Non-relativistic QED in different gauges

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Motivation

1. Classical Electrodynamics (CED) teaches that a change of gauge $A'_\mu = A_\mu + \partial_\mu f$ does not matter. After all $F_{\mu\nu} = F'_{\mu\nu}$.
2. In Quantum Electrodynamics (QED) canonical quantization requires gauge fixing.
3. In spite of the freedom promised by (CED) the list of different choices appearing in the literature is rather short:

$$\partial_i A^i = 0, \quad \partial_\mu A^\mu = 0, \quad e_\mu A^\mu = 0, \ldots$$

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Goal of this talk

I will argue that different gauge fixing prescriptions may give theories which are not unitarily equivalent.

Strategy:

1. Consider spacelike asymptotic flux of the electric field

$$\Phi(n) := \lim_{r\to\infty} r^2 n \cdot E(rn), \quad n \in S^2.$$

   This is a superselection rule which traditionally plays a role in discussions of infrared problems in QED.

2. Different gauge fixing prescriptions in the quantization procedure may lead to different Φ and therefore unitarily inequivalent reps of QED. (Cf. [Buchholz 82]).

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   But the limit may depend on state in which it is taken.

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Classical Maxwell-Newton equations

The classical Maxwell-Newton system:

$$\begin{aligned}
\partial_t B(t,x) &= -\nabla \times E(t,x), \\
\partial_t E(t,x) &= \nabla \times B(t,x) - j(t,x), \\
\nabla \cdot E(t,x) &= \rho(t,x), \\
\nabla \cdot B(t,x) &= 0, \\
m\ddot{q}_j(t) &= e\big(E_\varphi(t,q(t)) + \dot{q}(t) \times B_\varphi(t,q(t))\big).
\end{aligned}$$

where

$$\begin{aligned}
\rho(t,x) &:= e\varphi(x-q(t)), \\
j(t,x) &:= e\varphi(x-q(t))\dot{q}_j(t), \\
E_\varphi(t,q(t)) &:= \int d^3x\, \varphi(q(t)-x)E(t,x),
\end{aligned}$$

and φ is the charge distribution of the electron, hence $e\hat{\varphi}(0)$ is the charge.

Quantum Maxwell-Newton system in Coulomb gauge

1. Hilbert space $\mathcal{H} := L^2(\mathbb{R}^3) \otimes \mathcal{F}$.
2. Time-zero fields:

$$\begin{aligned} A(x) &:= 1 \otimes A_\perp(x), \text{ so that } \nabla \cdot A(x) = 0, \\ E(x) &:= 1 \otimes E_\perp(x) + E_{||}(x) \otimes 1, \\ B(x) &:= 1 \otimes (\nabla_x \times A_\perp(x)), \end{aligned}$$

where

$$A_\perp(x) = \sum_{\lambda=1,2} \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{1}{2|k|}} e_\lambda(k) (\mathrm{e}^{ikx} a(k,\lambda) + \mathrm{e}^{-ikx} a^*(k,\lambda)),$$

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$$E_{||}(x) = -\nabla_x \int e\varphi(x') \frac{1}{4\pi|x'+q-x|} d^3x'.$$

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3. Hamiltonian

$$\begin{aligned} H &= \frac{1}{2m}(p \otimes 1 - eA_{\perp,\varphi}(q))^2 \\ &+ \frac{1}{2}\int d^3x \, \{ :(1 \otimes E_{\perp}(x))^2 : + :(1 \otimes \nabla_x \times A_{\perp}(x))^2 : \} \end{aligned}$$

4. Time-dependent quantities

$$E(t,x) := e^{itH}E(x)e^{-itH}, \ B(t,x) := e^{itH}B(x)e^{-itH}, \ q(t) := \mathrm{e}^{itH}q\mathrm{e}^{-itH}.$$

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m\dot{v}(t) &= eE_{\varphi}(t,q(t)) + \frac{1}{2}e\Big(v(t) \times B_{\varphi}(t,q(t)) - B_{\varphi}(t,q(t)) \times v(t) \Big),
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where

$$v := \frac{1}{m}(p \otimes 1 - eA_{\perp,\varphi}(q)), \quad v(t) := e^{itH} v e^{-itH}, \quad \dot{v}(t) := i[H, v(t)],$$

$$\rho(t,x) := e\varphi(x - q(t)),$$

$$j(t,x) := \frac{e}{2}\Big(\varphi(x - q(t))v(t) + v(t)\varphi(x - q(t)) \Big).$$

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Changing gauge

1. $U := e^{ief_{\varphi}(q)}$ where f is a function with values in operators on $\mathcal{F}$.
2. $A'(x) := UA(x)U^* + \nabla f(x)$,
3. $H' := UHU^*$,
4. $f(t,x) := e^{itH'} f(x) e^{-itH'}$,

From this one computes:

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3. $v' = UvU^* = \frac{1}{m}(p \otimes 1 - eA_{\varphi}'(q))$,
4. $E'(t,x) = UE(t,x)U^*$,
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The transformed system satisfies again the Maxwell-Newton equations.

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Example: regularized axial gauge

For a function h on S^2 s.t. $\int d\Omega(n)h(n) = 1$, we set

$$f(x) = (-)\int d\Omega(n)h(n)\frac{1}{(n\cdot\nabla_x)}(n\cdot A_\perp(x)),$$

1. For $h = const$ we get $f = 0$ and we stay in the Coulomb gauge.
2. For $h(n) = \delta(n-\hat{n})$ we get the axial gauge in the direction $\hat{n}$.
3. For h supported in small sets we obtain similar potentials to [Mund-Schroer-Yngvason 04]
4. I will use $h(n) = |Y_{1,0}(n)|^2 = c_Y^2\cos^2\theta_n, \quad c_Y^2 := 3/(4\pi)$

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2. U is a unitary if $\hat\varphi(k)\sim|k|^\alpha$ near zero. (Zero charge).
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3. Change of the flux:

$$\Delta\Phi(\tilde n) := \lim_{r\to\infty} r^2\tilde n\cdot\Delta E(\tilde n r) = -2e(2\pi)^{-3/2}(1 - c_Y^2)\frac{\pi}{4} \neq 0$$

for $\tilde n = (0,0,1)$, $\hat\varphi(k) = e^{-|k|}$.

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Conjecture

1. *$H', E'(f), B'(f)$ are well-defined self-adjoint operators for $f \in C_0^\infty(\mathbb{R}^3)_{\mathbb{R}}$.*
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Supporting argument for part 2: Up to domain questions

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