Rigidity of 2-step Carnot groups

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H-type and pseudo H-type algebras

Investigation of H(eisenberg)-type algebras started with A. Kaplan's work (1980). These are special type 2-step nilpotent Lie algebras: $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1} = U \oplus V$ with the bracket $b : \Lambda^2 V \to U$.

To define (more generally pseudo) H-type algebras let us consider a non-degenerate inner product $\langle , \rangle_{\mathfrak{n}} = \langle , \rangle_{\mathfrak{n}_{-2}} \oplus \langle , \rangle_{\mathfrak{n}_{-1}}$ that has nondegenerate restrictions to U and V, and derive the linear representation $J : \mathfrak{n}_{-2} \to \mathrm{End}(V)$ given by

$$\langle J_z x, y \rangle_{\mathfrak{n}_{-1}} = \langle z, [x, y] \rangle_{\mathfrak{n}_{-2}} \quad \text{for all} \quad x, y \in \mathfrak{n}_{-1}, \quad z \in \mathfrak{n}_{-2}.$$

The metric algebra $(\mathfrak{n}, \langle , \rangle)$ is said to be of pseudo H-type if this extends to a representation of the Clifford algera $J : C\ell(\mathbb{R}^{r,s}) \to V$, where we identify $\mathfrak{n}_{-2} = \mathbb{R}^{r,s}$ equipped with the metric

$$\langle v, w \rangle_{r,s} = \sum_{i=1}^{r} v_i w_i - \sum_{j=r+1}^{r+s} v_j w_j, \quad v = (v_i), w = (w_i) \in \mathbb{R}^{r+s}.$$

Generalizations of H-type algebras

In other words, the J-maps satisfy the Clifford relations

$$J_{z_i}J_{z_j}+J_{z_j}J_{z_i}=-2\langle z_i,z_j\rangle_{\mathfrak{n}_{-2}}\mathrm{Id}_{\mathfrak{n}_{-1}},\quad i,j=1,\dots,m,$$

for a basis $\{z_1,\dots,z_m\}$ of $\mathfrak{n}_{-2}$.

Chosing the basis orthonormal, we obtain the following generalization. The metric algebra $(\mathfrak{n},\langle,\rangle)$ is said to be of J-type if for some orthonormal basis $z_1,\dots,z_m$ of $\mathfrak{n}_{-2}$

$$J_{z_i}^2=\pm\,\mathrm{Id}_{\mathfrak{n}_{-1}},\qquad i=1,\dots,m.$$

Another generalization of H-type condition is Métivier's hypothesis (H), which states that $\omega_\alpha(x,y)=\alpha([x,y])$ is a non-degenerate 2-form on $\mathfrak{n}_{-1}$ for any non-zero $\alpha\in\mathfrak{n}^*_{-2}$. It is equivalent to

- $\mathrm{ad}_x\colon\mathfrak{n}_{-1}\to\mathfrak{n}_{-2}$ is surjective for any non-zero $x\in\mathfrak{n}_{-1}$.
- The map J_z is non-degenerate for any non-zero $z\in\mathfrak{n}_{-2}$.

Specifications of H-type algebras

In the case V is an irreducible $C\ell(\mathbb{R}^{r,s})$-representation, the Lie algebra $\mathfrak{n}$ is called the Poincaré translation algebra.

These (and their extended versions) were classified in 90's by D. Alekseevesky and V. Cortés, the problem of rigidity was studied in 2014 by A. Altomani and A. Santi.

They also investigated the corresponding problems for super-Poincaré algebras. In this case the Lie algebra structure (of the translation part) is given by a linear map $S^2V \to U$.

The general problem that I will discuss in the classical case has not been studied in the super-case yet.

Given a graded nilpotent Lie algebra $\mathfrak{n} = \bigoplus_{i=1}^{s} \mathfrak{n}_{-i}$ generated by $\mathfrak{n}_{-1}$, there is an algebraic procedure to compute symmetries of the Carnot structure $(\exp \mathfrak{n}, \mathfrak{n}_{-1})$. The output - Tanaka prolongation - is the maximal graded Lie algebra $\mathfrak{g} = \hat{\mathfrak{n}}$ with $\mathfrak{g}_{<0} = \mathfrak{n}$:

$$\mathfrak{g} = \mathfrak{n}_{-s} \oplus \cdots \oplus \mathfrak{n}_{-1} \oplus \mathfrak{n}_0 \oplus \dots$$

A graded nilpotent Lie algebra $\mathfrak{n}$ is *rigid* or of *finite type* if $\dim \hat{\mathfrak{n}} < \infty$. Otherwise $\mathfrak{n}$ is of *infinite type*.

We will assume the $\mathfrak{n}$ is *fundamental* (equivalently: *stratified*), meaning that $\mathfrak{n}_{-1}$ generates $\mathfrak{n}$, and $x \in \mathfrak{n}_{-1}$, $[x, \mathfrak{n}] = 0 \Rightarrow x = 0$. In the 2-step case the latter means that $\mathfrak{n}_{-1}$ contains no central elements: $\mathfrak{z}(\mathfrak{n}) = \mathfrak{n}_{-2}$.

Equivalently, denoting $\mathfrak{n}^p = \sum_{i=-s}^{p} \mathfrak{n}_i$, we have $\mathfrak{n}_p = H^1(\mathfrak{n}, \mathfrak{n}^{p-1})_p$ for $p \geq 0$, and $H^1(\mathfrak{n}, \hat{\mathfrak{n}}) = 0$. This can be also used in the Lie super-algebra case as well (V. Kac).

Rigidity criterion

The following corank one criterion for *complex* graded Lie algebras is derived from the original Tanaka's criterion by B. Doubrov-O. Radko (2010) and A. Otazzi-B. Warhurst (2011).

GNLA $\mathfrak{n}$ is of infinite type iff there exists $0 \neq x \in \mathfrak{n}_{-1}$ and a hyperplane $\Pi \subset \mathfrak{n}$ such that $[x, \Pi] = 0$.

For real graded nilpotent Lie algebras $\mathfrak{n}$ the criterion fails in its sufficient part and the complexification is required:

Let $\mathfrak{n}$ be a real graded nilpotent Lie algebra of any step, and let $\mathfrak{n}^{\mathbb{C}}$ be its complexification. Then $\hat{\mathfrak{n}}^{\mathbb{C}} = \widehat{\mathfrak{n}^{\mathbb{C}}}$.

Consequently GNLA $\mathfrak{n}$ and $\mathfrak{n}^{\mathbb{C}}$ are either simultaneously rigid or simultaneously of infinite type.

Reformulation of the criterion for metric 2-step algebras

The corank one criterion can be conveniently rewritten in the case when the adjoint map of $\mathfrak{n}$ is encoded by the representation J. Indeed, it is equivalent to the existence of $0 \neq x \in \mathfrak{n}_{-1}$ with

$$J_{\mathfrak{n}_{-2}} x \in \Pi^{\perp}, \quad \text{or} \quad \dim J_{\mathfrak{n}_{-2}} x = \dim(\Pi^{\perp}) = 1.$$

This means that $J_{\mathfrak{n}_{-2}}$ is a one dimensional complex subspace of $\mathrm{End}(\mathfrak{n}_{-1})$, and therefore, there exists

$$L \subset \mathfrak{n}_{-2}, \quad \operatorname{codim} L = 1, \quad \text{such that} \quad J_L x = 0.$$

The last condition is equivalent to the property $x \in \bigcap_{z \in L} \ker(J_z)$, so we conclude (everything is over $\mathbb{C}$):

$\mathfrak{n}$ is of infinite type if and only if there exists a subspace

$$L \subset \mathfrak{n}_{-2}, \quad \operatorname{codim}\ L = 1, \quad \text{such that} \quad \bigcap_{z \in L} \ker(J_z) \neq \{0\}.$$

Application I

As a simple applcation of the criterion let us re-prove with it the result by B. Doubrov-O. Radko and A. Otazzi-B. Warhurst.

Proposition

Let $\mathfrak{n} = \mathfrak{n}_{-1} \oplus \mathfrak{n}_{-2}$ *be a 2-step nilpotent graded Lie algebra. If* $\dim \mathfrak{n}_{-2} \leq 2$*, then* $\mathfrak{n}$ *is of infinite type.*

Indeed, the case $\dim \mathfrak{n}_{-2} = 1$ is obvious, so consider $\dim \mathfrak{n}_{-2} = 2$. Use the complexification and choose a basis z_1, z_2 of $\mathfrak{n}_{-2}$. For some $\lambda \in \mathbb{C}$ we have $\det J_{z_1+\lambda z_2} = 0$. Then taking $z = z_1 + \lambda z_2$, $L = \mathrm{span}\{z\}$ the claim follows from $\mathrm{Ker}(J_z) \neq 0$.

Application II

Theorem

Assume there exist three linearly independent vectors $z_1, z_2, z_3 \in \mathfrak{n}_{-2}$ such that the maps J_{z_i} are non-degenerate, and $J_{z_i} J_{z_j} = \sigma_{ij} J_{z_j} J_{z_i}$, for all $i, j \in \{1, 2, 3\}$, where $\sigma_{ij} \in \{-1, 1\}$ and $\sigma_{12}\sigma_{13}\sigma_{23} = -1$. Then $\mathfrak{n}$ is rigid.

Indeed, define $K = \mathrm{span}\{z_1, z_2, z_3\} \subset \mathfrak{n}_{-2}$ (in complexification). Suppose $\mathfrak{n}$ is of infinite type, and let $L \subset \mathfrak{n}_{-2}$ be the codimension one subspace from the criterion. Then $\dim(K \cap L) \geq 2$ and $K \cap L \supset \mathrm{span}\{z_1 - s_1 z_3, z_2 - s_2 z_3\}$ for some $s_1, s_2 \in \mathbb{C}$. There exists a non-zero $x \in \mathfrak{n}_{-1}$ such that

$$(J_{z_1} - s_1 J_{z_3})(x) = (J_{z_2} - s_2 J_{z_3})(x) = 0.$$

Then $s_1, s_2 \neq 0$ by non-degeneracy of J_{z_k}, and

$$J_{z_1} J_{z_2} x = s_2 J_{z_1} J_{z_3} x = \sigma_{13} s_2 J_{z_3} J_{z_1} x = \sigma_{13} s_1 s_2 J_{z_3}^2 x,$$

and analogously, $J_{z_2} J_{z_1} x = \sigma_{23} s_1 s_2 J_{z_3}^2 x$. Thus, it follows that $\sigma_{23} = \sigma_{12}\sigma_{13}$. This contradicts $\sigma_{12}\sigma_{13}\sigma_{23} = -1$.

Application III

Corollary

Any pseudo H-type algebra with $\dim \mathfrak{n}_{-2} \geq 3$ is rigid.

This is due to H. Reimann (2001) in the case of H-type algebras and to A. Altomiani-A. Santi (2014) for Poincaré algebras.

Corollary

Let $\mathfrak{n}$ be a pseudo J-type algebra, and A be the subalgebra of $\mathrm{End}(\mathfrak{n}_{-1})$ generated by the set $\{J_z J_w : z, w \in \mathfrak{n}_{-2}\}$. If $\mathfrak{n}$ is of infinite type, then A has a common eigenvector over $\mathbb{C}$.

For an orthonormal basis $\{z_1, \dots, z_m\}$ of $\mathfrak{n}_{-2}$ and non-zero $s_i \in \mathbb{C}$:

$$J_{z_i} - s_i J_{z_m} \in J_L,$$

where L is the subspace given in the criterion. If $0 \neq x \in \mathfrak{n}_{-1}^{\mathbb{C}}$ is in the kernel of J_L, then $J_{z_i} J_{z_j} x = c_{ij} x$ for some constants $c_{ij} \in \mathbb{C}$. Consequently, for any $I = P(\dots, J_{z_i} J_{z_j}, \dots) \in A$ we get $Ix = P(\dots, J_{z_i} J_{z_j}, \dots)x = P(\dots, c_{ij}, \dots)x$.

J^2 condition for 2-step algebras

Next we generalize J^2 condition studied by B. Kostant, M. Cowling, A. Dooley, A. Korányi, F. Ricci, and P. Ciatti in the context of Iwasawa decompositions for rank one Lie algebras and H-type.

Definition

A pseudo J-type algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ satisfies the J^2-condition if for every $x \in \mathfrak{n}_{-1}$, $\langle x, x\rangle_{\mathfrak{n}_{-1}} \neq 0$, and for every orthogonal pair $z, z' \in \mathfrak{n}_{-2}$, there exists $z'' \in \mathfrak{n}_{-2}$ satisfying

$$J_z J_{z'} x = J_{z''} x.$$

The general J^2-condition is the same with no condition on $x \in \mathfrak{n}_{-1}$.

This condition implies that the space $\mathbb{R}x \oplus J_{\mathfrak{n}_{-2}} x$ is invariant under the action of J_z for every $z \in \mathfrak{n}_{-2}$. The converse statement is true for pseudo H-type algebras, as follows from the identity

$$\langle J_z x, J_{z'} x\rangle_{\mathfrak{n}_{-1}} = \langle z, z'\rangle_{\mathfrak{n}_{-2}} \langle x, x\rangle_{\mathfrak{n}_{-1}}.$$

Implications of J^2

Observe that for (not pseudo) J-type algebras the general J^2 condition is equivalent to the usual J^2 condition, and that in any case the general J^2 condition implies the usual one. Note also that the (general) J^2-condition is trivially fulfilled for $\dim(\mathfrak{n}_{-2}) = 0, 1$.

Lemma

The J^2-condition is never satisfied for $\dim \mathfrak{n}_{-2} = 2$.

Importance of the J^2 condition for rigidity problem is in the following.

Theorem

A pseudo J-type algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$, $\dim(\mathfrak{n}_{-2}) \geq 3$, satisfying the general J^2-condition is rigid.

Definition

Let $J : C\ell(U, \langle\cdot,\cdot\rangle_U) \to \mathrm{End}(V)$ be a Clifford algebra representation. The module V is called admissible if there is a bilinear form $\langle\cdot,\cdot\rangle_V$ such that the endomorphisms J_z are skew: $\langle J_z x, y\rangle_V = -\langle x, J_z y\rangle_V$.

If $\langle\cdot,\cdot\rangle_U$ is positive definite, then the module V is admissible. In particular, any irreducible module is admissible.

In the case $\langle\cdot,\cdot\rangle_U$ is indefinite, the module V is not always admissible, sometimes only the direct sum $V \oplus V$ is. In this case V has to be neutral.

Let us note that we can describe dimensions of minimal admissible modules $V^{r,s}$ corresponding to the signature (r, s) of $\langle\cdot,\cdot\rangle_U$ only for $0 \le r, s \le 8$, the others being obtained by Bott's periodicity.

In this table we describe dimensions of the minimal admissible Clifford modules $V^{r,s}$, $r, s \leq 8$. The bold integers denote minimal admissible modules that are direct sums of two irreps. Two minimal admissible modules (non-iso irreps) are denoted as $_{\times 2}$.

8	16	32	64	$64_{\times 2}$	128	128	128	$128_{\times 2}$	256
7	16	32	**64**	64	**128**	**128**	**128**	128	256
6	16	$16_{\times 2}$	32	32	**64**	$\mathbf{64}_{\times 2}$	**128**	128	256
5	**16**	16	16	16	**32**	**64**	**128**	128	**256**
4	8	8	8	$8_{\times 2}$	16	32	64	$64_{\times 2}$	128
3	**8**	**8**	**8**	8	16	32	**64**	64	**128**
2	**4**	$\mathbf{4}_{\times 2}$	**8**	8	16	$16_{\times 2}$	32	32	**64**
1	**2**	**4**	**8**	8	**16**	16	16	16	**32**
0	1	2	4	$4_{\times 2}$	8	8	8	$8_{\times 2}$	16
s/r	0	1	2	3	4	5	6	7	8

J^2 condition for H-type

Let us assume that $\mathfrak{n}$ satisfies the J^2-condition. If $x \in \mathfrak{n}_{-1}$ and $z, z' \in \mathfrak{n}_{-2}$ is any orthogonal pair satisfying

$$\langle J_{\tilde{z}} x, J_z J_{z'} x \rangle_{\mathfrak{n}_{-1}} = 0, \qquad \forall \ \tilde{z} \in \mathfrak{n}_{-2},$$

then $\langle x, x \rangle_{\mathfrak{n}_{-1}} = 0$. Indeed

$$0 = \langle J_{\tilde{z}} x, J_z J_{z'} x \rangle_{\mathfrak{n}_{-1}} = \langle J_{\tilde{z}} x, J_{z''} x \rangle_{\mathfrak{n}_{-1}} = \langle \tilde{z}, z'' \rangle_{\mathfrak{n}_{-2}} \langle x, x \rangle_{\mathfrak{n}_{-1}}.$$

Using this observation and the table of the minimal modules we get:

Theorem

Only the following pseudo H-type algebras satisfy the J^2-condition:

(1) $\dim \mathfrak{n}_{-2} = 0$: $\mathbb{R}^n$ – *any module (vector space) over* $\mathbb{R}$.

(2) $\dim \mathfrak{n}_{-2} = 1$: $\mathfrak{n}^{1,0}$ *and* $\mathfrak{n}^{0,1}$ *for any admissible module.*

(3) $\dim \mathfrak{n}_{-2} = 3$: $\mathfrak{n}^{3,0}$ *and* $\mathfrak{n}^{1,2}$ *for any isotypic module.*

(4) $\dim \mathfrak{n}_{-2} = 7$: $\mathfrak{n}^{7,0}$ *and* $\mathfrak{n}^{3,4}$ *for the mininal admissible modules.*

Generic rigidity of 2-step nilpotent algebras

Denote $N(m,n)$ the space of 2-step graded nilpotent Lie algebras with bi-dimensions $(m,n) = (\dim \mathfrak{n}_{-2}, \dim \mathfrak{n}_{-1})$, $0 \le m \le \binom{n}{2}$.

This space is an algebraic manifold of dimension md, $d = \binom{n}{2} - m$, with the isomorphism to the Grassmanian $\mathrm{Gr}_d(\Lambda^2\mathfrak{n}_{-1})$ given by associating $Z = \ker(\Lambda^2\mathfrak{n}_{-1} \to \mathfrak{n}_{-2})$ to the bracket on $\mathfrak{n}$.

Reciprocally, $\mathfrak{n}$ is restored by letting $\mathfrak{n}_{-2} = \Lambda^2\mathfrak{n}_{-1}/Z$. In particular, the notion of a generic Lie algebra structure $\mathfrak{n} \in N(m,n)$ is given by the notion of a Zariski generic point $Z \in \mathrm{Gr}_d(\Lambda^2\mathfrak{n}_{-1})$.

Several authors have studied automorphisms of generic 2-step Carnot structures. P. Pansu (1989) proved that that in general, the automorphism group is generated by translations ad_x, $x \in \mathfrak{n}_{-1}$, and the standard homothety $e \in \mathfrak{n}_0 \subset \hat{\mathfrak{n}}$, provided that $n \in 2\mathbb{Z}$, $n \ge 10$, $3 \le m < 2n-3$. P. Eberlein (2004) relaxed the restrictions to $\bar{d} = \min\{m, \binom{n}{2} - m\} \ge 3$ excluding the cases $n \le 6$ for $\bar{d} = 3$.

When $\mathfrak{n}_0 = \langle e \rangle$, then the Tanaka prolongation is trivial $\mathfrak{n}_+ = 0$ (because $[\mathfrak{n}_{-1}, \mathfrak{n}_1] \subset \mathfrak{n}_0$ is traceless, so $\mathfrak{n}_1 = 0$). Thus in bi-degrees specified by Pansu and Eberlein generically $\mathfrak{n}$ is rigid.

We generalize this.

Theorem

A generic algebra $\mathfrak{n} \in N(m,n)$ is rigid for $m \geq 3$, $n \geq 3$.

Clearly, if $m < 3$ or $n < 3$, then any algebra in $N(n,m)$ is of infinite type.

Proof 1: Absence of rank 1 elements in the family $\{J_z : 0 \neq z \in \mathfrak{n}_{-2}\} \subset \mathrm{End}(\mathfrak{n}_{-1})$ is a Zariski open condition, so proving there exists one rigid Lie algebra structure on $\mathfrak{n}$ in bi-dimensions (m,n) implies the same for a generic one. Hence we construct one rigid representative...

Proof 2:

There exist three linearly independent operators $J_1, J_2, J_3 \in \mathfrak{so}(n)$ such that any 2-dimensional subspace of their span generates the whole Lie algebra $\mathfrak{so}(n)$.

Let $J_{\mathfrak{n}_{-2}}$ contain such three operators $J_{z_1}, J_{z_2}, J_{z_3}$, but still $\dim \mathfrak{g} = \infty$. Then the intersection $L^\dagger$ of the three-dimensional span$\{z_1, z_2, z_3\} \subset \mathfrak{n}_{-2}$ with the hyperplane L, used in the rigidity criterion, has dimension 2 or 3. Since $L^\dagger \subset L$, there exists a non-zero vector $x \in \mathfrak{n}_{-1}$ satisfying $J_z x = 0$ for all $z \in L^\dagger$. However $L^\dagger$ generates the Lie algebra $\mathfrak{so}(n)$, and thus, we get $Ax = 0$ for all $A \in \mathfrak{so}(n)$, whence $x = 0$. This contradiction proves the result.

Moduli of 2-step nilpotent algebras

The moduli space of nilpotent Lie structures on $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$, is the quotient $N(m,n)/\operatorname{GL}(n)$ (the action induced from $\mathfrak{n}_{-1}$).

This is no longer a manifold due to existence of singular orbits. Since the action is algebraic, and has a good quotient on a Zariski open stratum by Rosenlicht's theorem, the moduli space is rational.

It has positive dimension in the following cases:

- $\bar{d} = \min\{m, \binom{n}{2} - m\} \geq 3$, $n \geq 6$, because $\dim \mathfrak{sl}(n) < \dim \operatorname{Gr}_m(\mathfrak{so}(n))$ and the action is locally free.
- $\bar{d} = 2$, $n \in 2\mathbb{Z}$, $n > 6$, because of the continuous invariant coming from considering pensils of non-degenerate skew-symmetric operators on $\mathfrak{n}_{-1}$.

In all other cases, there is an open orbit, and thus, no generic moduli for 2-step structures $\mathfrak{n} \in N(m,n)$.

Let's start with low dimensions. The first non-trivial case is $\dim \mathfrak{n}_{-2} = 3$, then $\dim \mathfrak{n}_{-1} \geq 3$. In the case of equality $\mathfrak{n}_{-2} = \Lambda^2 \mathfrak{n}_{-1}$ and $\hat{\mathfrak{n}} = B_3$ (Yamaguchi 1993),$\mathfrak{n} = so(7)/\mathfrak{p}_3$. Thus the only algebra $\mathfrak{n}$ in $N(3,3)$ is rigid.

The situation changes when $\dim \mathfrak{n}_{-2} = 3$ and $\dim \mathfrak{n}_{-1} = 4$:

Proposition

The algebra $\mathfrak{n}$ with $(\dim \mathfrak{n}_{-2}, \dim \mathfrak{n}_{-1}) = (3,4)$ is of finite type if and only if it is of pseudo H-type.

The corresponding statement does not hold for $n = \dim \mathfrak{n}_{-1} > 4$.

Theorem

Generic (resp. generic rigid) algebras $\mathfrak{n} \in N(m,n)$ in the range $m > 1$ (resp. for $m > 2$) except for $(m,n) \in \{(2,4),(3,4)\}$ are not of pseudo H-type.

Open problems.

1. Bound the prolongations in the rigid cases
(very little is known except for H-type)

2. Generalize this (conditions, rigidity, bounds) to super-algebras
(in super-Poincaré case done: A. Altomani-A. Santi).

3. Find out when there exists a cocompact lattice $\Gamma \subset G$ to get
nilmanifold $K = \Gamma \backslash G$ with nonholonomic metric structure
(for pseudo-H type it exisits: K. Furutani-I. Markina).

4. Study the metric properties of K...