb{C}^{2n}$,
2. $\phi^*\gamma = \phi^*(i\Omega(\cdot, \bar{\cdot}))$ is non-degenerate.

Simply transitive action of $\mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ on set of KLIs.

Definition

If $(z, w) = \phi : U \subset M \to \mathbb{C}^{2n}$ is a KLI, we call a holomorphic function $F : U \to \mathbb{C}$ a *prepotential* of ϕ if $dF = w_i \, dz^i$.

Remark

1. Locally, if z, w are coordinates, then $z(M) \subset \mathbb{C}^n$, $\phi = dF$.
2. $\mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ does **not** act on F!

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Group extension

- Define $G_{SK} := \mathrm{Sp}(\mathbb{R}^{2n}) \ltimes \mathrm{Heis}_{2n_1}(\mathbb{C})$.
- G_{SK} is a central extension:

$$1 \to \mathbb{C} \to G_{SK} \xrightarrow{\bar{\rho}} \mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n}) \to 1$$

Proposition

We can 'lift' $\bar{\rho}$ to **linear** *representation on* $\mathbb{C}^{2n+2} = \mathbb{C}^2 \times \mathbb{C}^{2n}$

$$\rho : G_{SK} \to \mathrm{Sp}(\mathbb{C}^{2n+2})$$
$$x = (X, s, v) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ -2s & 1 & \hat{v}^t \\ v & 0 & X \end{pmatrix}, \hat{v} := X^t \Omega v,$$

where $X \in \mathrm{Sp}(\mathbb{R}^{2n})$, $s \in \mathbb{C}$, $v \in \mathbb{C}^{2n}$.

Proposition

G_{SK} *acts simply transitively on set* $\mathcal{F}(U)$ *of special Kähler pairs* (ϕ, F)

$$x \cdot (\phi, F) := (\bar{\rho}(x) \circ \phi, x \cdot F)$$
$$x \cdot F := F - \frac{1}{2} z^j w_j + \frac{1}{2} z'^j w'_j + \frac{1}{2} (\bar{\rho}(x) \circ \phi)^* \Omega(\cdot, v) - s,$$

where $(z, w) = \phi$, $(z', w') = \bar{\rho}(x) \circ \phi$.

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- Given a special Kähler pair (ϕ, F) on U, we define

$$\begin{aligned}\Phi : \hat{U} := \mathbb{C}^* \times U &\to \mathbb{C}^{2n+2} \\ (Z^0, p) &\mapsto Z^0(1, f(p), \phi(p)),\end{aligned}$$

where $f = 2F - z^j w_j$, $(z, w) = \phi$.

- We call Φ the conification of (ϕ, F), write $\mathrm{con}(\phi, F) := \Phi$.

Theorem

1. *Φ is a Lagrangian immersion iff F is a prepotential of ϕ.*
2. *Equivariance:* $\mathrm{con}(x \cdot (\Phi, F)) = \rho(x) \circ \mathrm{con}(\phi, F)$.
3. *If Φ is Kählerian and $\mathrm{Im}(f + \bar{z}^j w_j) \neq 0$, then Φ induces CASK structure on $\hat{U}$ (and, hence, PSK structure on $\bar{U} = \hat{M}/\mathbb{C}^*$).*
4. *CASK structure depends only on equivalence class in $\mathcal{F}(U)/G$ where $G = \mathrm{Sp}(\mathbb{R}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{R}) \subset G_{SK}$.*

- The (local) ASK/PSK-correspondence assigns to U given (ϕ, F) non-degenerate (i.e., $\mathrm{con}(\phi, F)$ induces CASK structure) a projective special Kähler structure on $\overline{U} \cong U$.

Example

- Let (ϕ, F) be a non-degenerate SK pair on U, $(z, w) := \phi$. Then the ASK metric on U is given by the Kähler potential

$$K = \mathrm{Im}(\bar{z}^j w_j).$$

- The PSK metric of the ASK/PSK-correspondence is then given by the Kähler potential

$$K' = -\log|\mathrm{Im}\, f + K|,$$

for $f = 2F - z^j w_j$.

Theorem

Applying the (local) ASK/PSK correspondence to

$$(\phi_c, F_c) := (dF, F - 2ic), c \in \mathbb{R},$$

with $F(z) = -h(z)$, $c \in \mathbb{R}$, *on* $M_c \subset M = \mathbb{R}^n + iU$, *gives a PSK manifold* $(\bar{M}_c, \bar{g}_c)$ *for each* $c \in \mathbb{R}$.

- *If* $c = 0$ *we recover the sugra r-map metric.*
- *If* $cc' > 0$, *then* $(\bar{M}_c, \bar{g}_c) \cong (\bar{M}_{c'}, \bar{g}_{c'})$.
- *If the PSR manifold defined by* h *is complete, then* $(\bar{M}_c, \bar{g}_c)$ *is complete for* $c < 0$.
- *Correction to metric can be interpreted as an* α' *correction.*

- We show that every ASK manifold admits a flat principal G_{SK}-bundle $\pi : P \to M$, connection θ, with fibers consisting of germs of special Kähler pairs.
- If $u := (\phi, F)$ is non-degenerate (i.e. $\mathrm{con}(\phi, F)$ induces CASK structure), define $\mathrm{dom}(u) \subset M$ to be the set on which analytic continuation of u is non-degenerate.
- Analytic continuation corresponds to parallel transport of u in P.

Theorem (ASK/PSK-correspondence)

If $\mathrm{Hol}(\theta) \subset G$ *then* $\hat{M}_u = \mathbb{C}^* \times \mathrm{dom}(u)$ *has a CASK-structure (and, hence, a PSK structure on* $\overline{M}_u = \hat{M}_u/\mathbb{C}^*$*).*

Thank you for your attention!

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