Completeness in Projective Special Geometry

Stefan Suhr (Hamburg),
joint work with Vicente Cortés, Malte Dyckmanns (Hamburg)
and Marc Nardmann (Kiel)

April 25, 2017

Problem: Construct complete quaternionic Kähler manifolds.

Motivation: Quaternionic Kähler manifolds are target spaces for sigma models in $N = 2$ supersymmetric theories of gravity.

Definition

A $4n$-dimensional Riemannian manifold (M, g) is **quaternionic Kähler** if the holonomy group, up to conjugacy, is contained in $Sp(n) \cdot Sp(1)$, but not in $Sp(n)$.

Subproblem 1: Construct quaternionic Kähler manifolds.

Subproblem 2: When are these manifolds geodesically complete?

Problem 1 (Means to construct quaternionic Kähler manifolds)

Ferarra-Sabharwal'90:

$$\text{supergravity } c\text{-map}\colon \{\text{PSK manifolds}\} \to \{\text{QK manifolds}\}$$

("PSK" = projective special Kähler and "QK" = quaternionic Kähler)

deWit-Van Proyen'92:

$$\text{supergravity } r\text{-map}\colon \{\text{PSR manifolds}\} \to \{\text{PSK manifolds}\}$$

("PSR" = projective special real)

Local special Kähler geometry

A **conical special Kähler domain** (M, g, F) is a $\mathbb{C}^*$-invariant domain $M \subset \mathbb{C}^{n+1} \setminus \{0\}$ endowed with a holomorphic function $F \colon M \to \mathbb{C}$ such that

(i) F is homogeneous of degree 2,

(ii) the real matrix $(N_{IJ}(z) := 2\mathrm{Im} \frac{\partial^2 F}{\partial z^I \partial z^J}(z))_{I,J=0,\dots,n}$ is of signature $(1, n)$ for all $z \in M$,

(iii) $f(z) := \sum N_{IJ}(z) z^I \bar{z}^J > 0$ for all $z \in M$.

and the pseudo-Riemannian metric

$$g = \sum N_{IJ} dz^I d\bar{z}^J.$$

Definition

A **projective special Kähler domain** $(\overline{M}, \bar{g})$ is the quotient $\overline{M}$ of a conical special Kähler domain M by the natural $\mathbb{C}^*$-action, endowed with its canonical Kähler metric $\bar{g}$.

The supergravity c-map

$$(\overline{M}, \bar{g}) \mapsto (\overline{N} = \overline{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}, g_{\overline{N}})$$

with

$$\begin{aligned} g_{\overline{N}} &= \bar{g} + g_G, \\ g_G &= \frac{1}{4\rho^2} d\rho^2 + \frac{1}{4\rho^2}\left(d\tilde{\phi} + \sum \left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right)^2 \\ &\quad + \frac{1}{2\rho} \sum \mathcal{I}_{IJ} d\zeta^I d\zeta^J \\ &\quad + \frac{1}{2\rho} \sum \mathcal{I}^{IJ} (d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K)(d\tilde{\zeta}_J + \mathcal{R}_{JL} d\zeta^L), \end{aligned}$$

where $(\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I) \in \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$, $I = 0, 1, \ldots, n$, and

$$\mathcal{R}_{IJ} + i\mathcal{I}_{IJ} := \overline{\frac{\partial^2 F}{\partial z^I \partial z^J}} + i \frac{\sum_K N_{IK} z^K \sum_L N_{JL} z^L}{\sum_{IJ} N_{IJ} z^I z^J}.$$

Special real geometry

Let $U \subset \mathbb{R}^n$ be an open cone (i.e. $\mathbb{R}^{>0} \cdot U \subset U$) and $h\colon \mathbb{R}^n \to \mathbb{R}$ be a homogeneous cubic polynomial with $h|_U > 0$. Denote with $\mathcal{H} := \{h|_U \equiv 1\}$. Assume that $g_{\mathcal{H}} := \iota^*(-\partial^2 h) > 0$ for the inclusion $\iota\colon \mathcal{H} \to U$.

Definition

The smooth Riemannian manifold $(\mathcal{H}, g_{\mathcal{H}})$ is a **projective special real manifold**.

The supergravity r-map

$$(\mathcal{H}, g_{\mathcal{H}}) \mapsto (\overline{M} := \mathbb{R}^n \times \sqrt{-1} \cdot U, \bar{g})$$

with $\bar{g} = -\frac{1}{4} \sum_{i,j} \frac{\partial^2}{\partial x^i \partial x^j}(\log h)(dy^i dy^j + dx^i dx^j)$.

Problem 2 (Means to determine completeness)

Theorem (Cortés/Han/Mohaupt'12)

The supergravity r-map and the supergravity c-map preserve geodesic completeness.

$\rightsquigarrow$ **Subproblem 2.5:** When are projective special real and projective special Kähler manifolds complete?

The projective special real case

Theorem (Cortés/Nardmann/.'16)

The projective special real manifold $(\mathcal{H}, g_{\mathcal{H}})$ *is geodesically complete if, and only if* $\mathcal{H} \subset \mathbb{R}^n$ *is closed.*

Corollary

Let $h\colon \mathbb{R}^n \to \mathbb{R}$ *be a cubic polynomial and* $\mathcal{H}$ *a locally strictly convex component* $(\Leftrightarrow \iota^* \partial^2 h < 0)$ *of* $\{h \equiv 1\}$*. Then the complete projective special real manifold* $(\mathcal{H}, g_{\mathcal{H}})$ *defines a complete quaternionic Kähler manifold of dimension* $4n + 4$ *(via the r- and the c-map).*

Question: Is the theorem true for every homogeneous polynomial of any degree?

True for

- polynomials of degree 2 or 3.
- polynomials $h\colon \mathbb{R}^2 \to \mathbb{R}$ of any degree.
- generic polynomials in any number of variables and of any degree.

Not true in general for

- for rational functions, e.g.

$$h\colon \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \left(\frac{xy}{x+y}\right)^k,$$

$(k > 1)$.

The projective special Kähler case

Definition

The conical affine special Kähler domain (M, g, F) has **regular boundary behaviour** if the affine Kähler potential

$$f = \sum N_{IJ}(z) z^I \bar{z}^J$$

extends to a smooth function (denoted again by f) on some neighborhood of $\mathrm{cl}(M) \setminus \{0\} \subset \mathbb{C}^{n+1}$ with

- $f(p) = 0$, $df_p \neq 0$ and
- $g_p \leq 0$ on $T_p\partial M \cap J(T_p\partial M)$ with kernel $\mathbb{C}p \subset T_p\mathbb{C}^{n+1}$

for all boundary points $p \in \partial M \setminus \{0\}$.

Definition

The projective special Kähler domain $(\overline{M}, \bar{g})$ has **regular boundary behaviour** if the underlying conical special Kähler domain (M, g, F) has regular boundary behaviour.

Theorem (Cortés/Dyckmanns/-'17)

Every projective special Kähler domain with regular boundary behaviour is geodesically complete.

Theorem (Cortés/Dyckmanns/-'17)

Let $(\overline{M}, \bar{g})$ be a projective special Kähler domain with regular boundary behaviour and $(\overline{N}, g^c_{FS})$ the one-loop deformed Ferrara-Sabharwal (quaternionic Kähler) manifold associated to $(\overline{M}, \bar{g})$. Then $(\overline{N}, g^c_{FS})$ is complete for all $c \geq 0$.

References:

- V. Cortés, M. Nardmann, -. *Completeness of hyperbolic centroaffine hypersurfaces*, Comm. Anal. Geom., Vol. 24, no. 1 (2016), 59-92.
- V. Cortés, M. Dyckmanns,-. *Completeness of projective special Kähler and quaternionic Kähler manifolds*, to appear in "New perspectives in differential geometry: special metrics and quaternionic geometry" in honour of Simon Salamon (Rome, 16-20 November 2015).