Hidden Symmetries of Black Holes in Maximal Supergravities

Geometry, Gravity, and Supersymmetry, Mainz

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Symmetries of Kerr–Newman

Uniqueness:

4d Einstein–Maxwell gravity black hole = Kerr–Newman solution

Symmetry		Constant of geodesic motion
Killing vector	$\partial/\partial t$	E
Killing vector	$\partial/\partial \phi$	J_z
Killing–Yano 2-form	Y_{ab}	Carter constant ("$\mathbf{J}^2$")

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Killing–Yano form (Floyd 74, Penrose 73) explains special properties:

- Geodesics completely integrable
- Separability of massive fields of spins $s = 0, \frac{1}{2}$ (also massless $s = 1, \frac{3}{2}, 2$ for Kerr) (Teukolsky 73, ...)

$\implies$ detailed study of physical properties, e.g. stability, scattering

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Generalizations in vacuum Einstein gravity well-established:

- $D > 4$, $\Lambda \neq 0$, NUT charges (Chen Lü Pope 06)
- Killing tensor structure generalizes (e.g. Kubizňák 08, Yasui Houri 11)

Black hole solutions in supergravity

Consider charged and rotating black holes that are:

- Solutions of maximal (ungauged) supergravities
- Asymptotically flat (also works with NUT charges)
- Spacetime dimension $4 \leq D \leq 9$
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Generating solution:

- most general metric
- global symmetry acts on matter fields $\implies$ general solutions (mass, charges, rotations independent)
- U-duality $\implies$ few electromagnetic charges (Cvetič Hull 96)

Dimension	Charges	Rotations	
6, 7, 8, 9	2 electric	$\lfloor \frac{D-1}{2} \rfloor$	(Cvetič Youm 96)
5	3 electric	2	(Cvetič Youm 96)
4	5 electromagnetic	1	(DC Compère 13)

Killing tensors for general black holes?

Does the Killing–Yano 2-form of 4d Kerr–Newman generalize to general asymptotically flat black holes in maximal supergravities?

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Based on:

- 1511.09310, 1608.05052, to appear
- Earlier works in supergravity (Wu 09, Kubizňák Kunduri Yasui 09, Houri Kubizňák Warnick Yasui 10, Kubizňák Warnick Krtouš 10, Chervonyi Lunin 15)
- Earlier works in higher-dimensional Einstein gravity (Frolov, Kubizňák, . . .)

Killing tensors

2 equivalent definitions of Killing vectors:

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Each definition generalizes simply to higher ranks:

1. Antisymmetric Killing–Yano (KY) p-form (Yano 52):

$$\boxed{\nabla_a Y_{b_1 \ldots b_p} = \nabla_{[a} Y_{b_1 \ldots b_p]}} \qquad Y_{a_1 \ldots a_p} = Y_{[a_1 \ldots a_p]}$$

2. Symmetric rank-p Killing–Stäckel (KS) tensor (Stäckel 1893):

$$\boxed{\nabla_{(a} K_{b_1 \ldots b_p)} = 0} \qquad K_{a_1 \ldots a_p} = K_{(a_1 \ldots a_p)}$$

- Constant of geodesic motion: $K^{a_1 \ldots a_p} P_{a_1} \ldots P_{a_p}$
- Trivial rank-2 examples: metric g_{ab}, symmetrized Killing vectors $k_{(a} l_{b)}$

Antisymmetric generalization: CKY p-forms

Conformal Killing vector k_a:

$$\boxed{\nabla_a k_b = \nabla_{[a} k_{b]} + g_{ab} \widetilde{k}}$$

$$\widetilde{k} = \frac{1}{D} \nabla_a k^a$$

Antisymmetric generalization: CKY p-forms

Conformal Killing–Yano (CKY) p-forms (Tachibana 69, Kashiwada 68):

$$\boxed{\nabla_a k_{b_1 b_2 \dots b_p} = \nabla_{[a} k_{b_1 b_2 \dots b_p]} + p g_{a[b_1} \widetilde{k}_{b_2 \dots b_p]}}$$

$$k_{a_1 \dots a_p} = k_{[a_1 \dots a_p]}, \qquad \widetilde{k}_{b_2 \dots b_p} = \frac{1}{D-p+1} \nabla_a k^a{}_{b_2 \dots b_p}$$

In differential form notation:

$$\nabla_X k = \frac{1}{p+1} X \lrcorner \, \mathrm{d}k - \frac{1}{D-p+1} X^\flat \wedge \delta k$$

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From general decomposition of $\nabla_a k_{b_1 b_2 \dots b_p}$ into $\mathrm{SO}(D)$ irreps:

$$\nabla_a k_{b_1 b_2 \dots b_p} \equiv \nabla_{[a} k_{b_1 b_2 \dots b_p]} + p g_{a[b_1} \widetilde{k}_{b_2 \dots b_p]} + \cancel{(k_{a\perp})_{b_1 \dots b_p}}$$

1-form $\otimes$ p-form $\quad$ $(p+1)$-form $\quad$ $(p-1)$-form $\quad$ Cartan product

exterior derivative $\quad$ divergence $\quad$ twistor operator

Antisymmetric generalization: KY and CCKY p-forms

2 special cases of CKY p-forms:

1. Killing–Yano (KY) p-form $(\delta k = 0)$ (Yano 52):

$$\boxed{\nabla_a k_{b_1 \dots b_p} = \nabla_{[a} k_{b_1 \dots b_p]}}$$

2. Closed conformal Killing–Yano (CCKY) p-form $(\mathrm{d}k = 0)$:

$$\boxed{\nabla_a k_{b_1 \dots b_p} = p g_{a[b_1} \widetilde{k}_{b_2 \dots b_p]}}$$

Hodge dual: KY $(D-p)$-form $\stackrel{\star}{\longleftrightarrow}$ CCKY p-form

Symmetric generalization: KS tensors

Rank-p Killing–Stäckel (KS) tensor $K_{a_1 \dots a_p} = K_{(a_1 \dots a_p)}$ (Stäckel 1893):

$$\boxed{\nabla_{(a} K_{b_1 \dots b_p)} = 0}$$

Constant of motion: $K^{a_1 \dots a_p} P_{a_1} \dots P_{a_p}$

KY p-form "squared" = rank-2 KS tensor:

$$(Y \bullet Y)_{ab} := \frac{1}{(p-1)!} Y^{c_1 \dots c_{p-1}}{}_a Y_{c_1 \dots c_{p-1} b} = K_{ab}$$

Converse not true: rank-2 KS tensor $\not\Longrightarrow$ KY "square root"

Rank-2 KS tensor $\sim$ Separability of: HJE for geodesic motion,
Klein–Gordon equation

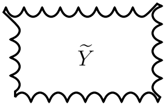
KY p-forms $\sim$ Separability of Dirac equation

Tower of Killing tensors for Kerr(–NUT–AdS)

Dimension $D = 2n + \varepsilon$ ($\varepsilon = 0, 1$)

$n + \varepsilon$ Killing vectors

CCKY 2-form



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CCKY 2-form $\qquad$ CCKY 4-form $\qquad$ CCKY 6-form $\qquad$ CCKY $2n$-form

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\wedge"] & \widetilde{Y}^{(2)} = \widetilde{Y} \wedge \widetilde{Y} \arrow[r, "\wedge"] & \widetilde{Y}^{(3)} = \widetilde{Y} \wedge \widetilde{Y} \wedge \widetilde{Y} \arrow[r, dashed, "\wedge"] & \widetilde{Y}^{(n)} = \widetilde{Y}^{\wedge n} \end{tikzcd}$$

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Y = \star\widetilde{Y} & Y^{(2)} = \star\widetilde{Y}^{(2)} & Y^{(3)} = \star\widetilde{Y}^{(3)} & Y^{(n)} = \star\widetilde{Y}^{(n)} \\
\text{KY } (D-2)\text{-form} & \text{KY } (D-4)\text{-form} & \text{KY } (D-6)\text{-form} & \text{scalar (trivial)/ Killing vector}
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K_{ab} = (Y \bullet Y)_{ab} & K^{(2)}{}_{ab} = (Y^{(2)} \bullet Y^{(2)})_{ab} & K^{(3)}{}_{ab} = (Y^{(3)} \bullet Y^{(3)})_{ab} & & \text{scalar (trivial)/ Killing vector} \\
\text{rank-2 KS tensor} & \text{rank-2 KS tensor} & \text{rank-2 KS tensor} & &
\end{tikzcd}
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$g_{ab} + (n+\varepsilon)$ Killing vectors $+ (n-1)$ KS tensors $\Longrightarrow$ D constants

3 modifications

3 generic fields of string theory:

1. φ scalar dilaton
2. H Kalb–Ramond 3-form field strength
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3 corresponding modifications:

1. Killing tensors in *string frame*
   $\Longrightarrow$ *conformal* Killing tensors in other (e.g. Einstein) frames
2. Torsionful connection associated with 3-form H:

   $$\Gamma^{H\,a}{}_{bc} = \underbrace{\Gamma^a{}_{bc}}_{\text{Levi-Civita}} + \underbrace{\tfrac{1}{2}H^a{}_{bc}}_{\text{torsion}}$$

   Define Killing–Yano p-forms with torsion (KYT p-forms) by replacing $\Gamma \to \Gamma^H$ (KS tensors unchanged)
3. Kaluza–Klein lift to higher (e.g. 10) dimensions
   $\Longrightarrow$ different perspectives of viewing same solution

$D \geq 6$ supergravity

D-dimensional bosonic theory for generating solution (Einstein frame):

$$\begin{aligned}\mathcal{L}_D = {} & R \star 1 - \tfrac{1}{2}\textstyle\sum_{i=1}^{2} \star \mathrm{d}\varphi_i \wedge \mathrm{d}\varphi_i - \tfrac{1}{2}\sum_{I=1}^{2} X_I^{-2} \star F_I \wedge F_I \\ & - \tfrac{1}{2}(X_1 X_2)^{-2} \star H \wedge H\end{aligned}$$

$$H = \mathrm{d}B - \tfrac{1}{2}(A_1 \wedge F_2 + A_2 \wedge F_1),$$

$$X_1 = \mathrm{e}^{-\varphi_1/\sqrt{2(D-2)} - \varphi_2/\sqrt{2}}, \qquad X_2 = \mathrm{e}^{-\varphi_1/\sqrt{2(D-2)} + \varphi_2/\sqrt{2}}$$

Lifts to $(D+1)$-dimensional theory (string frame):

$$\begin{aligned}\mathcal{L}_{D+1} &= \mathrm{e}^{\varphi\sqrt{(D-1)/2}}(R \star 1 + \tfrac{D-2}{2} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \tfrac{1}{2} \star H \wedge H) \\ H &= \mathrm{d}B\end{aligned}$$

$D = 5$: same theory, but $\star H \sim F_3 \implies$ 3 gauge fields

$D = 4$: more general theory (STU supergravity) needed

Killing tensors for $D \geq 6$, 2 equal charges

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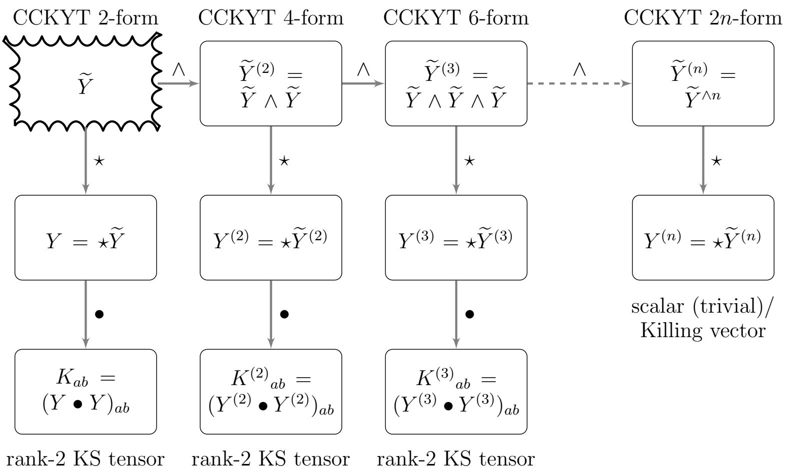

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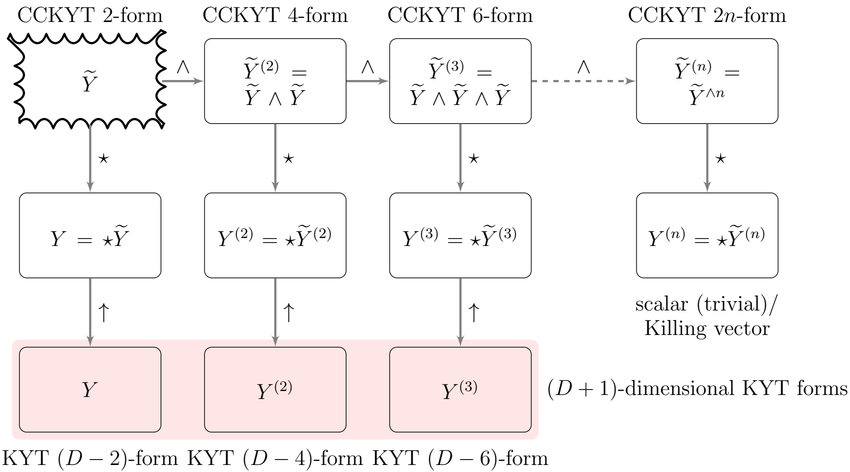
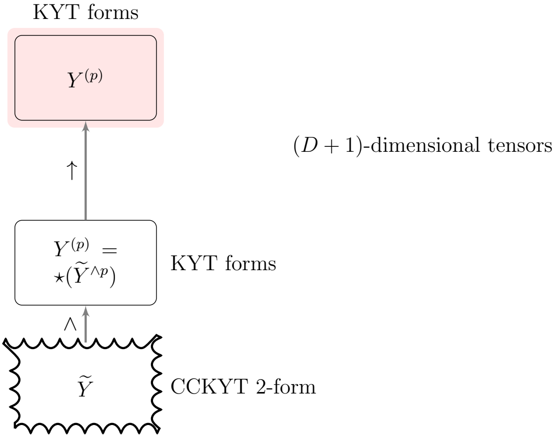


Killing tensors for $D \geq 6$, 2 independent charges

2 equal charges

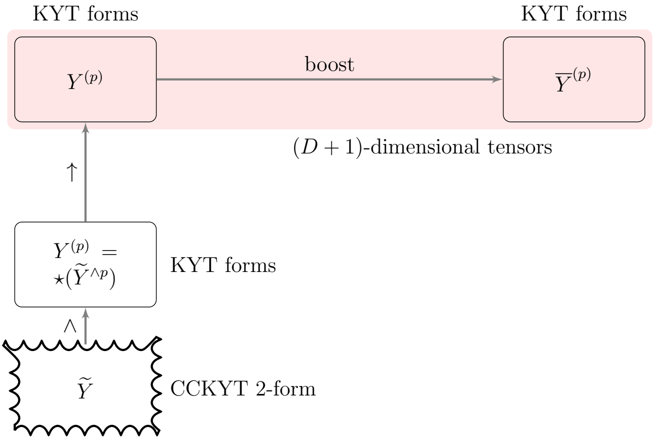


1. $\uparrow$: KYT forms lift to $D+1$ dimensions (DC 15)

Killing tensors for $D \geq 6$, 2 independent charges

2 equal charges

2 independent charges



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2. Lorentz boost: coordinate change $\implies$ 2 independent charges

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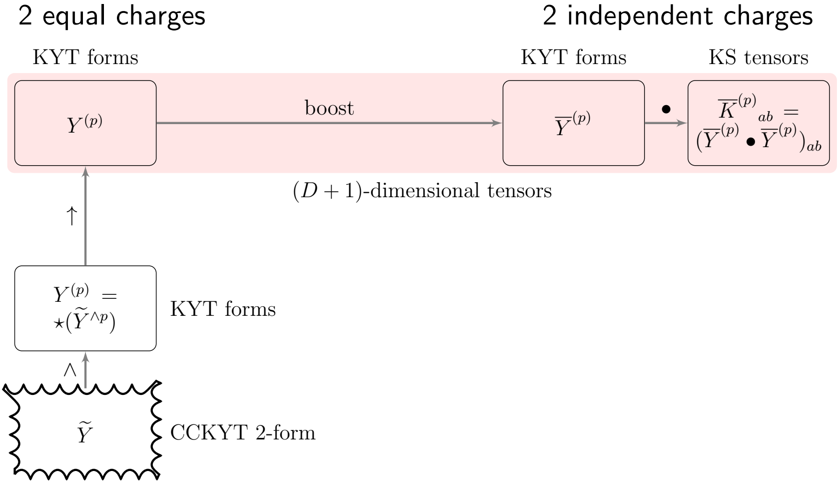

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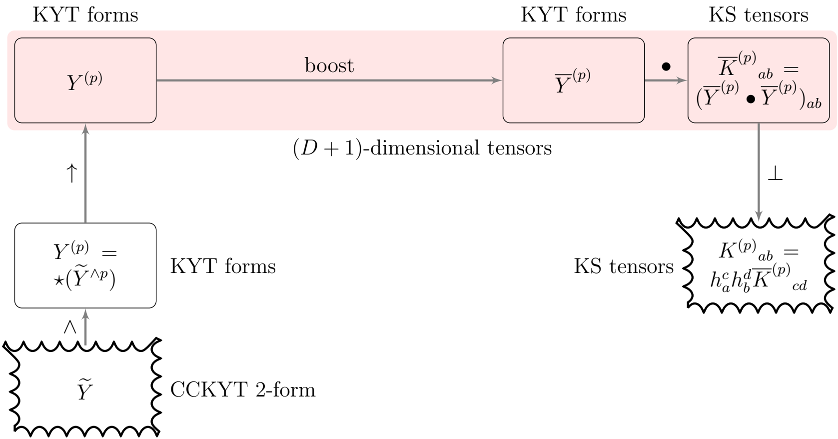
2 equal charges | 2 independent charges



1. $\uparrow$: KYT forms lift to $D+1$ dimensions (DC 15)
2. Lorentz boost: coordinate change $\implies$ 2 independent charges
3. $\perp$: projection h reduces KS forms to D dimensions (Carter Frolov 89)

Killing tensors for $D = 5, 4$

$D = 5$:

- Generating solution has 3 electric charges (Cvetič Youm 96)
- 5d KYT 3-form when $Q_1 = Q_2$
- 6d KYT 3-form for general charges

$D = 4$:

- Generating solution has 5 electromagnetic charges (DC Compère 13)
- 4d KYT 2-form when $Q_1 = Q_4$, $Q_2 = Q_3$
- 6d KYT 2-form for general electric charges
- Most general case — work in progress

First-order symmetry operators

Torsion-modified Dirac operator:

$$\mathcal{D} = \gamma^a \nabla_a - \frac{1}{24} T_{abc} \gamma^{abc}$$

Condition for first-order symmetry operator (ω (CC)KYT p-form) (Kubizňàk Warnick Krtouš 10):

$$\boxed{A_{(\mathrm{c})}(\omega) + A_{(\mathrm{q})}(\omega) - \mathrm{d}f + \delta\epsilon = 0}$$

$A_{(\mathrm{c})}$: $(p+2)$-form, $A_{(\mathrm{q})}$: $(p-2)$-form, f : scalar, ϵ : D-form

$$\begin{array}{lllll}
p \neq 3, D-3: & A_{(\mathrm{c})} = 0 & A_{(\mathrm{q})} = 0 & f = 0 & \epsilon = 0 \\
p = 3: & A_{(\mathrm{c})} = 0 & A_{(\mathrm{q})} = \mathrm{d}f & & \epsilon = 0 \\
p = D-3: & A_{(\mathrm{c})} = -\delta\epsilon & A_{(\mathrm{q})} = 0 & f = 0 &
\end{array}$$

$p = 3$, $p = D-3$ cases relevant for KYT 3-form and CCKYT 2-form in $D = 5$ black hole solutions

Known solutions: ✓

First-order symmetry operators

"classical" $(p+2)$-form and "quantum" $(p-2)$-form anomalies:

$$A_{(\mathrm{c})}(\omega) = \frac{\mathrm{d}(\mathrm{d}^T\omega)}{p+1} - \frac{T \wedge \delta^T\omega}{D-p+1} - \frac{1}{2}\mathrm{d}T \wedge_1 \omega,$$

$$A_{(\mathrm{q})}(\omega) = \frac{\delta(\delta^T\omega)}{D-p+1} - \frac{T \wedge_3 \mathrm{d}^T\omega}{p+1} + \frac{1}{2}\mathrm{d}T \wedge_3 \omega.$$

$$\mathrm{d}^T\omega = \mathrm{d}\omega - T \wedge_1 \omega, \qquad \delta^T\omega = \delta\omega - T \wedge_2 \omega$$

$$(\alpha \wedge_n \beta)_{c_1 \dots c_{p+q-2n}} \sim \alpha^{a_1 \dots a_n}{}_{[c_1 \dots c_{p-n}}\beta_{|a_1 \dots a_n| c_{p-n+1} \dots c_{p+q-2n}]}$$

"classical/quantum" duality:

$$A_{(\mathrm{c})}(\star\omega) = -\star A_{(\mathrm{q})}(\omega)$$

Symmetry operator graded commutes with $\mathcal{D}$:

$$L = 2X^a \lrcorner\, \omega \nabla_a + \frac{p}{p-1}\mathrm{d}\omega - \frac{p-1}{2(p+1)}T \wedge_1 \omega - T \wedge_2 \omega + \frac{1}{2}T \wedge_3 \omega + f$$

Conclusion

- 3 modifications to generalize Kerr–Newman Killing tensors
- Analogous results for black holes in maximal supergravities

To do:

- Further situations without natural 3-form H or toroidal reduction:
  - 11d supergravity has 4-form F
  - 5d gauged supergravity lifts on S^5 to type IIB with 5-form F
- Killing–Yano-like forms $\leftrightsquigarrow$ Discovering exact solutions, particularly in AdS?
- Hidden conformal symmetry? (near horizon, subtracted geometries, ...)