Andrea Santi

Killing superalgebras and high supersymmetry

(joint works with J. Figueroa-O'Farrill)

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Eleven-dimensional supergravity

Let (M, g, F) be Lorentzian mnfd (M, g), $\dim M = 11$, with closed $F \in \Omega^4(M)$ and endowed with spin bundle $S(M) \longrightarrow M$ (the fiber $S(M)_x \simeq S = \mathbb{R}^{32}$). The bosonic eqs of supergravity are two coupled PDE [Cremmer-Julia-Scherk '78]:

$$\left.\begin{aligned} \mathrm{Ric}(X,Y) &= -\tfrac{1}{2} g(i_X F, i_Y F) + \tfrac{1}{6} g(X,Y) \|F\|^2 \\ d * F &= \tfrac{1}{2} F \wedge F \end{aligned}\right\} \quad (*)$$

Supersym. transf. $\delta_\epsilon \Psi = D\epsilon + O(\Psi)$ of gravitino Ψ gives superconnection on $S(M)$:

$$D_X \epsilon = \nabla_X \epsilon - \frac{1}{24} [X \cdot F - 3F \cdot X] \cdot \epsilon$$

where $X \in \mathfrak{X}(M)$ and $\epsilon \in \Gamma(S(M))$.

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Def. A symmetry of a solution of $(*)$ is pair (ξ, ε) given by

(i) *a Killing vector field* for (g, F), *i.e.* a v.f. ξ s.t $\mathcal{L}_\xi g = \mathcal{L}_\xi F = 0$;

(ii) *a Killing spinor*, *i.e.* a section ε of $S(M)$ s.t. $D\varepsilon = 0$.

Killing superalgebras

Thm[Figueroa-O'Farrill, Meessen, Philip '05] The v.s. $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ of symmetries of (M, g, F) has natural structure of Lie superalgebra, called Killing superalgebra.

The Flat Model. (M, g) Minkowski, $F = 0$. In this case $D = \nabla$, $\mathfrak{k}_{\bar{1}} \simeq S$, $\mathfrak{k}_{\bar{0}} \simeq \mathfrak{so}(V) \oplus V$ and $\mathfrak{k}$ is the Poincaré superalgebra $\mathfrak{p} = (\mathfrak{so}(V) \oplus V) \oplus S$.

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The Killing superalgebra is useful invariant of a supergravity bkgd:

- late '90s: first general check of AdS/CFT correspondence;
- early 2000s: it contracts under Penrose limit;
- mid 2000s: homogeneity conjecture by Meessen, i.e. if

$$\dim(\mathfrak{k}_{\bar{1}}) > \frac{1}{2} \dim S = 16$$

  then bkgd is locally homogeneous;
- it is useful to constructing bkgds with prescribed infinit. automorphisms.

Supergravity solutions

- Local expressions for metric and 4-form of low supersymmetric bkgds have been derived solving the Killing spinor eqs: the G-structure method [Gauntlett, Gutowski, Pakis '03] and the spinorial geometry method [Gillard, Gran, Papadopoulos '05].
- There are (M, g) with parallel spinors that are not Ricci flat [Bryant '00].
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- Other approaches like exceptional generalized geometry that apply for special compactifications [Coimbra, Strickland-Constable, Waldram '14].
- The classification of highly supersymmetric bkgds is largely open. There is classification of maximally supersymmetric bkgds [Figueroa-O'Farrill, Papadopoulos '03] and non-existence results for 31 and 30 Killing spinors [Gran, Gutowski, Papadopoulos, Roest '07 & '10].
- There is one bkgd with 26 Killing spinors [Michelson '02] and also bkgds with $24, 22, 20, 18$ [Gauntlett, Hull '02].

Structural results for highly supersymmetric solutions

The homogeneity thm[Figueroa-O'Farrill, Hustler '12] If $\dim(\mathfrak{k}_{\bar{1}}) > 16$ then bkgd is locally homogeneous.

On S there is $\mathfrak{so}(V)$-invariant symplectic form $\langle -, - \rangle$ and transpose of Clifford multiplication $V \otimes S \to S$ gives a way to square spinors: the Dirac current

$$
\begin{aligned}
&k : \odot^2 S \to V \; , \\
&\eta(k(s,s), v) = \langle s, v \cdot s \rangle \qquad\qquad v \in V \; , \;\; s \in S \; .
\end{aligned}
$$

It turns out that $k|_{\odot^2 S'} : \odot^2 S' \to V$ is surjective for all subspaces $S' \subset S$ with $\dim S' > 16$ so that v.f. $\xi \in \mathfrak{k}_{\bar{0}}$ already span $T_x M$ at all $x \in M$.

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Rem. Supergravity eqs for homogeneous bkgds are algebraic and simpler than PDEs. Checking supersymmetry is additional problem — there exist homog. bkgds which are not (highly) supersymmetric.

Killing superalgebras are filtered deformations

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The nonzero Lie brackets of $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}} = (\mathfrak{so}(V) \oplus V) \oplus S$ are

$$[A, B] = AB - BA \ , \quad [A, s] = As \ , \quad [A, v] = Av \ , \quad [s, s] = k(s, s) \ ,$$

for all $A, B \in \mathfrak{so}(V)$, $s \in S$, $v \in V$. There exists a compatible $\mathbb{Z}$-grading

$$\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0$$

where $\mathfrak{p}_{-2} = V$, $\mathfrak{p}_{-1} = S$ and $\mathfrak{p}_0 = \mathfrak{so}(V)$. Compatibility means that

(i) $[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_{i+j}$ for all $i, j \in \mathbb{Z}$;

(ii) $\mathfrak{p}_{\bar{0}} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_0$ and $\mathfrak{p}_{\bar{1}} = \mathfrak{p}_{-1}$.

Killing superalgebras are filtered deformations

We shall be interested in graded subalgebras $\mathfrak{a} \subset \mathfrak{p}$, i.e.

$$\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 = V' \oplus S' \oplus \mathfrak{h} \; ,$$

where $V' \subset V$, $S' \subset S$ and $\mathfrak{h} \subset \mathfrak{so}(V)$. If $\dim S' > 16$ then $V' = V$ (this is the algebraic fact underlying homogeneity thm). The Lie brackets of $\mathfrak{a}$ are:

$$\begin{aligned} [A, B] &= AB - BA \\ [A, v] &= Av \\ [A, s] &= As \\ [s, s] &= \kappa(s, s) \\ [v, s] &= 0 \\ [v, w] &= 0 \end{aligned}$$

$A, B \in \mathfrak{h}$, $s \in S'$, $v, w \in V'$.

Killing superalgebras are filtered deformations

There is a natural filtration $\mathfrak{a}^\bullet$ on $\mathfrak{a}$, i.e.

$$\mathfrak{a} = \mathfrak{a}^{-2} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^{-1} = \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^{0} = \mathfrak{a}_0 \supset \mathfrak{a}^{1} = 0 \ .$$

Def. A filtered deformation of $\mathfrak{a}$ is a Lie superalgebra $\mathfrak{g}$ with same underlying vector space as $\mathfrak{a}$ and a new Lie bracket $[-,-]$ which satisfies:

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(i) $[\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_{i+j} \oplus \mathfrak{a}_{i+j+1} \oplus \cdots$,

(ii) components of $[-,-]$ of zero degree coincide with original bracket.

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$$\begin{aligned}
[A,B] &= AB - BA \\
[A,v] &= Av + t\delta(A,v) \\
[A,s] &= As \\
[s,s] &= \kappa(s,s) + t\gamma(s,s) \\
[v,s] &= t\beta(v,s) \\
[v,w] &= t\alpha(v,w) + t^2\rho(v,w)
\end{aligned}$$

for some maps $\delta : \mathfrak{h} \otimes V' \to \mathfrak{h}$, $\gamma : \odot^2 S' \to \mathfrak{h}$, $\beta : V' \otimes S' \to S'$, $\alpha : \Lambda^2 V' \to V'$ and $\rho : \Lambda^2 V' \to \mathfrak{h}$ subject to the Jacobi identities for all values of the parameter t, which has been introduced to keep track of the filtration degree.

Main Motivations and Questions

Motivations.

- Idea: instead of studying directly bkgds, we set to study filt. def.
- The problem of classifying filt. def. of $\mathbb{Z}$-graded Lie (super)algebras is well-defined mathematically [Sternberg, Guillemin mid '60s] and also subject of recent investigations [Kac, Cantarini, Cheng '00s] and [Kruglikov, The '14].
- According to Klein's Erlangen program, any geometry should be described by its transformation group. We aim to set up a "supergravity Erlangen program", systematising the search for supergravity bkgds using filt. def.

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Questions.

- Is every filt. def. a Killing superalgebra?
- Any 11-dimensional (M, g) with closed $F \in \Omega^4(M)$ has an associated Killing superalgebra. In supergravity, filt. def. are further constrained?

Spencer cohomology

Filtered deformations are governed by Spencer cohomology, a bi-graded refinement of the usual Chevalley-Eilenberg cohomology of a Lie (super)algebra and its adjoint representation to the case of $\mathbb{Z}$-graded Lie (super)algebras. The space of q-cochains for Poincaré superalgebra is $C^q(\mathfrak{p}_-,\mathfrak{p}) = \mathfrak{p}\otimes\Lambda^q\mathfrak{p}_-^*$, where $\mathfrak{p}_- = \mathfrak{p}_{-2}\oplus\mathfrak{p}_{-1} = V\oplus S$. It decomposes $C^q(\mathfrak{p}_-,\mathfrak{p}) = \bigoplus C^{p,q}(\mathfrak{p}_-,\mathfrak{p})$ in components of different $\deg = p$.

	q				
p	0	1	2	3	4
0	$\mathfrak{so}(V)$	$S\to S$ $V\to V$	$\odot^2 S\to V$		
2		$V\to\mathfrak{so}(V)$	$\Lambda^2 V\to V$ $V\otimes S\to S$ $\odot^2 S\to\mathfrak{so}(V)$	$\odot^3 S\to S$ $\odot^2 S\otimes V\to V$	$\odot^4 S\to V$
4			$\Lambda^2 V\to\mathfrak{so}(V)$	$\odot^2 S\otimes V\to\mathfrak{so}(V)$ $\Lambda^2 V\otimes S\to S$ $\Lambda^3 V\to V$	$\odot^4 S\to\mathfrak{so}(V)$ $\odot^3 S\otimes V\to S$

Spencer cohomology and Killing spinors

Thm[Figueroa-O'Farrill, A.S.] $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$, $H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \simeq \Lambda^4 V$.

We recover the 4-form of supergravity through cohomology! But there is more: the β-component (remember $\beta : V \otimes S \longrightarrow S$) of the Spencer cocycle is

$$\beta(v, s) = v \cdot \varphi \cdot s - 3\varphi \cdot v \cdot s \ ,$$

where $\varphi \in \Lambda^4 V$. In other words, it indicates what are the relevant Killing spinor eqs.

Rem. The Killing spinor eqs in 11-dimensional supergravity encode *all* the information about bosonic bkgds. Indeed the Clifford trace

$$\sum_i e^i \cdot \mathcal{R}(e_i, -) : TM \longrightarrow \mathrm{End}\,(S(M))$$

of curvature $\mathcal{R} : \Lambda^2 TM \longrightarrow \mathrm{End}\,(S(M))$ of D vanishes iff $dF = 0$ and the field eqs are satisfied [Gauntlett, Pakis '03].

Maximal supersymmetry

We classified maximally supersymmetric filt. def., that is filt. def. $\mathfrak{g}$ of subalgebras $\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0$ of $\mathfrak{p}$ with $\mathfrak{a}_{-2} = V$, $\mathfrak{a}_{-1} = S$ and $\mathfrak{a}_0 = \mathfrak{h}$ subalgebra of $\mathfrak{so}(V)$. The fact that $S' = S$ means we have maximal supersymmetry, whereas $V' = V$ (which is forced) means we are describing (locally) homogeneous geometries. We bootstrapped the computation of $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ from $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ and obtained:

Thm[Figueroa-O'Farrill, A.S.] The maximally supersymmetric filt. def. are exactly the Killing superalgebras of maximally supersymmetric bkgds and nothing else:

(i) $\mathfrak{p}$ itself for Minkowski spacetime;

(ii) $\mathrm{osp}(8|4)$ for $AdS_4 \times S^7$ [Freund, Rubin '80];

(iii) $\mathrm{osp}(6,2|4)$ for $S^4 \times AdS_7$ [Pilch, van Nieuwenhuizen, Townsend '84];

(iv) the Killing superalgebra of max. susy pp-wave [Kowalski-Glikman '84].

In all cases $\mathfrak{h} = \mathfrak{so}(V) \cap \mathrm{stab}(\varphi)$ where $\varphi \in \Lambda^4 V$.

High supersymmetry

Thm[Figueroa-O'Farrill, A.S.] Let (M, g) be 11-dimensional Lorentzian mnfd with closed $F \in \Omega^4(M)$. If space $\mathfrak{k}_{\bar{1}}$ of Killing spinors has $\dim \mathfrak{k}_{\bar{1}} > 16$, then (M, g, F) satisfies the Einstein and Maxwell eqs of supergravity.

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Sketch proof. The Jacobi identity in $[S'S'V]$ gives

$$\tfrac{1}{2}R(v, \kappa(s,s))w = \kappa((X_v\beta)(w,s),s) - \kappa(\beta_v(s), \beta_w(s)) - \kappa(\beta_w\beta_v(s), s) \quad (1)$$

for all $s \in S'$ and $v, w \in V$, for some map $X : V \to \mathfrak{so}(V)$. As $\kappa(S', S') = V$, this fully determines the curvature R and, by a further contraction, the Ricci tensor:

$$\begin{aligned}\mathrm{Ric}(v, \kappa(s,s)) = \tfrac{1}{2}F^2_{ab}v^a \left\langle \Gamma^b s, s\right\rangle - \tfrac{1}{6}\|F\|^2 \langle v \cdot s, s\rangle \\ + \tfrac{1}{6}\langle (v \wedge F \wedge F + 2\iota_v \delta F - v \wedge dF) \cdot s, s\rangle . \quad (2)\end{aligned}$$

We then showed that the terms which depend on forms of different degree in

$$\odot^2 S' \subset \odot^2 S = \Lambda^1 V \oplus \Lambda^2 V \oplus \Lambda^5 V \quad (3)$$

satisfy the eqs separately (not immediate: embedding (3) is in general diagonal)■

High supersymmetry

The theorem is sharp — there exist (M, g, F) with $\dim \mathfrak{k}_{\bar{1}} = 16$ which do *not* satisfy supergravity eqs. The theorem allows also to establish a reconstruction result for highly supersymmetric bkgds.

Def. A filtered subdeformation $\mathfrak{g}$ of $\mathfrak{p}$ is aligned if $\dim \mathfrak{g}_{\bar{1}} > 16$ and it is constructed out of a closed 4-form $\varphi \in \Lambda^4 V$. It is strongly aligned if in addition $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$.

Thm[Figueroa-O'Farrill, A.S.] Any aligned filtered subdeformation $\mathfrak{g}$ of $\mathfrak{p}$ is (a subalgebra of) the Killing superalgebra $\mathfrak{k}$ of a highly supersymmetric bkgd. In particular Killing superalgebras $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ of highly supersymmetric bgkds, up to local equivalence, are in a one-to one correspondence with maximal aligned filtered subdeformations of $\mathfrak{p}$, up to isomorphism of filtered subdeformations. The analogous statement holds the Killing ideals $[\mathfrak{k}_{\bar{1}}, \mathfrak{k}_{\bar{1}}] \oplus \mathfrak{k}_{\bar{1}}$ generated by the Killing spinors and strongly aligned deformations.

References

- J. Figueroa-O'Farrill, A. S., *Spencer cohomology and* 11*-dimensional supergravity*, Commun. Math. Phys. **349** (2017), 627–660.
- J. Figueroa-O'Farrill, A. S., *Eleven-dimensional supergravity from filtered subdeformations of the Poincaré superalgebra*, J. Phys A: Math. Theor. **49** (2016).
- J. Figueroa-O'Farrill, A. S., *On the algebraic structure of Killing superalgebras*, preprint arXiv:1608.05915 (2016).

References

– J. Figueroa-O'Farrill, A. S., *Spencer cohomology and* 11*-dimensional supergravity*, Commun. Math. Phys. **349** (2017), 627–660.

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– J. Figueroa-O'Farrill, A. S., *On the algebraic structure of Killing superalgebras*, preprint arXiv:1608.05915 (2016).

– P. de Medeiros, J. Figueroa-O'Farrill, A. S., *Killing superalgebras for Lorentzian four-manifolds*, J. High Energy Phys. **6** (2016).

Thanks!