Non-Linear Anti-Involutive Symmetries of Black Hole Entropy

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see also (other applications)

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Summary

Maxwell-Einstein-Scalar Gravity Theories

Symmetric Scalar Manifolds :
Application to (**Super)Gravity** and **Extremal Black Holes**

Attractor Mechanism

Freudenthal Duality

Groups of type E_7

E.m. Duality Orbits and **Pre-Homogeneous Vector Spaces**

Hints for the Future…

Maxwell-Einstein-Scalar Theories

[dubbed ESM theories in Shahbazi's talk]

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\, \partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\, \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}$$

D=4 Maxwell-Einstein-scalar system (with no potential)

[may be the bosonic sector of **D=4** (ungauged) sugra]

$$H := \left(F^\Lambda, G_\Lambda\right)^T;$$

$$^*G_{\Lambda|\mu\nu} := 2\frac{\delta\mathcal{L}}{\delta F^{\Lambda|\mu\nu}}.$$

Abelian 2-form field strengths

static, spherically symmetric, asympt. flat, **extremal BH**

$$ds^2 = -e^{2U(\tau)}dt^2 + e^{-2U(\tau)}\left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}\left(d\theta^2 + \sin\theta d\psi^2\right)\right]$$

$$\tau := -1/r$$

$$\mathcal{Q} \;:\; = \int_{S^2_\infty} H = \left(p^\Lambda, q_\Lambda\right)^T;$$

$$p^\Lambda \;:\; = \frac{1}{4\pi}\int_{S^2_\infty} F^\Lambda, \quad q_\Lambda = \frac{1}{4\pi}\int_{S^2_\infty} G_\Lambda.$$

dyonic vector of e.m. fluxes (**BH charges**)

$$S_{D=1} = \int [(U')^2 + g_{ij}\varphi'^i \varphi'^j + e^{2U} V_{BH}(\varphi(\tau), \mathcal{Q})] d\tau$$

$$' \equiv \frac{d}{d\tau}$$

reduction D=4 →D=1 :effective 1-dimensional (radial) Lagrangian

$$V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T \mathcal{M}(\varphi)\, \mathcal{Q},$$

BH effective potential

Ferrara,Gibbons,Kallosh

eoms

$$\begin{cases} \frac{d^2U}{d\tau^2} = e^{2U} V_{BH}; \\ \frac{d^2\varphi^i}{d\tau^2} = g^{ij} e^{2U} \frac{\partial V_{BH}}{\partial \varphi^j}. \end{cases}$$

in N=2 ungauged sugra, hyper mults. decouple, and we thus disregard them : scalar fields belong to vector mults.

Attractor Mechanism : $\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau \to -\infty} \varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$

conformally flat geometry $AdS_2 \times S^2$ near the horizon

$$ds^2_{\text{B-R}} = \frac{r^2}{M^2_{\text{B-R}}} dt^2 - \frac{M^2_{\text{B-R}}}{r^2}\left(dr^2 + r^2 d\Omega\right)$$

near the horizon, the scalar fields are **stabilized** purely in terms of charges

$$S = \frac{A_H}{4} = \pi \left. V_{BH} \right|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T \mathcal{M}_H \mathcal{Q}$$

Bekenstein-Hawking entropy-area formula for extremal dyonic BH

Symmetric Scalar Manifolds

A remarkable class of Einstein-Maxwell-scalar theories is endowed with scalar manifolds which are **symmetric cosets G/H**

[in presence of local SUSY :**N>2** : general, **N=2** : particular, **N=1** : special cases]

H = isotropy group = *linearly* realized; scalar fields sit in an **H**-repr.

G = (global) **electric-magnetic duality** group, on-shell symmetry

General Features in D=4

The 2-form field strengths (F,G) vector and the BH e.m. charges sit in a **G**-repr. **R** which is **symplectic** :

$$\exists ! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R};$$

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \equiv \mathcal{Q}_1^M \mathcal{Q}_2^N \mathbb{C}_{MN} = - \langle \mathcal{Q}_2, \mathcal{Q}_1 \rangle$$

$$\mathbb{C} = \begin{pmatrix} \mathbf{0}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbf{0}_n \end{pmatrix}$$

symplectic product

$$\begin{aligned} G &\subset Sp(2n, \mathbb{R}); \\ \mathbf{R} &= \mathbf{2n} \end{aligned}$$

in physics : **Gaillard-Zumino** embedding (generally maximal, but not symmetric)

[application of a Th. of Dynkin; more on this later…]

Let's reconsider the starting **Maxwell-Einstein-scalar** Lagrangian density

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\, \partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\, \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}$$

…and introduce the following real 2n x 2n matrix :

$$\mathcal{M} = \begin{pmatrix} \mathbb{I} & -R \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -R & \mathbb{I} \end{pmatrix} = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}$$

$$\mathcal{M} = \mathcal{M}(R, I) = \mathcal{M}\left(\mathrm{Re}(\mathcal{N}), \mathrm{Im}(\mathcal{N})\right).$$

$$\mathcal{M}^T = \mathcal{M} \qquad \mathcal{M}\mathbb{C}\mathcal{M} = \mathbb{C}$$

$$\mathcal{M} = -\left(\mathbf{L}\mathbf{L}^T\right)^{-1} = -\mathbf{L}^{-T}\mathbf{L}^{-1},$$

L = element of the **Sp(2n,R)**-bundle over the scalar manifold
(= *coset representative* for homogeneous spaces **G/H**)

By virtue of this matrix, one can introduce a (scalar-dependent) **anti-involution in *any*** Maxwell-Einstein-scalar gravity theory in **D=4**:

$$\mathcal{S}(\varphi) \quad : \quad = \mathbb{C}\mathcal{M}(\varphi)$$

$$\mathcal{S}^2(\varphi) \quad = \quad \mathbb{C}\mathcal{M}(\varphi)\mathbb{C}\mathcal{M}(\varphi) = \mathbb{C}^2 = -\mathbb{I},$$

Ferrara,AM,Yeranyan; Borsten,Duff,Ferrara,AM

In turn, this allows to define an **anti-involution** on the dyonic charge vector Q, which has been named (scalar-dependent) **Freudenthal duality (F-duality)**

$$\mathfrak{F}(\mathcal{Q}) := -\mathcal{S}(\varphi)(\mathcal{Q}).$$

$$\mathfrak{F}^2 \quad = \quad -Id.$$

By recalling $V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\varphi)\mathcal{Q},$

F-duality is the **symplectic gradient** of the **effective BH potential** :

$$\boxed{\mathfrak{F} : \mathcal{Q} \rightarrow \mathfrak{F}(\mathcal{Q}) := \mathbb{C}\frac{\partial V_{BH}}{\partial \mathcal{Q}}.}$$

All this enjoys a nice physical interpretation when evaluated **at the BH horizon** :

Attractor Mechanism $\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau \to -\infty} \varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$

Bekenstein-Hawking entropy $S = \frac{A_H}{4} = \pi V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2} \mathcal{Q}^T \mathcal{M}_H \mathcal{Q}$

By evaluating the matrix M at the horizon : $\lim_{\tau \to -\infty} \mathcal{M}(\varphi(\tau)) = \mathcal{M}_H(\mathcal{Q})$

one can define the **horizon F-duality** as:

$$\lim_{\tau \to -\infty} \mathfrak{F}(\mathcal{Q}) =: \mathfrak{F}_H(\mathcal{Q}) = -\mathbb{C}\mathcal{M}_H \mathcal{Q} = \frac{1}{\pi}\mathbb{C}\frac{\partial S_{BH}}{\partial \mathcal{Q}} =: \tilde{\mathcal{Q}},$$

$$\mathfrak{F}^2_H(\mathcal{Q}) = \mathfrak{F}_H(\tilde{\mathcal{Q}}) = -\mathcal{Q}$$

It is a **non-linear (scalar-independent) anti-involutive map** on Q (hom of degree 1)

Bek.-Haw. entropy is **invariant** under its own **non-linear symplectic gradient** (*i.e.*, **F-duality**) :

$$\boxed{S(\mathcal{Q}) = S(\mathfrak{F}_H(\mathcal{Q})) = S\left(\frac{1}{\pi}\mathbb{C}\frac{\partial S}{\partial \mathcal{Q}}\right) = S(\tilde{\mathcal{Q}})}$$

This can be extended to include *at least* **all quantum corrections** with **homogeneity 2** or **0** in the BH charges Q

Ferrara, AM, Yeranyan
(and late Raymond Stora)

Lie groups **of type E_7** : (G,**R**)

Brown (1967);
Garibaldi; Krutelevich;
Borsten,Duff *et al.*
Ferrara,Kallosh,AM;
AM,Orazi,Riccioni

- the (ir)repr. **R** is **symplectic** :

$$\exists ! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}; \qquad \langle Q_1, Q_2 \rangle \equiv Q_1^M Q_2^N \mathbb{C}_{MN} = -\langle Q_2, Q_1 \rangle ;$$

symplectic product

- the (ir)repr. admits a completely symmetric **invariant rank-4** tensor

$$\exists\ K_{MNPQ} = K_{(MNPQ)} \equiv \mathbf{1} \in (\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R})_s$$ (K-tensor)

G-invariant quartic polynomial

$$I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q =: \epsilon\, |I_4| , \qquad \boxed{S_{BH} = \pi \sqrt{|I_4|}}$$

- defining a **triple map** in **R** as

$$T : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R} \qquad \langle T(Q_1, Q_2, Q_3), Q_4 \rangle \equiv K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q$$

it holds $\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle\, K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q$

this third property makes a **group of type E_7** amenable to a description as automorphism group of a **Freudenthal triple system (FTS)**

Evidence : all electric-magnetic duality groups of D=4 ME(S)GT's with **symmetric** scalar manifolds (and *at least* **8** supersymmetries) are **of type E_7**

$N = 2$

G	R
$U(1,n)$	$(\mathbf{1+n})$
$SL(2,\mathbb{R}) \times SO(2,n)$	$(\mathbf{2,2+n})$
$SL(2,\mathbb{R})$	$\mathbf{4}$
$Sp(6,\mathbb{R})$	$\mathbf{14'}$
$SU(3,3)$	$\mathbf{20}$
$SO^*(12)$	$\mathbf{32}$
$E_{7(-25)}$	$\mathbf{56}$

N	G	R
3	$U(3,n)$	$(\mathbf{3+n})$
4	$SL(2,\mathbb{R}) \times SO(6,n)$	$(\mathbf{2,6+n})$
5	$SU(1,5)$	$\mathbf{20}$
8	$E_{7(7)}$	$\mathbf{56}$

(E_7, 912 – embedding tensor N=8/N=2 exc, D=4) satisfies the first two Brown's axioms, *but **not** the third one!*

"degenerate" groups **of type E_7**

$$I_4(p,q) = (I_2(p,q))^2$$

$$S_{BH} = \pi\sqrt{|I_4(p,q)|} = \pi\,|I_2(p,q)|\,.$$

In D=4 sugras with the previous electric-magnetic duality group **of type $\mathbf{E_7}$**, the **G**-invariant **K-tensor** determining the extremal BH Bekenstein-Hawking entropy

$$\boxed{S_{BH} = \pi\sqrt{|I_4|}}$$

$$I_4 := K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q =: \epsilon\,|I_4|\,,$$

can generally be expressed as adjoint-trace of the product of G-generators (dim **R** = 2n, and dim **Adj** = d)

$$K_{MNPQ} = -\frac{n\,(2n+1)}{6d}\left[t^{\alpha}_{MN}t_{\alpha|PQ} - \frac{d}{n(2n+1)}\mathbb{C}_{M(P}\mathbb{C}_{Q)N}\right]$$

The **horizon F-duality** can be expressed in terms of the **K-tensor**

$$\mathfrak{F}_H(\mathcal{Q})_M = \tilde{\mathcal{Q}}_M = \frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}^M} = \epsilon\frac{2}{\sqrt{|I_4(\mathcal{Q})|}}K_{MNPQ}\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q$$

Borsten,Dahanayake,Duff,Rubens

and the **invariance** of the BH entropy under **horizon F-duality** can be recast as

$$\boxed{I_4\left(\mathcal{Q}\right) = I_4(\mathbb{C}\tilde{\mathcal{Q}}) = I_4\left(\mathbb{C}\frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}}\right)}$$

Are there other relevant symplectic matrices at the horizon ? **YES!**

$$\left(M_{-}^{H}\left(\mathcal{Q}\right)\right)^{T}\mathbb{C}M_{-}^{H}\left(\mathcal{Q}\right)=\epsilon\mathbb{C}$$ $$\epsilon:=I_{4}\left(\mathcal{Q}\right)/\left|I_{4}(\mathcal{Q})\right|$$

$$\left(M_{-}^{H}\left(\mathcal{Q}\right)\right)^{T}=M_{-}^{H}\left(\mathcal{Q}\right)$$ $$\mathcal{Q}^{T}M_{-}^{H}\left(\mathcal{Q}\right)\mathcal{Q}=-2\sqrt{\left|I_{4}(\mathcal{Q})\right|}$$

$$\boxed{M_{-|MN}^{H}=-\partial_{M}\partial_{N}\sqrt{\left|I_{4}(\mathcal{Q})\right|}=-\frac{1}{\pi}\partial_{M}\partial_{N}S_{BH}}$$

This matrix is nothing but (the opposite) of the **Hessian matrix** of the **BH entropy**

The transformation property under **horizon F-duality** reads $\mathfrak{F}_{H}\left(M_{-}^{H}\left(\mathcal{Q}\right)\right)=\epsilon M_{-}^{H}$

Analogous to the transformation property of the matrix defining the **horizon F-duality** :

$$\mathfrak{F}_{H}\left(\mathcal{M}_{H}\left(\mathcal{Q}\right)\right)=\epsilon\mathcal{M}_{H}\left(\mathcal{Q}\right)$$

This matrix is the (opposite of) the metric of a non-compact, pseudo-Riemannian, **rigid special Kaehler manifold** related to the **e.m. duality orbit** of BH e.m. charges, Which in turn is an example of **pre-homogeneous vector space** (**PVS**)

1st example : ***"large"* BPS e.m. dualiy orbit** in **maximal N=8, D=4 sugra**

$$N=8, D=4 : scalar\ manifold\ \mathrm{M}_{N=8} = \frac{E_{7(7)}}{SU(8)},\ dim_{\mathbb{R}} = 70,\ rank = 7$$

$$I_4 > 0 : \frac{1}{8}-BPS\ E_{7(7)}-orbit\ in\ \mathbf{56}\ repr.space : \mathcal{O}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)}}$$

$$\boxed{M^H_- = -\partial^2\sqrt{I_4} : metric\ of\ \mathcal{O}_{I_4>0} \times \mathbb{R}^+ = \frac{E_{7(7)}}{E_{6(2)}} \times \mathbb{R}^+;\ \ (n_+, n_-) = (30, 26)}$$

2nd example : ***"large"* non-BPS e.m. duality orbit** in **maximal N=8, D=4 sugra**

$$I_4 < 0 : non - BPS\ E_{7(7)}{-}orbit\ in\ \mathbf{56}\ repr.space : \mathcal{O}_{I_4<0} = \frac{E_{7(7)}}{E_{6(6)}}$$

$$\boxed{M^{H}_{-} = -\partial^2\sqrt{-I_4} : metric\ of\ \mathcal{O}_{I_4<0} \times \mathbb{R}^+ = \frac{E_{7(7)}}{E_{6(6)}} \times \mathbb{R}^+;\ \ (n_+, n_-) = (28, 28)}$$

zero character (holding for all $\mathbf{I_4<0}$ U-orbits)

$$\frac{E_{7(7)}}{E_{6(2)}} \times \mathbb{R}^+$$ and $$\frac{E_{7(7)}}{E_{6(6)}} \times \mathbb{R}^+$$ are non-compact, real forms of $$\frac{E_7}{E_6} \times GL(1)$$

Regular **Pre-Homogeneous Vector Space** (**PVS**) of type (29) in the classification by Sato and Kimura (‘77) :

(29) $(GL(1) \times E_7,\ \square \otimes \Lambda_6,\ V(1) \otimes V(56))$.

(i) $H \sim E_6$, (ii) $\deg f = 4$, (iii) $f(X) = T(x^\#, y^\#) - \xi N(x) - \eta N(y) - \frac{1}{4}(T(x,y) - \xi\eta)^2$ (see (1.16), or Proposition 52 in § 5).

A **PVS** is a finite-dimensional vector space ***V*** together with a subgroup ***G*** of **GL(*V*)** such that ***G*** has an **open**, **dense orbit** in ***V*** [Sato,Kimura; Knapp]

PVS are subdivided into two types, according to whether there exists a *homogeneous* polynomial **f** on ***V*** which is **invariant** under the semisimple part of ***G***.

In this case : **V = 56** (fundamental irrep. of **G=E$_7$**), **f** = **quartic** invariant polynomial **I$_4$**

H= isotropy (stabilizer) group = **E$_6$**

Manifestly E$_6$ -invariant expression of the quartic invariant **I$_4$** of the **56** of **E$_7$** :

much before (**’77** = almost contemporary to sugra) the expression introduced by Ferrara & Gunaydin (**‘97**) **!**

$$I_4\left(p^0, p^i, q_0, q_i\right) = -\left(p^0 q_0 + p^i q_i\right)^2 + 4\left[q_0 I_3(p) - p^0 I_3(q) + \left\{\frac{\partial I_3(p)}{\partial p}, \frac{\partial I_3(q)}{\partial q}\right\}\right]$$

Thank You!