Topological charges and spacelike linearity in QED

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based on two joint works [LMP 16, LMP 17] with
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Outline

The linear e.m. quantum field

The universal C^*-algebra

Spacelike linearity, topological charges and quantum currents

Topological charges and non-Abelian gauge theories

Introduction

- The universal C^*-algebras $\mathfrak{U}_{em}$ of the e.m. quantum field is **model independent tool** for the analysis of fundamental properties of the e.m. quantum field. In particular is represented in **any** theory of the e.m. quantum field.
- **Main result:** appearance of a new kind of topological charges.
  We describe regular representations of $\mathfrak{U}_{em}$ in which topological charges are not trivial also in presence of an electric current.
  The e.m. field in these representations **fails to be linear** on test functions but satisfies a weak, but physically reasonable, form of linearity: the spacelike linearity.
- Such topological charges appears also in non-Abelian gauge theories: the field in this case is linear. This will be discussed by an example.

n-Forms on Minkowski spacetime

- *Minkowski spacetime*: $\mathbb{R}^4$ with signature $(+,-,-,-)$. $\perp$ *spacelike separation*.
- $\mathcal{D}_k$ set smooth k-forms with compact support in the Minkowski spacetime. f, h are spacelike separated, $f \perp h$, whenever

$$\mathrm{supp}(f) \perp \mathrm{supp}(h) \ .$$

- $d : \mathcal{D}_k \to \mathcal{D}_{k+1} \ , \ d^2 = 0$ *differential operator*
- $\star : \mathcal{D}_k \to \mathcal{D}_{4-k} \ , \ \star\star = (-)^{k+1} \, id_k$ *Hodge dual*
- $\delta : \mathcal{D}_{k+1} \to \mathcal{D}_k \ , \ \delta := - \star d \star$ *co-differential (gen. divergence)*

$$\delta^2 = 0 \quad , \quad \Box = \delta d + d\delta$$

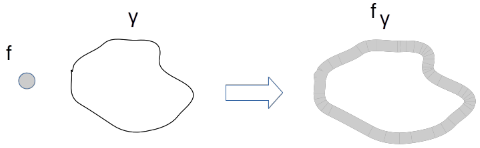
- Of particular importance: $\mathcal{C}_k$ set of co-closed k-forms (divergence-free):

$$\mathcal{C}_1 := \{f \in \mathcal{D}_1 \mid \delta f = 0\}$$

Example: smearing loops, $f \in \mathcal{D}_0$ a scalar function, γ a closed curve

$$f_\gamma^\mu := \int_0^1 f(x - \gamma(t))\,\dot{\gamma}(t)\,dt$$

then $\delta f_\gamma = f_{\partial\gamma} = 0 \;\Rightarrow\; f_\gamma \in \mathcal{C}_1$ and $\mathrm{supp}(f_\chi) \subseteq \mathrm{supp}(f) + \chi$



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The e.m. quantum field F linear mapping $F : \mathcal{D}_2 \ni h \to F(h) \in \mathscr{A}$ to some $*$-algebra $\mathscr{A}$

(i) Locality: $h_1 \perp h_2 \quad \Rightarrow \quad [F(h_1), F(h_2)] = 0 \quad ,$

(ii) 1^{st} Maxwell equation: $dF(\tau) := F(\delta\tau) = 0 \quad , \tau \in \mathcal{D}_3$.

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We get

- Covariance: $F(h) \mapsto F(h_P)$ with $P \in \mathcal{P}_+^\uparrow$
- 2^{nd} Maxwell equation

$$J(f) := \delta F(f) = F(df) \;, \qquad f \in \mathcal{D}_1$$

J is a conserved current: $\delta J = \delta^2 F = 0$.

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Observation: The above relation defines A on $\delta(\mathcal{D}_2) \subseteq \mathcal{C}_1$

$$F(h) = dA(h) = A(\delta h) \ , \qquad h \in \mathcal{D}_2$$

Local Poincaré lemma: for any $f \in \mathcal{C}_1$ and any double cone $\mathcal{O}$ containing the support of f, there exists a co-primitive $\widehat{f} \in \mathcal{D}_2$ (i.e. $\delta\widehat{f} = f$) whose support is contained in $\mathcal{O}$ i.e.

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So restricting to divergence-free 1-forms we may define

$$A(f) := F(\widehat{f}) \ , \qquad f \in \mathcal{C}_1 \ ,$$

well defined: by 1^{st}-Maxwell eq. independent of the choice of the co-primitive $\widehat{f}$

Equivalent description of the em field in terms of A.

The intrinsic (gauge independent) vector potential is a linear mapping $\mathcal{C}_1 \ni f \mapsto A(f) \in \mathscr{A}$ s.t.

(i) Locality

$$f_1 \bowtie f_2 \quad \Rightarrow \quad [A(f_1), A(f_2)] = 0$$

where $f_1 \bowtie f_2$ means that the supports of f_1 and f_2 are contained, respectively, in two contractible and spacelike separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$ (for instance double cones). We shall refer to $\bowtie$ as strong spacelike separation.

- Covariance: as $\mathcal{C}_1 \ni f \to f_P \in \mathcal{C}_1$ we have an action $A(f) \mapsto A(f_P)$ for any $P \in \mathcal{P}_+^\uparrow$.
- The e.m. field $F = dA$
- The 1^{st} Maxwell equation $dF = d^2 A = 0$
- The conserved current: $J = \delta dA = \delta F \;\Rightarrow\; \delta J = 0$.

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The em field strength F linear mapping $F : \mathcal{D}_2 \ni h \to F(h) \in \mathscr{A}$ s.t.

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Basic question: we have seen that A is local wrt the strong spacelike separation

$$f_1 \bowtie f_2 \quad \Rightarrow \quad [A(f_1), A(f_2)] = 0 \ .$$

but

$$f_1 \perp f_2 \quad \Rightarrow \quad [A(f_1), A(f_2)] = \ ? \ .$$

- Clearly $f_1 \bowtie f_2 \Rightarrow f_1 \perp f_2$
- The converse does not hold in general:

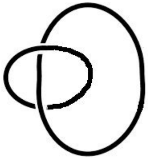

Figure: Spacelike separated linked curves at the subspace $t = 0$

Causal Poincaré Lemma and centrality

Causal Poincaré Lemma: given a double cone $\mathcal{O}$ and $f \in \mathcal{C}_1$ with $\mathrm{supp}(f) \perp \mathcal{O}$, there is $\widehat{f} \in \mathcal{D}_2$ with $\delta\widehat{f} = f$ and $\mathrm{supp}(\widehat{f}) \perp \mathcal{O}$.

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$\Rightarrow$ Translation invariance

$$f_1 \perp f_2 \quad \Rightarrow \quad [A(f_{1,x}), A(f_{2,x})] = [A(f_1), A(f_2)] \ , \qquad \forall x \in \mathbb{R}^4$$

$\Rightarrow$ Dilation invariance

$$f_1 \perp f_2 \quad \Rightarrow \quad [A(\tau_\lambda(f_1)), A(\tau_\lambda(f_2))] = \lambda^{-6} [A(f_1), A(f_2)] \ , \qquad \forall \lambda > 0$$

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By translation invariance

Thm. Centrality-topological charges

$$\boxed{f_1 \perp f_2 \quad \Rightarrow \quad [[A(f_1), A(f_2)] \ , \ A(f)] = 0 \ , \qquad \forall f \in \mathcal{C}_1}$$

Outline

The universal C*-algebra of the e.m. quantum field

Heuristic idea: proceed to abstract unitary operators

$$U(a,f) \;\text{"}=\text{"}\; e^{iaA(f)} \qquad a \in \mathbb{R},\; f \in \mathcal{C}_1$$

Let $\mathcal{U}$ be the group generated by $U : \mathbb{R} \times \mathcal{C}_1 \ni (a,f) \to U(a,f)$ s.t.

(i) $U(a,f)^* = U(-a,f) \;\;,\;\; U(0,f) = 1 \;\;,\;\; U(a,f)\,U(b,f) = U(a+b,f)$;

(ii) $f_1 \bowtie f_2 \;\;\Rightarrow\;\; U(a_1,f_1)\,U(a_2,f_2) = U(1, a_1 f_1 + a_2 f_2)$;

(iii) $f_1 \perp f_2 \;\;\Rightarrow\;\; \lfloor U(a,f), \lfloor U(a_1,f_1), U(a_2,f_2) \rfloor \rfloor = 1$

where $\lfloor , \rfloor$ is the group commutator.

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Action of the Poincaré group: $PU(a,f) := U(a,f_P)$ for any $P \in \mathcal{P}_+^\uparrow$.

The universal C*-algebra of the e.m. field $\mathfrak{U}_{em}$ is the full group C^*-algebra of $\mathcal{U}$.

States and representations: recovering the intrisic vector potential

A regular vacuum state of the algebra $\mathfrak{U}$ is a pure and Poincaré invariant state ω s.t.

- strong regularity: $a_1, \ldots, a_n \mapsto \omega(U(a_1, f_1) \cdots U(a_n, f_n))$ are smooth with tempered derivatives at 0
- $\mathcal{P}_+^\uparrow \ni P \to \omega(A\alpha_P(B))$ continuous ;
- spectrum condition $\mathbb{R}^4 \ni p \to \int e^{ipx}\omega(A\alpha_x(B))\, d^4x \in \overline{V}_+$

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Consequences:

ω is a regular vacuum state; $(\Omega, \pi, \mathcal{H})$ be the GNS of ω.

- By spectrum condition the functional

  $$\mathcal{C}_1 \ni f \to \omega(U(1, f)) \ ,$$

  is a generating functional i.e. determines the value of ω on all $\mathfrak{U}$.
- By strong regularity exist selfadjoint operators $A_\pi(f)$ with common stable core $\mathcal{D} \subseteq \mathcal{H}$ such that

  $$\pi(U(a, f)) = e^{iaA_\pi(f)} \ .$$

  A_π satisfies all the properties defining the intrisic vector potential excepts linearity on test functions.

If a regular vacuum state ω satisfies condition L i.e.

$$\frac{d}{dt}\omega(VU(t,f_1)U(t,f_2)U(-t,f_1+f_2)W)|_{t=0} = 0$$

then

$$a_1 A_\pi(f_1) + a_2 A_\pi(f_2) = A_\pi(a_1 f_1 + a_2 f_2) \ on\ \mathcal{D}$$

i.e. $\mathcal{C}_1 \ni f \mapsto A_\pi(f)$ is the intrisic vector potential and satisfies all the Wightaman axioms

Meaningful states.

- Zero current $J = 0$. ω_0 reg. vacuum state with proprty L and s.t.

$$J_\pi(f) = A_\pi(\delta df) = 0 \ , \qquad \forall f \in \mathcal{C}_1 \ .$$

  then

$$\omega_0(U(1, f)) = e^{-W(f,f)/2} \ , \qquad f \in \mathcal{C}_1$$

  where $W(f, f)$ is the 2-point function of the free electromagnetic field i.e. A_π free electromagnetic field in Fock representation

- Classical current (central current). ω reg. vacuum state with proprty L and s.t.

$$[J_\pi(g), A_\pi(f)] = 0 \ , \qquad g \in \mathcal{D}_1, \ f \in \mathcal{C}_1$$

  then

$$\omega(U(1, f)) = e^{iJ_\pi(G_0(f))}\omega_0(U(1, f))$$

  where G_0 Green's function of $\Box$ (we recover the results by Streater [RJMP 14])

Questions

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- Does there exists representations carrying non-trivial topological charges ? More precisely, we have seen that

  $$f_1 \perp f_2 \;\Rightarrow\; \lfloor U(1, f_1), U(1, f_2) \rfloor \text{ is central}$$

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  Does exists regular vacuum states of the universal C^*-algebra $\mathfrak{U}_{em}$ s.t. the above commutator is non-trivially represented ?
- Positive answer if the intrinsic vector potential A_π in the representation defined by a regular vacuum state ω **is not linear on test functions** (violate property L) but satisfies a weak form of linearity, i.e. spacelike linearity: it is homogeneous and

$$f_1 \bowtie f_2 \;\Rightarrow\; A_\pi(f_1) + A_\pi(f_2) = A_\pi(f_1 + f_2)$$

Outline

Non existence of topological charges in case of linearity

Let ω be the regular vacuum state with **property L** of the algebra $\mathfrak{U}_{em}$. The corresponding intrinsic vector potential A_π is a Wightman field (linear on test functions in particular)

Thm. Let γ_1, γ_2 be simple closed curves and $\mathcal{O}_1$,$\mathcal{O}_2$ double cones such that

$$\mathcal{O}_1 + \gamma_1 \quad \perp \quad \mathcal{O}_2 + \gamma_2$$

$\forall\ f_1, f_2 \in \mathcal{C}_1$ with $\mathrm{supp}(f_1) \subset \mathcal{O}_1 + \gamma_1$ and $\mathrm{supp}(f_2) \subset \mathcal{O}_2 + \gamma_2$ we have

$$[A_\pi(f_1), A_\pi(f_2)] = [A_\pi(f_2), A_\pi(f_1)] \quad \Rightarrow \quad [A_\pi(f_1), A_\pi(f_2)] = 0$$

Thopological charges and spacelike linearity: the zero current case

Let ω_0 be the regular vacuum state with property L of the algebra $\mathfrak{U}_{em}$ with zero conserved current:

$$\omega_0(U(1,g)) = e^{iA_0(g)} \ , \qquad g \in \mathcal{C}_1$$

A_0 the free e.m. intrinsic potential in the Fock space.

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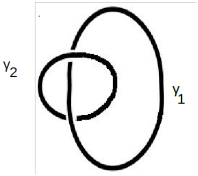
Basic idea: define a new potential by exchanging magnetic with the electric component of a test function: Note



γ_1, γ_2 spacelike separated and **nontrivially** linked closed curves. $f \in \mathcal{D}_0$ with $\int f\, d^4x \neq 0$ and support small enough that the smearing loops f_{γ_i} for $i = 1, 2$ verifies $f_{\gamma_1} \perp f_{\gamma_2}$. Roberts [78]

$$[F_0(\widehat{f}_{\gamma_1}), \star F_0(\widehat{f}_{\gamma_2})] = [A_0(f_{\gamma_1}), A_0(\delta\star\widehat{f}_{\gamma_2})] = \left(\int f\, d^4x\right)^2 c1\ ,\ c \neq 0$$

the Hodge $\star$ exchange the magnetic with the electric component.

To select magnetic and electric functions we use the invariant: given $f \in \mathcal{C}_1$ and a co-primitive $\widehat{f} \in \mathcal{D}_2$, $\delta\widehat{f} = f$

$$\lambda_f^2 := \lambda_f^{\mu\nu}\lambda_{f,\mu\nu} \quad , \quad \lambda_f^{\mu\nu} := \int \widehat{f}^{\mu\nu}(x)d^4x \quad ,$$

λ_f^2 independent of the co-primitive $\widehat{f}$ of f; Poincaré invariant.

- f of Electric type if $\lambda_f^2 > 0$;
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Def. Given $f \in \mathcal{C}_1$ with connected support, let $\widehat{f}$ be any co-primitive of f:

$$A_T(f) := \theta_+(\lambda_f^2)A_0(\delta\widehat{f}) + \theta_-(\lambda_f^2)A_0(\delta \star \widehat{f})$$

θ_+ step function and $\theta_-(t) = \theta_+(-t)$.

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θ_+ select electric type 1-forms, θ_- select magnetic type 1-forms and $\star$ change magnetic $\to$ electric:

$$A_T(f) = \begin{cases} A_0(\delta \widehat{f}) \ , & \lambda_f^2 > 0; \\ 0 \ , & \lambda_f^2 = 0; \\ A_0(\delta \star \widehat{f}) \ , & \lambda_f^2 < 0 \end{cases}$$

- **The definition is well posed**: A_T does not depend on the co-primitive $\widehat{f}$: $\delta\hat{f} = f$. By local Poincaré lemma $\exists\tau \in \mathcal{D}_3$, $\delta\tau = \widetilde{f} - \widehat{f}$. Since $J_0 = 0$ we have

$$
\begin{aligned}
A_0(\delta * \widehat{f}) &= A_0(\delta * \widetilde{f}) - A_0(\delta * \delta\tau) = A_0(\delta * \widetilde{f}) + A_0(\delta d \star \tau) \\
&= A_0(\delta * \widetilde{f}) + J_0(\star\tau) = A_0(\delta * \widetilde{f})
\end{aligned}
$$

- A_T is covariant and local

$$
f_1 \bowtie f_2 \quad \Rightarrow \quad [A_T(f_1), A_T(f_2)] = 0
$$

with zero conserved current

$$
J_T(f) := \delta d A_T(f) = A_T(\delta df) = 0 \ , \qquad f \in \mathcal{D}_1 \ .
$$

Topological charges:

- $f_1 \perp f_2$ then

$$[A_T(f_1), A_T(f_2)] = \left(\theta_+(\lambda_{f_1}^2)\theta_-(\lambda_{f_2}^2) - \theta_-(\lambda_{f_1}^2)\theta_+(\lambda_{f_2}^2)\right)[A_0(f_1), A_0(\delta \star \widehat{f_2})]$$

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- The 1-forms used in Roberts calculation are all of electric type, so the above commutator in that case vanishes. However, there are 1-forms f_1, f_2, obtained by a perturbation of the Roberts example such that

$$f_1 \perp f_2 \quad \Rightarrow \quad [A_T(f_1), A_T(f_2)] = c \cdot \mathbb{1} \quad , \quad c \neq 0$$

so this representation describes nontrivial topological charges.

Topological charges:

- $f_1 \perp f_2$ then

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However A_T, because of θ functions, is homogeneous but not additive!. This is why we restrict at the beginning to 1-forms with a connected support.

The support of a generic $f \in \mathcal{C}_1$ has in general infinite countable connected components. It turns out that f decomposes into a sum

$$f = f^e + f^0 + f^m \quad , \quad f^e, f^m, f^0 \in \mathcal{C}_1$$

where f^e, f^m and f^0 have disjoint supports and are of **electric, magnetic and null type** respectively.

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Using this decomposition the intrinsic vector potential is defined by

$$\boxed{A_T(f) := A_0(\delta \widehat{f}^\sharp) \quad , \qquad f \in \mathcal{C}_1}$$

where $\widehat{f}^\sharp := \widehat{f}^e + \star \widehat{f}^m$ and $\widehat{f}^e$ and $\widehat{f}^m$ are any two co-primitives of of f^e and f^m.

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Thm. The functional

$$\omega_T(U(a, f)) := \omega_0(e^{iaA_T(f)}) = \omega_0(e^{iaA_0(\delta \widehat{f}^{\sharp})}) \ , \qquad a \in \mathbb{R}, f \in \mathcal{C}_1$$

defines a regular vacuum state of the algebra $\mathfrak{U}_{em}$ and topological charges appear in this representation.

Topological charges and spacelike linearity: the quantum current case

Aim: For a given conserved current J find a regular vacuum state of $\mathfrak{U}_{em}$ s.t. the intrinsic potential A verifies $\delta dA = J$.

Let J be a local, covariant conserved current

$$\delta J(g) = J(dg) = 0 \ , \qquad g \in \mathcal{D}_0$$

which is a Wightman field on a Hilbert space H_J, covariant wrt U_J a unitary rep of the Poincaré group, with a vacuum vector Ω_J(for instance, the conserved current associated with the free Dirac field.)

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Basic idea: use the pre-image of the operator

$$\delta d : \mathcal{D}_1 \to \mathcal{C}_1$$

Using the pre-image of δd leads to non linearity. So we start by considering divergence-free 1-forms.

Def. For any $f \in \mathcal{C}_1$ with **connected support** we let

$$A_J(f) := \begin{cases} J(f^\circ) \ , & \text{if } \exists f^\circ \in \mathcal{D}_1, \ \delta d f^\circ = f \\ 0 \ , & \textit{otherwise} \end{cases}$$

- **Well posedness**: local Poincaré Lemma and conservation law of J

$$\tilde{f} \in \mathcal{D}_1 \ , \ \delta d \tilde{f} = f \quad \Rightarrow \quad J(f^\circ) = J(\tilde{f})$$

- **Well localized**: $\mathrm{supp}(f^\circ) \subset V^+(\mathrm{supp}(f)) \cap V^-(\mathrm{supp}(f))$ in particular for any double cone $\mathcal{O}$ with $\mathrm{supp}(f) \subset \mathcal{O}$, then

$$\mathrm{supp}(f^\circ) \subset \mathcal{O}$$

- With the above definition A_J is homogenous but **not additive on 1-forms**, (reason why we restrict to 1-forms with a connected support).
  However, similarly to the previous case (but technically more hard), A_J canonically extends to all divergence free 1-forms.

Summing up the intrinsic vector potential A_J satisfies the following properties

- Covariance: $U_J(P)\, A_J(f)\, U_J^*(P) = A_J(f_P)$
- Locality: $f_1 \bowtie f_2 \;\Rightarrow\; [A_J(f_1), A_J(f_2)] = 0$
- The conserved current is J:

$$\boxed{\delta d A_J(f) = A_J(\delta d f) = J(f) \qquad f \in \mathcal{C}_1}$$

- A_J is not linear but spacelike linear and

$$f_1 \perp f_2 \;\Rightarrow\; [A_J(f_1), A_J(f_2)] = 0 \;\; \text{No topological charges}$$

Thm. The functional

$$\omega_J(U(a,f)) := (\Omega_j, e^{iaA_J(f)}\Omega_j)\ , \qquad f \in \mathcal{C}_1$$

is a regular vacuum state of $\mathfrak{U}_{em}$ with conserved current J but there are no topological charges.

Finally: take ω_T and ω_J the states with topological charges and quantum current defined before.

Thm. The functional

$$\omega_{TJ}(U(a,f)) := \omega_T(U(a,f)) \cdot \omega_J(U(a,f))$$

defines a regular vacuum state of $\mathfrak{U}_{em}$ with nontrivial topological charges and quantum conserved current J.

Outline

The non-Abelian example and linearity

We consider the space of test functions $\mathcal{C}_1 \oplus \mathcal{C}_1$: $\vec{f} = \begin{pmatrix} f^u \\ f^d \end{pmatrix}$ with $f^u, f^d \in \mathcal{C}_1$

For any $-1 < \zeta < 1$ there corresponds a sesquilinear form :

$$\langle \vec{f}_1, \vec{f}_2 \rangle_\zeta := \langle f_1^u, f_2^u \rangle_0 + \langle f_1^d, f_2^d \rangle_0 + \zeta \langle f_1^u, \delta \star \widehat{f_2^d} \rangle_0 - \zeta \langle f_1^d, \delta \star \widehat{f_2^u} \rangle_0$$

where $\langle , \rangle_0$ denotes the standard scalar product of the free electromagnetic field

$$\langle f, g \rangle_0 := \int_{V_+} \frac{d^3 p}{\boldsymbol{p}} \, \hat{f}^\mu(p)^* \, \hat{g}_\mu(p) \; , \qquad f, g \in \mathcal{C}_1$$

This is a positive semi-definite scalar product: the relations $\langle f, \delta \star \widehat{g} \rangle_0 = \langle g, \delta \star \widehat{f} \rangle_0$ and $\langle \delta \star \widehat{f}, \delta \star \widehat{f} \rangle_0 = \langle f, f \rangle_0$, for any $f, g \in \mathcal{C}_1$ imply

$$\langle \vec{f}, \vec{f} \rangle_\zeta \geq \langle f^u, f^u \rangle_0 + \langle f^d, f^d \rangle_0 - |\zeta| \sqrt{\langle f^u, f^u \rangle_0 \cdot \langle f^d, f^d \rangle_0} \geq 0$$

Invariant under the Poincaré group and $SO(2)$ (global gauge group)

$$R(\theta) := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \; , \quad \vec{f} \mapsto R(\theta)\vec{f} = \begin{pmatrix} \cos\theta f^u + \sin\theta f^d \\ -\sin\theta f^u + \cos\theta f^d \end{pmatrix}$$

Using this scalar product we define the Weyl algebra

$$U_\zeta(\vec{f}_1)\, U_\zeta(\vec{f}_2) = e^{\frac{i}{2}\Im\left\langle \vec{f}_1, \vec{f}_2 \right\rangle_\zeta} \cdot U_\zeta(\vec{f}_1 + \vec{f}_2)$$

and a Poincaré invariant Fock state

$$\omega_\zeta(U_\zeta(\vec{f})) = e^{-\frac{1}{2}\left\langle \vec{f}, \vec{f} \right\rangle_\zeta}$$

defining a Fock representation of the algebra and a representation of $SO(2)$ and a Poincaré representation satisfying the spectrum condition (i.e. ω_ζ is a vacuum state).

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The generator A_ζ i.e. $\pi_\zeta(U(\vec{f})) = \exp(iA_\zeta(\vec{f}))$ is a Wightman field satisfying the commutation relations

$$[A_\zeta(\vec{f}_1), A_\zeta(\vec{f}_2)] = \left(\langle \vec{f}_1, \vec{f}_2\rangle_\zeta - \langle \vec{f}_2, \vec{f}_1\rangle_\zeta\right) \cdot \mathbb{1}$$

In particular

$$\vec{f}_1 \perp \vec{f}_2 \;\Rightarrow\; [A_\zeta(\vec{f}_1), A_\zeta(\vec{f}_2)] = 2\zeta \cdot \Im\left(\langle f_1^u, \delta \star \widehat{f_2^d}\rangle_0 - \langle f_1^d, \delta \star \widehat{f_2^u}\rangle_0\right) \cdot \mathbb{1}$$

The last terms are the same appearing in the Roberts calculations: so if we take f_{γ_1} and f_{γ_2} as before and set $\vec{f_1} = \begin{pmatrix} f_{\gamma_1} \\ 0 \end{pmatrix}$ and $\vec{f_2} = \begin{pmatrix} 0 \\ f_{\gamma_2} \end{pmatrix}$ we have that

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So the model describes toplogical charges.

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Note: this model describes a pair of free e.m. fields

$$A^u(f) := A_\zeta \begin{pmatrix} f \\ 0 \end{pmatrix} \quad , \quad A^d(f) := A_\zeta \begin{pmatrix} 0 \\ f \end{pmatrix} \quad , \quad f \in \mathcal{C}_1$$

satisfying the usual commutation relations:

$$[A^u(f), A^u(g)] = 2\Im\langle \vec{f}, \vec{g} \rangle_0 = [A^d(f), A^d(g)]$$

Topological charges appear as a nontrivial coupling between these two fields.

Conclusions

1. Topological charges appears, also in the presence of the electric current, in a theory describing the quantum electromagnetic field provided that the field **is not linear** but spacelike linear.
   • This is a limitation from the mathematical point of view but not from the physical point of view: only spacelike linearity can be tested in an hypothetical experiment.
   • **Testing toplogical charges**:coherent photons, traversing a loop in the complement of another electromagnetic loop, would exhibit interference patterns, akin to the AharonovBohm effect.
2. Topological charges arise in a model with a non-Abelian gauge group without the restriction of spacelike linearity. These kind of models appears in asymptotically free gauge theories.