Decomposable $(4, 7)$ solutions in 11-dimensional supergravity

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Equations of 11D supergravity

The (bosonic) 11-dimensional supergravity background is defined as an 11-dimensional Lorentzian spin manifold $(\mathcal{M}, g_{\mathcal{M}})$ which admits a 4-form $\mathcal{F}$ satisfying the following system of equations

$$\left\{\begin{array}{rcl} d\,\mathcal{F} & = & 0, \\ d\star\mathcal{F} & = & (1/2)\mathcal{F}\wedge\mathcal{F}, \\ \mathrm{Ric}^{g_{\mathcal{M}}}(X,Y) & = & (1/2)\langle X\lrcorner\mathcal{F}, Y\lrcorner\mathcal{F}\rangle_{\mathcal{M}} - (1/6)g_{\mathcal{M}}(X,Y)\|\mathcal{F}\|^2_{\mathcal{M}}. \end{array}\right.$$

Here

$$\begin{array}{rcl} \langle X\lrcorner\mathcal{F}, Y\lrcorner\mathcal{F}\rangle_{\mathcal{M}} & = & \frac{1}{3!}g_{\mathcal{M}}(X\lrcorner\mathcal{F}, Y\lrcorner\mathcal{F}), \\ \|\mathcal{F}\|^2_{\mathcal{M}} & = & \frac{1}{4!}g_{\mathcal{M}}(\mathcal{F},\mathcal{F}). \end{array}$$

Remind that: $\langle \mathrm{vol}^g, \mathrm{vol}^g\rangle = (-1)^q$ where (p,q) is the signature of a metric g.

Two types of assumptions: homogeneity and decomposability.
Homogeneity: $G \subset \mathrm{Aut}(\mathcal{M}^{11}, \mathrm{g}_{\mathcal{M}}, \mathcal{F})$ acts transitively.
Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and $(g_{\mathcal{M}}, \mathcal{F})$ is defined by
$g_{\mathfrak{m}} \in S^2(\mathfrak{m}^*)$ and $F_{\mathfrak{m}} \in \Lambda^4(\mathfrak{m}^*)$

$(d, 11-d)$ **decomposability**:
$(\mathcal{M}^{11} = \tilde{M}^d \times M^{11-d}, \, g_{\mathcal{M}} = \tilde{g} \oplus g)$.
In case $d = 4$, $\mathcal{F}^4 = \tilde{F} \oplus F$(complete decompoability).

SUGRA equations for $(4,7)$ (completely) decomposable case

Our aim is describe some solutions of SUGRA equations in $(4,7)$ (completely) decomposable case:

$$(\mathcal{M}^{11} = \widetilde{M}^4 \times M^7, g_{\mathcal{M}} = \widetilde{g} + g, \mathcal{F} = \widetilde{F} + F = \lambda \tilde{\mathrm{vol}} + F),$$

in particular, under assumption that the internal manifold (M^7, g, F) is compact and homogeneous.

The SUGRA equations are

$$
\begin{aligned}
dF &= 0,\ \lambda = const && (C)\\
d * F &= \lambda F, && (M)\\
\mathrm{Ric}^{\widetilde{g}} &= -\tfrac{1}{3}(\lambda^2 + |F|^2)\widetilde{g} && (E_1)\\
\mathrm{Ric}^{g}(X, Y) &= \tfrac{1}{6}(\lambda^2 - |F|^2)g(X, Y) + \tfrac{1}{2}\langle X \lrcorner F, Y \lrcorner F\rangle && (E_2)
\end{aligned}
$$

We set

$$\phi := \star_7 F.$$

Since $\star_7^2|_{\Omega^p(M^7)} = (-1)^{p(7-p)} \,\mathrm{Id}_{\Omega^p(M^7)}$, and $\star_7 \phi = F^4$. Moreover, $|F|^2 = |\phi|^2$ and a computation shows that

$$\langle X \lrcorner F, \lrcorner Y F \rangle = |\phi|^2 g(X, Y) - \langle X \lrcorner \phi, \lrcorner Y \phi \rangle.$$

The Einstein equation becomes

$$\mathrm{Ric} := \mathrm{Ric}^g = \frac{1}{6}(\lambda^2 + 2|\phi|^2) g - \frac{1}{2} q_\phi \quad \textbf{(ES)}$$

with $q(X, Y) \equiv q_\phi(X, Y) := \langle X \lrcorner \phi, \lrcorner Y \phi \rangle$.

Definition. α) A 3-form $\phi(= \star F) \in \Omega^3(M)$ on a Riemannian 7-manifold (M, g)) is called **special** if it is co-closed ($d * \phi = 0$) and satisfy the following Maxwell equation

$$d\phi = \lambda * \phi \quad (\text{ Main equation })$$

β) The Riemannian 7- manifold (M, g, ϕ) with a **special** 3-form ϕ is called a **(generalized) special Einstein manifold** if (g, ϕ) is a solution of the (generalized) Einstein equation

$$\mathrm{Ric} = \frac{1}{6}(\lambda^2 + 2|\phi|^2)g - \frac{1}{2}q_\phi. \quad (ES)$$

The following result show that any (4,7) decomposable background is a direct product of a Lorentz Einstein 4-manifold of negative type and a **generalized Einstein manifold** (M^7, g, ϕ) **with special 3-form** ϕ.

Theorem. Any (4,7)-decomposable SUGRA solution $(\mathcal{M}^{11}, g_{\mathcal{M}}, \mathcal{F})$ is a product of Lorentzian Einstein manifold $(\widetilde{M}, \widetilde{g})$ with negative Einstein constant and a (Riemannian) (generalized) Einstein 7-manifold (M^7, g) with special 3-form $\phi \in \Omega^3(M^7)$. The flux 4-form is given by

$$\mathcal{F} = \lambda \, \mathrm{vol}_{\widetilde{M}} + \star_7 \phi = \lambda \, \mathrm{vol}_{\widetilde{M}} + F^4,$$

Three classes of Riemannian 7-manifolds with a special 3-form ϕ

We consider three classes of special 3-forms on a Riemannian 7-manifolds and discuss the problem of solution of (generalized) Einstein equation (ES) for such manifolds.

A) Zero form $\phi = F = 0$.

B) Non zero harmonic form $\phi \neq 0, \lambda = 0$.

C) Non harmonic form $\phi \neq 0$, $\lambda \neq 0$.

(Note that in symmetric case any invariant form is parallel and case C) is impossible)

Case A) The Einstein equation **(ES)** reduces to the standard Einstein equation

$$\mathrm{Ric} = \frac{\lambda^2}{6} g.$$

and a SUGRA background is a product of a Lorentz Einstein 4-manifold $(\tilde{M}^4, \tilde{g})$ and a Riemannian Einstein 7-manifold (M^7, g) with Einstein constant $\frac{\lambda^2}{6}$ and the flux 4-form $\mathcal{F} = \lambda \tilde{\mathrm{vol}}$.

Case B) The Einstein equation **(ES)** for a special harmonic 3-form $\phi \neq 0$ on M^7 reduces to the equation

$$\mathrm{Ric} = \frac{1}{3}|\phi|^2 g - \frac{1}{2} q_\phi, \quad q_\phi(X,X) = ||X \lrcorner \phi||^2. \tag{1}$$

An example. Let $(M^7 = Q^3 \times P^4, g = g_Q + g_P)$ be a Riemannian product and $\phi = \mathrm{vol}_Q$ is the closed (and even parallel) volume 3-form of the first factor. Then $q_\phi = g_Q$ and the Einstein equation becomes

$$\mathrm{Ric} = \frac{1}{3}(g_P + g_Q) - \frac{1}{2}g_Q = \frac{1}{3}g_P - \frac{1}{6}g_Q.$$

or $\quad \mathrm{Ric}^{g_Q} = -\frac{1}{6}g_Q, \; \mathrm{Ric}^{g_P} = \frac{1}{3}g_P$ Assume that the metric g is complete. Then Q is a complete space of constant negative curvature (i.e. a quotient $\mathbb{R}H^3/\Gamma$ of Lobachevski space $\mathbb{R}H^3$ by a lattice) and P is a compact Einstein 4-manifold. In particular, we get background of **type B)**, with compact internal space .

Case C) The Einstein equation **(ES)** for a manifold (M^7, g, ϕ) , where ϕ is a special non harmonic 3-form is

$$\mathrm{Ric} = \frac{1}{6}(\lambda^2 + 2|\phi|^2)g - \frac{1}{2}q_\phi,\ q_\phi(X,X) =< X\lrcorner\phi, X\lrcorner\phi >\geq 0. \quad (2)$$

C1 **Case when $(M = G/H, g)$ is a weakly G_2 manifold**

Some basic facts about G_2 structure

- G_2 -structure on M^7 is a principal G_2-subbundle of the bundle $Fr(M)$ of frames.
- A manifold M^7 admits a G_2 structure if and only if it is **orientable** and **spin**.

G_2- structures defines a Riemannian metric $g = g_\phi$ and a 3-form ϕ which is **generic**.
Conversely, any generic 3-form $\phi \in \Lambda^3 V^*$ has the stability subgroup G_2 and determines G_2-structure as the set of frames where ϕ_x has a canonical form.

3-form ϕ and the associated G_2 -structure are ∇^{g_ϕ}- parallel, if and only if ϕ is harmonic, Then (M^7, g_ϕ) is called a **G_2-manifold**.
A **generic** 3-form ϕ which satisfies the main equation

$$d\phi = \lambda * \phi, \lambda = const \neq 0$$

is called **weak G_2 -form** and the Riemannian manifold (M, g_ϕ) is called **weak G_2-manifold**

$(4,7)$-decomposable SUGRA solutions associated to weak G_2-structures

Any weak G_2-structure admits real Killing spinor, hence is an Einstein manifold (Friedrich and Kath). This implies

Theorem. *Assume that the product* $(\mathcal{M}^{11} = \widetilde{M}^{3,1} \times M^7, g_{\mathcal{M}} = \widetilde{g} + g)$ *is endowed with the 4-form*

$$\mathcal{F}^4 := \lambda \cdot \mathrm{vol}_{\widetilde{M}^4} + F^4,$$

where F is a closed 4-form $F^4 \in \Omega^4_{\mathrm{cl}}(M^7)$ on M^7, such that the special 3-form $\phi := \star_7 F^4$ is a generic 3-form on M^7 Then, $(\mathcal{M}^{11}, g_{\mathcal{M}}, \mathcal{F}^4)$ gives rise to a supergravity background, **if and only if**, *$(M^7, g, \phi := \star_7 F^4)$ is a weak G_2-manifold and $\lambda = 2$.*

Corollary. *If $\lambda = 0$ and $\phi := \star_7 F^4$ is a generic 3-form on M^7, where $F^4 \in \Omega^4_{\mathrm{cl}}(M^7)$, then the Maxwell equation for flux form*

$$\mathcal{F}^4 := F^4,$$

implies that ϕ **is ∇^g-parallel**, *i.e. ϕ induces a parallel G_2-structure on M^7. In this case, the product*

$$(\mathcal{M}^{11} = \widetilde{M}^{3,1} \times M^7, g_{\mathcal{M}} = \widetilde{g} + g, F^4)$$

is not a $(4,7)$-decomposable supergravity background.

SUGRA (4,7)-decomposable background with compact homogeneous inner space (M^7, g, ϕ)

The description of such manifolds is divided into 3 steps.

1. Description of homogeneous 7-manifolds $M = G/H$ of compact group G.
2. Description of invariant special 3-forms ϕ of type A), B), C).
3. Solution of the Einstein equation (ES) for (g, ϕ) for an invariant metric g.

Step 1.

The description of simply connected homogeneous 7-manifolds $M = G/H$ manifold with almost effective action of G reduces to description of

1) (closed) subalgebras $\mathfrak{h}$ of the orthogonal Lie algebra $B_3 = \mathfrak{so}_7$
2) enumeration of compact Lie algebras $\mathfrak{g}$ of dimension $\dim \mathfrak{h} + 7$ and determination of all imbeddings of $\mathfrak{h} \to \mathfrak{g}$.

In **Table 1**, we describe all subalgebras of $\mathfrak{so}_7$ algebra up to a conjugation and calculate the dimension $\dim \mathfrak{g} = \dim \mathfrak{h} + 7$ of possible Lie algebras we may contains $\mathfrak{h}$.

Using description of all compact Lie algebras of appropriate dimension, we get a **Table 2**, which give a list of all effective pairs $(\mathfrak{h} \subset \mathfrak{g})$ and hance a description of all homogeneous 7-manifolds of compact Lie groups.

Table 1. Lie subalgebras $\mathfrak{h}$ of rank r of $\mathfrak{so}(7) = \mathfrak{b}_3$ and decomposition of $\mathfrak{h}$-module $V = \mathbb{R}^7$.

r	$\mathfrak{h}$	$\mathfrak{g}^{d+7}$	$\mathfrak{h}$-decomposition of V
0	$\mathfrak{h} = \{0\}$	$\mathfrak{g}^7$	$7\mathbb{R}$
1	$\mathfrak{so}_2$	$\mathfrak{g}^8$	$V^2 + 5\mathbb{R}$
	$\mathfrak{so}_2$	$\mathfrak{g}^8$	$2V^2 + 3\mathbb{R}$
	$\mathfrak{so}_2$	$\mathfrak{g}^8$	$3V^2 + \mathbb{R}$
	$\mathfrak{su}_2$	$\mathfrak{g}^{10}$	$V^4 + 3\mathbb{R}$
	$\mathfrak{su}_2^s$	$\mathfrak{g}^{10}$	$V^4 + 3\mathbb{R}$
	$\mathfrak{so}_3^{3,3}$	$\mathfrak{g}^{10}$	$V^3 + V^3 + \mathbb{R}$
	$\mathfrak{so}_3^{(5)}$	$\mathfrak{g}^{10}$	$V^5 + 2\mathbb{R}$
	$\mathfrak{so}_3^{\mathrm{irr}}$,	$\mathfrak{g}^{10}$	V^7
2	$(\mathfrak{so}_2 + \mathfrak{so}_2)^{diag} + \mathfrak{so}_2$	$\mathfrak{g}^9$	$3V^2 + \mathbb{R}$
	$\mathfrak{u}_2$	$\mathfrak{g}^{11}$	$V^4 + 3\mathbb{R}$
	$\mathfrak{u}_2^s$	$\mathfrak{g}^{11}$	$V^4 + +3\mathbb{R}$
	$\mathfrak{so}_4 + \mathfrak{so}_2$	$\mathfrak{g}^{13}$	$V^4 + V^2 + \mathbb{R}$
	$\mathfrak{su}_3$	$\mathfrak{g}^{15}$	$V^6 + \mathbb{R}$
	$\mathfrak{so}_5$	$\mathfrak{g}^{17}$	$V^5 + 2\mathbb{R}$
	$\mathfrak{g}_2$	$\mathfrak{g}^{21}$	V^7

... continuation of Table 1.

r	$\mathfrak{h}$	$\mathfrak{g}^{d+7}$	$\mathfrak{h}$-decomposition of V
3	$3\mathfrak{so}_2$	$\mathfrak{g}^{10}$	$3V^2 + \mathbb{R}$
	$\mathfrak{so}_4 + \mathfrak{so}(2)$	$\mathfrak{g}^{16}$	$V^4 + V^2 + \mathbb{R}$
	$\mathfrak{so}_4 + \mathfrak{so}(3)$	$\mathfrak{g}^{16}$	$V^4 + V^3$
	$\mathfrak{so}_6$	$\mathfrak{g}^{22}$	$V^6 + \mathbb{R}$
	$\mathfrak{so}_7$	$\mathfrak{g}^{28} = \mathfrak{d}_4$	V^7

Table 2. List of compact almost effective homogeneous 7-manifolds

1) $\mathrm{S}^7 \cong \frac{\mathsf{SO}_8}{\mathsf{SO}_7} \cong \frac{\mathsf{SU}_4}{\mathsf{SU}_3} \cong \frac{\mathsf{Spin}_7}{\mathsf{G}_2} \cong \frac{\mathsf{Sp}_2}{\mathsf{Sp}_1}$
$\cong \frac{\mathsf{Sp}_2 \times \mathsf{U}_1}{\mathsf{Sp}_1 \times \mathsf{U}_1} \cong \frac{\mathsf{Sp}_2 \times \mathsf{Sp}_1}{\mathsf{Sp}_1 \times \mathsf{Sp}_1}$

2) $\mathsf{S}^3 \times \mathrm{T}^4$

3) $\mathsf{S}^3 \times \mathsf{S}^3 \times \mathsf{S}^1$

4) $\mathsf{S}^3 \times \mathsf{S}^4$

5) $\mathsf{S}^4 \times \mathrm{T}^3$

6) $\mathsf{S}^5 \times \mathrm{T}^2$,

7) $\mathsf{S}^6 \times \mathsf{S}^1$

8) $\mathsf{S}^2 \times \mathsf{S}^2 \times \mathsf{S}^2 \times \mathsf{S}^1$

9) $\mathsf{S}^3 \times \mathsf{S}^2 \times \mathrm{T}^2$

10) $W_{k,l} := \frac{\mathsf{SU}_3}{\mathrm{T}^1_{k,l}}$

11) $\mathsf{S}^3 \times \mathbb{C}P^2$

12) $\mathbb{C}P^1 \times \mathrm{T}^5$

13) $\mathbb{C}P^3 \times \mathsf{S}^1$

14) $\mathbb{C}P^2 \times \mathrm{T}^3$

15) $\mathbb{V}_{5,2} := \frac{\mathsf{SO}_5}{\mathsf{SO}_3} \cong T^1\mathsf{S}^3$,

16) $N_{a,b} = \frac{(\mathsf{SU}_2 \times \mathsf{SU}_3)}{(\mathsf{SU}_2 \times \mathsf{S}^1)}$,

17) $M_{a,b,c} := \frac{(\mathsf{S}^3 \times \mathsf{S}^3 \times \mathsf{S}^3)}{\mathrm{T}^2}$,

18) $\mathbb{V}_{4,2} \times \mathrm{T}^2 \cong \frac{(\mathsf{SU}_2 \times \mathsf{SU}_2)}{\mathrm{T}^1} \tilde{\times} \mathrm{T}^2$

19) $B_{a,b} := \frac{(\mathsf{SU}_2 \times \mathsf{SU}_2)}{\mathrm{T}^2} \tilde{\times} \mathrm{T}^3$

20) $\mathbb{F}_{1,2} \times \mathsf{S}^1$

21) $B^7 := \frac{\mathsf{Sp}_2}{\mathsf{SU}_2^{\mathrm{ir}}} = \frac{\mathsf{SO}_5}{\mathsf{SO}_3^{\mathrm{ir}}}$

22) T^7

Special homogeneous Einstein manifolds $M^7 = G/H$ of type A)

- Homogeneous special Einstein manifolds of type A) are homogeneous Einstein 7-manifold $(M^7 = G/H, g))$ of positive type and we may assume that the group G is semisimple and compact.
- The direct product $\mathcal{M}^{11} = \tilde{M}^4 \times M^7$ of such manifold $(M^7 = G/H, g)$ with a Lorentz Einstein 4-manifold $(\tilde{M}, \tilde{g})$ give a solution of SUGRA.
- The simplest example of such Einstein manifold $\tilde{M}$ is the anti-de Sitter space AdS_4 and simplest examples of homogeneous Einstein 7-manifolds are symmetric spaces of compact type. There are 6 such spaces:

$$S^7, S^5 \times S^2, S^4 \times S^3, S^3 \times S^2 \times S^2, SU_3/SO_3 \times S^2, \mathbb{C}P^2 \times S^2.$$

The product of such manifolds with (symmetric) anti-de Sitter space AdS_4, give SUGRA solutions. These were found by J. Figueroa-O'Farrill in his classification of symmetric solutions of SUGRA. The sphere S^7 gives the famous Freund-Rubin solution.

The list of simply connected non-symmetric homogeneous Einstein 7-manifolds of positive type (Yu. Nikonorov (2004))

Table 3. List of homogeneous 7-manifolds $M^7 = G/H$ with invariant Einstein non-symmetric metric of positive type

	M^7	number of metrics
1.	$B^7 = \mathrm{Sp}_2 / \mathrm{SU}_2 \simeq \mathrm{SO}_5 / \mathrm{SO}_3^{\mathrm{irr}}$	1 isotropy irreducible case
2.	$\mathbb{V}_{3,5} = \mathrm{SO}_5 / \mathrm{SO}_3$	1
3.	$\mathrm{S}^7 = \mathrm{Sp}_2 / \mathrm{Sp}_1$	1 (Jensen metric)
4.	$N_{k,\ell} = (\mathrm{SU}_3 \times \mathrm{SU}_2)/(\mathrm{SU}_2 \times \mathrm{S}^1)$	1 for any k, ℓ
5.	$M_{k,\ell,m} = (\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2)/(\mathrm{S}^1 \times \mathrm{S}^1)$	1 for any $k \geq \ell \geq m \geq 0, \ell > 0$
6.	$W_{k,\ell} = \mathrm{SU}_3 / \mathrm{S}^1_{k,l}$	2 for any $k \geq \ell \geq 0, a > 0.$

A class of special homogeneous manifolds of Type C1) - homogeneous weak G_2- manifolds.

The Lie algebra $\mathfrak{g}_2 \subset \mathfrak{so}_7$ has three maximal subalgebras:

$$\mathfrak{su}_3,\ \mathfrak{so}_3^{\mathrm{irr}},\ \mathfrak{so}(V^3) \oplus \mathfrak{so}((V')^3) \simeq 2\mathfrak{so}_3,$$

where $V^7 = V^3 \oplus (V')^3 \oplus \mathbb{R}$.

Proposition. A homogeneous manifold $M = G/H$ has an invariant weak G_2-structure ϕ if and only if it belongs to the list on the right or is one of homogeneous representation of the sphere indicated below.

Table 4. Compact homogeneous G_2-manifolds vs weak G_2-manifolds

Invariant (non-weak) G_2-structures	weak G_2-structures
T^7	$W_{k,l} = \dfrac{\mathrm{SU}_3}{\mathrm{S}^1_{k,l}}$
$\mathrm{S}^3 \times \mathrm{T}^4$	$\mathbb{V}_{5,2} = \dfrac{\mathrm{SO}_5}{\mathrm{SO}_3}$
$\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1 = \dfrac{\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{S}^1}{\{e\}}$	$\mathrm{S}^7 = \dfrac{\mathrm{Sp}_2}{\mathrm{Sp}_1}$
$\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1 = \dfrac{(\mathrm{SU}_2 \times \mathrm{SU}_2)}{\Delta\,\mathrm{SU}_2} \times \dfrac{(\mathrm{SU}_2 \times \mathrm{SU}_2)}{\Delta\,\mathrm{SU}_2} \times \mathrm{S}^1$	$B^7 = \dfrac{\mathrm{SO}_5}{\mathrm{SO}_3^{\mathrm{irr}}}$
$\mathrm{S}^6 \times \mathrm{S}^1$	$M_{k,l,m} := \dfrac{(\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^3)}{(\mathrm{S}^1 \times \mathrm{S}^1)}$
$\mathbb{V}^{4,2} \times \mathrm{T}^2$	$N_{k,l} = \dfrac{(\mathrm{SU}_3 \times \mathrm{SU}_2)}{(\mathrm{SU}_2 \times \mathrm{S}^1)}$
$\mathrm{S}^5 \times \mathrm{T}^2$	
$\mathbb{F}_{1,2} \times \mathrm{S}^1$	
$\mathbb{C}P^3 \times \mathrm{S}^1$	

7-sphere as a homogeneous weak G_2 manifold.

$$\mathsf{S}^7 = \frac{\mathsf{Sp}_2 \times \mathsf{U}_1}{\mathsf{Sp}_1 \times \mathsf{U}_1}, \quad \mathsf{S}^7 = \frac{\mathsf{Sp}_2 \times \mathsf{Sp}_1}{\mathsf{Sp}_1 \times \mathsf{Sp}_1}, \quad \mathsf{S}^7 = \frac{\mathsf{SU}_4}{\mathsf{SU}_3}, \quad \mathsf{S}^7 = \frac{\mathsf{Spin}_7}{\mathsf{G}_2}.$$

List of homogeneous 7-manifolds $M = G/H$ of a semisimple group G with $\Omega^3(M)^H \neq 0$

1. $W_{k,\ell} = SU_3/T^1_{k,\ell}$ (Aloff-Wallach space)
2. $M_{k,\ell,m} = (SU_2 \times SU_2 \times SU_2)/T^2$
3. $B^5 = SO_5/SO_3^{irr}$, $\Omega^3(B^5)^H = \mathbb{R}\phi$ ϕ is a weak G_2 form.
4. $\mathbb{C}P^2 \times S^3 = SU_3/U_2 \times SU_2$,
5. $S^5 \times S^2 = Su_3/SU_2 \times SO_3/SO_2$,
6. $S^4 \times S^3 = SO_5/SO_4 \times SU_2$,

7.$S^7 = SO_7/G_2$, $\Omega^3(S^7)^H = \mathbb{R}\phi$, ϕ is a weak G_2 form.

8. $S^7 = Sp_2/Sp_1 = Sp_1 \times Sp_2/(Sp_1 \times Sp_1)$. It is sufficient to consider the first case.
9. $S^7 = SU_4/SU_3$