Special holonomy groups in supergeometry

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Holonomy groups of connections in vector bundles

Let $E \to M$ be a vector bundle over a smooth manifold M,
$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ a connection on E.

$\gamma : [a, b] \to M$ a curve in M

$\tau_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}$ the parallel transport along γ

The holonomy group at the point x:

$$\mathrm{Hol}_x(\nabla) := \{\tau_\gamma \big| \gamma \text{ is a loop at } x\} \subset \mathrm{GL}(E_x) \simeq \mathrm{GL}(m, \mathbb{R}).$$

The restricted holonomy group at the point x:

$$\mathrm{Hol}^0_x(\nabla) := \{\tau_\gamma \big| \gamma \text{ is a loop at } x, \gamma \sim \mathrm{pt}_x\} \subset \mathrm{Hol}_x(\nabla).$$

Fact: $\mathrm{Hol}_x(\nabla) \subset \mathrm{GL}(E_x)$ *is a Lie subgroup,*
$\mathrm{Hol}^0_x(\nabla)$ *is the identity component of* $\mathrm{Hol}_x(\nabla)$.

The holonomy algebra at the point x:

$$\mathfrak{hol}_x(\nabla) := \mathrm{LA}\,\mathrm{Hol}_x(\nabla) = \mathrm{LA}\,\mathrm{Hol}^0_x(\nabla) \subset \mathfrak{gl}(E_x) \simeq \mathfrak{gl}(m, \mathbb{R}).$$

Theorem. (Ambrose, Singer, 1952)

$$\mathfrak{hol}_x(\nabla) = \{(\tau_\gamma)^{-1} \circ R_{\gamma(b)}(X, Y) \circ \tau_\gamma \Big| \gamma(a) = x,\ X, Y \in T_{\gamma(b)}M\}.$$

The fundamental principle:

$\{$ parallel sections $X \in \Gamma(E)\} \longleftrightarrow \{X_x \in E_x \,|\, Hol_x X_x = X_x\}$

($X \in \Gamma(E)$ is parallel if $\nabla X = 0$, or for any $\gamma : [a,b] \to M$, $\tau_\gamma X_{\gamma(a)} = X_{\gamma(b)}$)

The fundamental principle:

Let ∇ be a connection on TM

$\{$ parallel tensor fields P of type (p,q) $\}$
$\longleftrightarrow \{P_x \in \otimes^p_q T_xM \,|\, \mathrm{Hol}_x P_x = P_x\}$

Example: (M^n,g), $\nabla g = 0 \Rightarrow \mathrm{Hol} \subset \mathrm{O}(n)$;
(M^{2m},g) is Kählerian ($\exists\ J$, $\nabla J = 0$) $\Leftrightarrow \mathrm{Hol} \subset \mathrm{U}(m)$.

∇ is flat if locally there exist m point-wise independent parallel sections of E.

Theorem. ∇ is flat $\Leftrightarrow R = 0 \Leftrightarrow \mathfrak{hol}(\nabla) = 0$.

Holonomy of supermanifolds

Let $(\mathcal{M}, \mathcal{O}_\mathcal{M})$ be a supermanifold.
Let $\mathcal{E}$ be a locally free sheaf of supermodules over $\mathcal{O}_\mathcal{M}$ of rank $p|q$.

For $x \in M$ consider the fiber at x: $\mathcal{E}_x := \mathcal{E}(U)/(\mathcal{O}_\mathcal{M}(U))_x\mathcal{E}(U)$,

where $x \in U$ and $(\mathcal{O}_\mathcal{M}(U))_x \subset \mathcal{O}_\mathcal{M}(U)$ are functions vanishing at x.

For $X \in \mathcal{E}(U)$ consider the value $X_x \in \mathcal{E}_x$

Example. $\mathcal{E} = \mathcal{T}_\mathcal{M} \Rightarrow (\mathcal{T}_\mathcal{M})_x = T_x\mathcal{M}$ and $(T_x\mathcal{M})_{\bar{0}} = T_xM$

Let $\mathcal{E}$ be a locally free sheaf of supermodules over $\mathcal{O}_{\mathcal{M}}$ of rank $p|q$.

Consider the vector bundle $E = \cup_{x \in M} \mathcal{E}_x \to M$.

We get the projection $\sim: \mathcal{E}(U) \to \Gamma(U, E), \quad X \mapsto \tilde{X}, \quad \tilde{X}_x = X_x$

Let (e_A) $A = 1, ..., p+q$ be a basis of $\mathcal{E}(U)$

$X \in \mathcal{E}(U) \Rightarrow X = X^A e_A$ $(X^A \in \mathcal{O}_{\mathcal{M}}(U)) \Rightarrow \tilde{X} = \tilde{X}^A \tilde{e}_A$

$X \in \mathcal{E}(U)$ is not defined by its values!

Connection on $\mathcal{E}$: $\quad \nabla : \mathcal{T}_{\mathcal{M}} \otimes_{\mathbb{R}} \mathcal{E} \to \mathcal{E} \qquad |\nabla_\xi X| = |\xi| + |X|$,

$$\nabla_{f\xi} X = f \nabla_\xi X \quad \text{and} \quad \nabla_\xi f X = (\xi f) X + (-1)^{|\xi||f|} f \nabla_\xi X$$

Locally: $\nabla_{\partial_a} e_B = \Gamma^A_{aB} e_A, \quad \Gamma^A_{aB} \in \mathcal{O}_{\mathcal{M}}(U)$

$\tilde{\nabla} = (\nabla|_{\Gamma(TM) \otimes \Gamma(E)})^{\sim} : \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E)$ is a connection on E

$\tilde{\Gamma}^A_{iB}$ are Cristoffel symbols of $\tilde{\nabla}$

$\gamma : [a, b] \subset \mathbb{R} \to M \qquad \tau_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}$ the parallel displac. along γ (defined by $\tilde{\nabla}$).

$\tau_\gamma : \mathcal{E}_{\gamma(a)} \to \mathcal{E}_{\gamma(b)}$ is an isomorphism of vector superspaces.

Problem: Define holonomy of ∇ (it must give information about all parallel sections of $\mathcal{E}$!)

Example: Purely odd supermanifold:

$$\mathcal{M} = (\{x\}, \Lambda(q)),$$

$$\mathcal{T}_{\mathcal{M}} = \mathfrak{vect}(0|q) = \Lambda(q) \otimes \Pi(\mathbb{R}^q), \qquad T_x M = \Pi(\mathbb{R}^q)$$

It is easy to construct a connection $\nabla : \mathcal{T}_{\mathcal{M}} \times \mathcal{T}_{\mathcal{M}} \to \mathcal{T}_{\mathcal{M}}$ with $R \neq 0$!

There is only one loop, which is trivial!

Parallel sections

$X \in \mathcal{E}(M)$ is called parallel if $\nabla X = 0$.

$\nabla X = 0 \Rightarrow \tilde{\nabla}\tilde{X} = 0 \quad (\not\Leftarrow!!!)$

Locally:

$$\nabla X = 0 \Leftrightarrow \begin{cases} \partial_i X^A + X^B \Gamma^A_{iB} = 0, \\ \partial_\gamma X^A + (-1)^{|X^B|} X^B \Gamma^A_{\gamma B} \quad = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (\partial_{\gamma_r} \dots \partial_{\gamma_1}(\partial_i X^A + X^B \Gamma^A_{iB}))^\sim = 0, & (*) \\ (\partial_{\gamma_r} \dots \partial_{\gamma_1}(\partial_\gamma X^A + (-1)^{|X^B|} X^B \Gamma^A_{\gamma B}))^\sim = 0 & (**) \end{cases} \quad r = 0, \dots, m$$

$$\tilde{\nabla}\tilde{X} = 0 \Leftrightarrow \partial_i \tilde{X}^A + \tilde{X}^B \tilde{\Gamma}^A_{iB} = 0$$

Proposition. A parallel section $X \in \mathcal{E}(M)$ is uniquely defined by its value at any point $x \in M$.

Proof. $\nabla X = 0 \Rightarrow \tilde{\nabla}\tilde{X} = 0$; $\tilde{X}_x = X_x$ uniquely determine $\tilde{X}$, i.e. we know the functions $\tilde{X}^A$.

Further, use $(**)$: $X^A_\gamma = -\tilde{X}^B \tilde{\Gamma}^A_{\gamma B}$,
$X^A_{\gamma\gamma_1} = -\tilde{X}^B \Gamma^A_{\gamma B \gamma_1} + X^B_{\gamma_1} \tilde{\Gamma}^A_{\gamma B}$ ... $\Rightarrow$ we know the functions X^A. $\square$

Definition (holonomy algebra)

$\mathfrak{hol}(\nabla)_x :=$

$$\left\langle \tau_\gamma^{-1} \circ \bar{\bar{\nabla}}^r_{Y_r,\dots,Y_1} R_y(Y,Z) \circ \tau_\gamma \,\middle|\, \begin{array}{l} r \geq 0, \;\; Y, Z, Y_i \in T_y\mathcal{M} \\ \bar{\bar{\nabla}}\text{: connect on } \mathcal{T}_{\mathcal{M}}|_U \end{array} \right\rangle \subset \mathfrak{gl}(\mathcal{E}_x)$$

Note: $\mathfrak{hol}(\tilde{\nabla})_x \subset (\mathfrak{hol}(\nabla)_x)_{\bar{0}} \qquad (\neq !)$

Lie supergroup $\mathcal{G} = (G, \mathcal{O}_{\mathcal{G}})$ is a group object in the category of supermanifolds; $\mathcal{G}$ is uniquely given by the Harish-Chandra pair $(G, \mathfrak{g})$, where $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra, $\mathfrak{g}_{\bar{0}}$ is the Lie algebra of G.

Denote by $\mathrm{Hol}(\nabla)^0_x$ the connected Lie subgroup of $\mathrm{GL}((\mathcal{E}_x)_{\bar{0}}) \times \mathrm{GL}((\mathcal{E}_x)_{\bar{1}})$ corresponding to $(\mathfrak{hol}(\nabla)_x)_{\bar{0}} \subset \mathfrak{gl}((\mathcal{E}_x)_{\bar{0}}) \oplus \mathfrak{gl}((\mathcal{E}_x)_{\bar{1}}) \subset \mathfrak{gl}(\mathcal{E}_x)$;

$\mathrm{Hol}(\nabla)_x := \mathrm{Hol}(\nabla)^0_x \cdot \mathrm{Hol}(\tilde{\nabla})_x \subset \mathrm{GL}((\mathcal{E}_x)_{\bar{0}}) \times \mathrm{GL}((\mathcal{E}_x)_{\bar{1}})$.

Def. Holonomy group: $\mathcal{H}ol(\nabla)_x := (\mathrm{Hol}(\nabla)_x, \mathfrak{hol}(\nabla)_x)$;

the restricted holonomy group: $\mathcal{H}ol(\nabla)^0_x := (\mathrm{Hol}(\nabla)^0_x, \mathfrak{hol}(\nabla)_x)$.

Theorem.

$$\{X \in \mathcal{E}(M),\ \ \nabla X = 0\} \longleftrightarrow \left\{ \begin{matrix} X_x \in \mathcal{E}_x \text{ annihilated by } \mathfrak{hol}(\nabla)_x \\ \text{and preserved by } \mathrm{Hol}(\tilde{\nabla})_x \end{matrix} \right\}$$

Connection ∇ is flat if $\mathcal{E}$ admit local basis of parallel sections.

Corollary ∇ is flat $\Longleftrightarrow R = 0 \Longleftrightarrow \mathfrak{hol}(\nabla) = 0$.

Linear connections

∇ a connection on $\mathcal{E} = \mathcal{T}_{\mathcal{M}}$,

$E = \cup_{y \in M} T_y\mathcal{M} = T\mathcal{M}, \quad E_{\bar{0}} = TM$

$\mathfrak{hol}(\nabla) \subset \mathfrak{gl}(n|m, \mathbb{R}), \qquad \mathrm{Hol}(\tilde{\nabla}) \subset \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(m, \mathbb{R})$

Theorem.

$$\left\{ \begin{array}{c} \text{Parallel tensor fields} \\ \text{of type } (p,q) \text{ on } \mathcal{M} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} A_x \in T_x^{p,q}\mathcal{M} \text{ annihilated by } \mathfrak{hol}(\nabla)_x \\ \text{and preserved by } \mathrm{Hol}(\tilde{\nabla})_x \end{array} \right\}$$

Another approach:

J. Groeger, Super Wilson Loops and Holonomy on Supermanifolds. Comm. Math. 22 (2014)

J. Groeger, The Twofold Way of Super Holonomy. Forum Mathematicum 28 (2016)

J. Groeger, On Complex Supermanifolds with Trivial Canonical Bundle, arXiv:1607.07686

Riemannian supermanifolds

$(\mathcal{M}, g)$, where g is a symmetric even nondegenerate metric on $\mathcal{T}_{\mathcal{M}}$.

g defines a pseudo-Riemannian metric $\tilde{g}$ (of signature (p, q)) on M.

On $(\mathcal{M}, g)$ exists a unique Levi-Civita connection ∇

$\mathfrak{hol}(\mathcal{M}, g) \subset \mathfrak{osp}(p, q|2k)$ and $\mathrm{Hol}(\tilde{\nabla}) \subset \mathrm{O}(p, q) \times \mathrm{Sp}(2k, \mathbb{R})$

Special geometries of Riemannian supermanifolds and the corresponding holonomies

type of $(\mathcal{M}, g)$	$\mathfrak{hol}(\mathcal{M}, g)$ is contained in	$\mathrm{Hol}(\tilde{\nabla})$ is contained in
Kählerian	$\mathfrak{u}(p_0, q_0 \mid p_1, q_1)$	$\mathrm{U}(p_0, q_0) \times \mathrm{U}(p_1, q_1)$
special Käh. (by def.)	$\mathfrak{su}(p_0, q_0 \mid p_1, q_1)$	$\mathrm{U}(1)(\mathrm{SU}(p_0, q_0) \times \mathrm{SU}(p_1, q_1))$
hyper-Käh.	$\mathfrak{hosp}(p_0, q_0 \mid 4k)$	$\mathrm{Sp}(p_0, q_0) \times \mathrm{SO}(k, \mathbb{H})$
quaternion.-Kählerian	$\mathfrak{sp}(1) \oplus \mathfrak{hosp}(p_0, q_0 \mid 4k)$	$\mathrm{Sp}(1)(\mathrm{Sp}(p_0, q_0) \times \mathrm{SO}(k, \mathbb{H}))$

$$\mathrm{Ric}(Y,Z) := \mathrm{str}\left(X \mapsto (-1)^{|X||Z|} R(Y,X)Z\right),$$

$$\mathrm{str}\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \mathrm{tr} A - \mathrm{tr} D$$

Proposition. Let $(\mathcal{M}, g)$ be a Kählerian supermanifold, then $\mathrm{Ric} = 0$ if and only if $\mathfrak{hol}(\mathcal{M}, g) \subset \mathfrak{su}(p_0, q_0|p_1, q_1)$. In particular, if $(\mathcal{M}, g)$ is special Kählerian, then $\mathrm{Ric} = 0$; if M is simply connected, $(\mathcal{M}, g)$ is Kählerian and $\mathrm{Ric} = 0$, then $(\mathcal{M}, g)$ is special Kählerian.

Purely odd case

$\mathcal{M} = (\{x\}, \Lambda(q)), \qquad \mathcal{T}_\mathcal{M} = \mathfrak{vect}(0|q), \qquad T_x M = \Pi(\mathbb{R}^q)$

$\mathfrak{g} \subset \mathfrak{osp}(0|2m) \simeq \mathfrak{sp}(2m, \mathbb{R}), \qquad \Lambda^2\Pi(\mathbb{R}^{2m}) = \odot^2\mathbb{R}^{2m}$

The space of skew-symmetric algebraic curvature tensors of type $\mathfrak{g}$:

$$\bar{\mathcal{R}}(\mathfrak{g}) = \left\{ R \in \odot^2(\mathbb{R}^{2m})^* \otimes \mathfrak{g} \,\middle|\, \begin{array}{c} R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \\ \text{for all } X, Y, Z \in \mathbb{R}^{2m} \end{array} \right\}$$

$\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R})$ is *a skew Berger algebra* if

$$\mathrm{span}\{R(X,Y) | R \in \bar{\mathcal{R}}(\mathfrak{g}),\ X, Y \in \mathbb{R}^{2m}\} = \mathfrak{g}$$

Irreducible skew Berger subalgebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{C}) = \mathfrak{sp}(V)$

$\mathfrak{g}$	V	restriction
$\mathfrak{sp}(2m, \mathbb{C})$	$\mathbb{C}^{2m}$	$n \geq 1$
$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(m, \mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^m$	$m \geq 3$
$\mathfrak{spin}(12, \mathbb{C})$	$\Delta_{12}^{+} = \mathbb{C}^{32}$	
$\mathfrak{sl}(6, \mathbb{C})$	$\Lambda^3\mathbb{C}^6 = \mathbb{C}^{20}$	
$\mathfrak{sp}(6, \mathbb{C})$	$V_{\pi_3} = \mathbb{C}^{14}$	
$\mathfrak{so}(n, \mathbb{C}) \oplus \mathfrak{sp}(2q, \mathbb{C})$	$\mathbb{C}^n \otimes \mathbb{C}^{2q}$	$n \geq 3,\ q \geq 2$
$G_2^{\mathbb{C}} \oplus \mathfrak{sl}(2, \mathbb{C})$	$\mathbb{C}^7 \otimes \mathbb{C}^2$	
$\mathfrak{so}(7, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$	$\mathbb{C}^8 \otimes \mathbb{C}^2$	

Possible irreducible holonomy algebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R}) = \mathfrak{sp}(V)$ of not symmetric odd Riemannian supermanifolds.

$\mathfrak{g}$	V	restriction
$\mathfrak{sp}(2m,\mathbb{R})$	$\mathbb{R}^{2m}$	$m\geq 1$
$\mathfrak{u}(p,q)$, $\mathfrak{su}(p,q)$	$\mathbb{C}^{p,q}$	$p+q\geq 2$
$\mathfrak{so}(n,\mathbb{H})$	$\mathbb{H}^n$	$n\geq 2$
$\mathfrak{sp}(1)\oplus\mathfrak{so}(n,\mathbb{H})$	$\mathbb{H}^n$	$n\geq 2$
$\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(p,q)$	$\mathbb{R}^2\otimes\mathbb{R}^{p,q}$	$p+q\geq 3$
$\mathfrak{spin}(2,10)$	$\Delta^+_{2,10}=\mathbb{R}^{32}$	
$\mathfrak{spin}(6,6)$	$\Delta^+_{6,6}=\mathbb{R}^{32}$	
$\mathfrak{so}(6,\mathbb{H})$	$\Delta^{\mathbb{H}}_6=\mathbb{H}^8$	
$\mathfrak{sl}(6,\mathbb{R})$	$\Lambda^3\mathbb{R}^6=\mathbb{R}^{20}$	
$\mathfrak{su}(1,5)$, $\mathfrak{su}(3,3)$	$\{\omega\in\Lambda^3\mathbb{C}^6 \mid *w=w\}$	
$\mathfrak{sp}(6,\mathbb{R})$	$\mathbb{R}^{14}\subset\Lambda^3\mathbb{R}^6$	
$\mathfrak{sp}(2m,\mathbb{C})$	$\mathbb{C}^{2m}$	$m\geq 1$
$\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{so}(m,\mathbb{C})$	$\mathbb{C}^2\otimes\mathbb{C}^m$	$m\geq 3$
$\mathfrak{spin}(12,\mathbb{C})$	$\Delta^+_{12}=\mathbb{C}^{32}$	
$\mathfrak{sl}(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6=\mathbb{C}^{20}$	
$\mathfrak{sp}(6,\mathbb{C})$	$V_{\pi_3}=\mathbb{C}^{14}$	

Classification of irreducible holonomy algebras

$$\mathfrak{g} \subset \mathfrak{osp}(p, q|2m)$$

of the form

$$\mathfrak{g} = (\oplus_i \mathfrak{g}_i) \oplus \mathfrak{z}$$

of not locally symmetric Riemannian supermanifolds :

$$\begin{aligned}
&\mathfrak{osp}(p, q|2m),\\
&\mathfrak{osp}(r|2k, \mathbb{C}),\\
&\mathfrak{u}(p_0, q_0|p_1, q_1),\\
&\mathfrak{su}(p_0, q_0|p_1, q_1),\\
&\mathfrak{hosp}(r, s|k),\\
&\mathfrak{hosp}(r, s|k) \oplus \mathfrak{sp}(1),\\
&\mathfrak{osp}^{sk}(2k|r, s) \oplus \mathfrak{sl}(2, \mathbb{R}),\\
&\mathfrak{osp}^{sk}(2k|r) \oplus \mathfrak{sl}(2, \mathbb{C}).
\end{aligned}$$

Joint work with Andrea Santi in progress

What about generalization of the exceptional holonomy groups $G_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$?

Candidates are exceptional Lie supergroups G_3 and F_4.

$$(\mathfrak{g}_3)_{\bar{0}} = \mathfrak{g}_2 \oplus \mathfrak{sl}(2,\mathbb{R}), \quad (\mathfrak{g}_3)_{\bar{1}} = \mathbb{R}^7 \otimes \mathbb{R}^2$$

$$(\mathfrak{f}_4)_{\bar{0}} = \mathfrak{so}(7) \oplus \mathfrak{sl}(2,\mathbb{R}), \quad (\mathfrak{f}_4)_{\bar{1}} = \mathbb{R}^8 \otimes \mathbb{R}^2$$

We should consider a proper representation!

Table 3.74: Dimensions of $G(3)$ irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\bar{0}}$	$\dim \mathcal{R}_{\bar{1}}$	decomposition under $sl(2) \oplus G(2)$
0;0,0	atp-1	1	1	0	$(1,1)^+$
2;0;0	atp-3	31	17	14	$(3,1)^+$ / $(2,7)^-$ / $(1,14)^+$
3;0,0	atp-4	95	46	49	$(4,1)^+$ / $(3,7)^-$ / $(2,14)^+$ $(2,7)^+$ / $(1,27)^-$ $(1,1)^-$
4;0,0	typ	192	96	96	$(5,1)^+$ / $(4,7)^-$ / $(3,14)^+$ $(3,7)^+$ / $(2,27)^-$ $(2,7)^-$ / $(1,27)^+$ $(1,1)^+$
2;0,1	atp-3	289	147	142	$(3,14)^+$ / $(2,64)^-$ $(2,7)^-$ / $(1,77)^+$ $(1,27)^+$ $(1,1)^+$
5;0,0	atp-6	321	160	161	$(6,1)^+$ / $(5,7)^-$ / $(4,14)^+$ $(4,7)^+$ / $(3,27)^-$ $(3,7)^-$ $(3,1)^-$ / $(2,27)^+$ $(2,7)^+$ $(2,1)^+$ / $(1,14)^-$ $(1,7)^-$
6;0,0	typ	448	224	224	$(7,1)^+$ / $(6,7)^-$ / $(5,14)^+$ $(5,7)^+$ / $(4,27)^-$ $(4,7)^-$ $(4,1)^-$ / $(3,27)^+$ $(3,7)^+$ $(3,1)^+$ / $(2,14)^-$ $(2,7)^-$ / $(1,7)^+$
3;1,0	typ	448	224	224	$(4,7)^+$ / $(3,27)^-$ $(3,14)^-$ $(3,1)^-$ / $(2,64)^+$ $(2,27)^+$ $(2,7)^+$ / $(1,77)^-$ $(1,14)^-$ $(1,7)^-$
7;0,0	typ	576	288	288	$(8,1)^+$ / $(7,7)^-$ / $(6,14)^+$ $(6,7)^+$ /

Table 3.73: Dimensions of $F(4)$ irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomposition under $sl(2) \oplus so(7)$
0;0,0,0	atp-1	1	1	0	$(1,1)^+$
2;0,0,0	atp-3	40	24	16	$(3,1)^+$ / $(2,8)^-$ / $(1,21)^+$
4;0,0,0	atp-6	296	152	144	$(5,1)^+$ / $(4,8)^-$ / $(3,21)^+$ $(3,7)^+$ / $(2,48)^-$ $(2,8)^-$ / $(1,35)^+$ $(1,27)^+$ $(1,1)^+$
2;0,1,0	atp-3	507	267	240	$(3,21)^+$ / $(2,112)^-$ $(2,8)^-$ / $(1,168)^+$ $(1,35)^+$ $(1,1)^+$
5;0,0,0	typ	512	256	256	$(6,1)^+$ / $(5,8)^-$ / $(4,21)^+$ $(4,7)^+$ / $(3,48)^-$ $(3,8)^-$ / $(2,35)^+$ $(2,27)^+$ $(2,7)^+$ / $(1,48)^-$
3;1,0,0	atp-4,5	756	368	364	$(4,8)^+$ / $(3,35)^-$ $(3,21)^-$ / $(2,112)^+$ $(2,48)^+$ $(2,8)^+$ / $(1,189)^-$ $(1,7)^-$
6;0,0,0	atp-8	769	385	384	$(7,1)^+$ / $(6,8)^-$ / $(5,21)^+$ $(5,7)^+$ / $(4,48)^-$ $(4,8)^-$ / $(3,35)^+$ $(3,27)^+$ $(3,7)^+$ $(3,1)^+$ / $(2,48)^-$ $(2,8)^-$ / $(1,21)^+$ $(1,7)^+$
4;0,0,1	atp-5	1036	508	528	$(5,7)^+$ / $(4,48)^-$ / $(3,105)^+$ $(3,27)^+$ / $(2,168)^-$ / $(1,77)^+$
4;1,0,0	typ	2048	1024	1024	$(5,8)^+$ / $(4,35)^-$ $(4,21)^-$ $(4,7)^-$

Adjoin representation $\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ is the holonomy of the symmetric superspace G.

Consider the representation $\mathfrak{g} \subset \mathfrak{gl}(\Pi\mathfrak{g})$, where Π is the parity changing functor.
The first prolongation: $\mathfrak{g}^{(1)} = \mathbb{R}\Pi$ is non-trivial!
Let ∇ be a flat connection on $\mathbb{R}^{\dim \Pi\mathfrak{g}}$ and

$$\hat{\nabla} = \nabla + f\Pi,$$

where f is an odd function. Then $\hat{\nabla}$ is torsion-free, not locally symmetric, and its holonomy algebra is $\mathfrak{g} \subset \mathfrak{gl}(\Pi\mathfrak{g})$.
(the idea is taken from Čap, A. AHS-structures and affine holonomies. Proc. Amer. Math. Soc. 137 (2008), no. 3, 1073–1080.)