Background Independence in Gauge Theories

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Foundational and structural aspects of gauge theories
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Overview

- Yang-Mills gauge theory is the dynamical theory of a G-connection $\mathcal{A}$ on a principal bundle with curvature F and the Yang-Mills action:

$$S_{\mathrm{YM}} = -\frac{1}{2}\int_M F \wedge *F.$$

- For perturbative quantization, one splits

$$\mathcal{A} = \bar{\mathcal{A}} + A,$$

into a background connection $\bar{\mathcal{A}}$ (is kept classical, is a c-number)
and a dynamical $\mathfrak{g}$-valued 1-form A (is quantized perturbatively around $\bar{\mathcal{A}}$)

- For quantum gravity $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ [yesterday talk by M. Reuter].
- S_{YM} is obviously independent of the split, i.e. it is background independent (but this is more subtle for the gauge-fixed theory).
- In QFT, $\bar{\mathcal{A}}$ enters the propagators, while A is treated as a quantum field.

- In which mathematically precise sense one can define background-independence at the quantum level?
- Are there obstructions to the background independence of renormalized quantum YM and perturbative QG?

A toy model: scalar field theory I

- Let $\Phi \in C^\infty(M)$ be a scalar field on a globally hyperbolic manifold (M, g).
- We split Φ

$$\Phi = \bar{\phi} + \phi, \tag{1}$$

into a background $\bar{\phi}$ which is assumed to be on-shell, i.e. satisfying

$$(\Box - m^2)\bar{\phi} + \tfrac{1}{3!}\lambda\bar{\phi}^3 = 0. \tag{2}$$

and a dynamical perturbation ϕ whose dynamics is governed by

$$S[\bar{\phi} + \phi] = \int_M \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + (m^2 + \frac{\lambda}{2}\bar{\phi}^2)\phi^2 + \frac{\lambda}{3!}\bar{\phi}\phi^3 + \frac{\lambda}{4!}\phi^4. \tag{3}$$

- $S[\bar{\phi} + \phi]$ is independent of the split, i.e. it has *split or shift symmetry*. Infinitesimally

$$(\frac{\delta}{\delta\bar{\phi}} - \frac{\delta}{\delta\phi})S = 0. \tag{4}$$

- We define shift-invariant or background independent observables as those local $\mathcal{O}$ which infinitesimally satisfy

$$(\frac{\delta}{\delta\bar{\phi}} - \frac{\delta}{\delta\phi})\mathcal{O} = 0. \tag{5}$$

The question of background independence:

Is the shift symmetry preserved at the quantum level?

A toy model: scalar field theory II

- In the framework *locally covariant field theory* [Hollands-Wald '01], [Brunetti-Fredenhagen-Verch '03], we construct QFT as a covariant assignment of local algebra of observables to backgrounds $(M, g, \bar{\phi})$ under

$$\psi : (M', g', \bar{\phi}') \to (M, g, \bar{\phi}). \tag{6}$$

- ψ is an embedding with $\psi^* g = g'$ preserving the causal str., and $\psi^* \bar{\phi} = \bar{\phi}'$.

Algebra of observables $\mathbf{W}_{\bar{\phi}}$ at a background $\bar{\phi}$

- The space of Wick powers w.r.t. a Hadamard 2-point function $\omega_{\bar{\phi}}(x, y)$ which is a bi-solution of $P_{\bar{\phi}} = \Box_g - m^2 - \frac{1}{2}\lambda\bar{\phi}^2$,
- equipped with a product $\star_{\bar{\phi}}$ s.t.

$$[\phi(x), \phi(y)]_{\star_{\bar{\phi}}} = i\hbar\Delta_{\bar{\phi}}(x, y), \tag{7}$$

- Locally covariant time-ordered products $T_{\bar{\phi},n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n) \in D'(M^n; \mathbf{W}_{\bar{\phi}})$,
- $T_{\bar{\phi},n}$ exist and are unique up to "renormalization ambiguity",
- Renormalized interacting fields $\mathcal{O}_{\bar{\phi}}$ are defined by

$$\mathcal{O}_{\bar{\phi}}(x) := T_{\bar{\phi}}(e_{\otimes}^{iS^{\text{int}}/\hbar})^{-1} \star T_{\bar{\phi}}(\mathcal{O}(x) \otimes e_{\otimes}^{iS^{\text{int}}/\hbar}) \in \mathbf{W}_{\bar{\phi}}[[\lambda]], \tag{8}$$

$$\left[\mathcal{O}_{\bar{\phi}}(x), \mathcal{O}_{\bar{\phi}}(y)\right]_{\star_{\bar{\phi}}} = 0, \qquad \text{if } x,\ y \text{ are causally separated} \tag{9}$$

A toy model: scalar field theory III

- Given an observables $\mathcal{O}_{\bar{\phi}}$ at a given background $\bar{\phi}$, can we uniquely extend this to all other backgrounds?
- Inspired by Fedosov quantization for finite dimensional symplectic manifolds [Fedosov '94] and its generalization to pert. QFT [Collini,2016], [Hollands, unpublished], we set up a geometric setting to formulate background independence:

Geometrical definition of background-independence

- Let $\mathcal{S}$ be the "manifold" of on-shell background field configurations, i.e. solutions to the non-linear equation $(\Box - m^2)\bar{\phi} + \frac{1}{3!}\lambda\bar{\phi}^3 = 0$.
- We patch all the algebras together to obtain the algebra bundle $\mathbf{W} = \bigsqcup_{\bar{\phi}} \mathbf{W}_{\bar{\phi}} \to \mathcal{S}$.
- Interacting fields $\mathcal{O}_{\bar{\phi}}$ are viewed as smooth sections of $\mathbf{W}$.
- $T_{\bar{\phi}}\mathcal{S}$ is the linear space of solutions to linearized equation $\left(\Box - m^2 - \frac{1}{2}\lambda\bar{\phi}^2\right)\delta\bar{\phi}(x) = 0$.
- One then constructs a flat connection $\nabla^{\hbar}_{\delta\bar{\phi}}$ on $\mathbf{W} \to \mathcal{S}$ which implements variation of sections $\mathcal{O}_{\bar{\phi}}$ in the direction of $\delta\bar{\phi}$.
- Background-independent observables are defined as flat sections w.r.t. $\nabla^{\hbar}_{\delta\bar{\phi}}$:

$$\nabla^{\hbar}_{\delta\bar{\phi}}\mathcal{O}_{\bar{\phi}} = 0. \tag{10}$$

- Flatness of $\nabla^{\hbar}_{\delta\bar{\phi}}$ is necessary to extent $\mathcal{O}_{\bar{\phi}}$ via the parallel transport of $\nabla^{\hbar}_{\delta\bar{\phi}}$ to all other backgrounds in a unique and consistent way.

A toy model: scalar field theory IV

- How to construct such a flat connection $\nabla^{\hbar}_{\delta\bar{\phi}}$?
- Classical background-independent observables $\mathcal{O}$ satisfy $(\frac{\delta}{\delta\bar{\phi}} - \frac{\delta}{\delta\phi})\mathcal{O} = 0$.
- However, the naive variation "$\frac{\delta}{\delta\bar{\phi}}$" does not make sense in QFT, since variations w.r.t. $\bar{\phi}$ requires algebras $\mathbf{W}_{\bar{\phi}}$ and $\mathbf{W}_{\bar{\phi}'}$ at backgrounds $\bar{\phi}$ and $\bar{\phi}'$ to be identified.
- This identification is achieved for $\bar{\phi}' = \bar{\phi} + \delta\bar{\phi}$ via the *retarded variation*

$$\delta^R_{\delta\bar{\phi}} : \mathbf{W}_{\bar{\phi}} \to \mathbf{W}_{\bar{\phi}'}.$$

- For scalar fields in 4-d, $\delta^R_{\delta\bar{\phi}}$ can be consistently implemented on $\mathcal{O}_{\bar{\phi}}$ (principle of perturbative agreement [Hollands-Wald '02])
- Therefore,

$$\nabla^{\hbar}_{\delta\bar{\phi}} := \delta^R_{\delta\bar{\phi}} - \langle \delta\bar{\phi}, \frac{\delta}{\delta\phi} \rangle \tag{11}$$

defines the desired flat connection:

$$\nabla^{\hbar}_{\delta\bar{\phi}} \mathcal{O}_{\bar{\phi}} = \big(\langle \delta\bar{\phi}, (\tfrac{\delta}{\delta\bar{\phi}} - \tfrac{\delta}{\delta\phi})\mathcal{O} \rangle \big)_{\bar{\phi}}, \tag{12}$$

$$([\nabla^{\hbar}_{\delta\bar{\phi}}, \nabla^{\hbar}_{\delta\bar{\phi}'}] - \nabla^{\hbar}_{[\delta\bar{\phi}, \delta\bar{\phi}']})\mathcal{O}_{\bar{\phi}} = 0. \tag{13}$$

Thus, classical and quantum background-independent observables are in 1-1 correspondence

- In the Yang-Mills theory more complications arise because of gauge-fixing. We split
$$\mathcal{A} = \bar{\mathcal{A}} + A$$
into a background connection $\bar{\mathcal{A}}$ and a dynamical $\mathfrak{g}$-valued 1-form A.
- Obviously, $S_{\mathrm{YM}} = S_{\mathrm{YM}}[\bar{\mathcal{A}} + A]$ is independent of the split (shift symmetry).
- However, for perturbative quantization one has to fix the gauge (only for A), and the gauge-fixed action is not shift-symmetric.
- Nevertheless, in the BRST formalism, it turns out that this violation is not physical.

BV-BRST formalism [Becchi-Rouet-Stora, '74, Tyutin'75], [Batalin, Vilkovisky, '81]

- BRST symmetry is the fermionic symmetry of the gauge-fixed YM action.
- Its action on enlarged field configuration (including ghosts) $\Phi = (A_\mu, C, \bar{C}, B)$ is given by a nilpotent differential s with $s^2 = 0$.
- Gauge-inv. observables of the original theory are recovered as the s -cohomology.
- Anti-fields $\Phi^\ddagger = (A_\mu^\ddagger, C^\ddagger, \bar{C}^\ddagger, B^\ddagger)$ are added as a source for $s\Phi$.

Background convariant gauge-fixing

- Gauge fixing is done via adding a BRST-exact term $s\psi$ to S_{YM}.
- We choose the so-called "background covariant (Feynman) gauge"

$$\psi = \int \mathrm{Tr}\bar{C}(\bar{\nabla}^{\mu} A_{\mu} - \tfrac{1}{2} B), \tag{14}$$

where $\bar{\nabla}^{\mu}$ is the covariant derivative w.r.t. the background connection $\bar{\mathcal{A}}$.
- The full action $S = S_{\mathrm{YM}} + s\psi + s\Phi \cdot \Phi^{\ddagger}$ satisfies

$$sS = 0, \qquad (S, S) = 0, \tag{15}$$

where $(-, -)$ is the anti-bracket with $(\Phi(x), \Phi^{\ddagger}(y)) = \delta(x, y)$ and satisfies

$$s(\mathcal{O}_1, \mathcal{O}_2) = (s\mathcal{O}_1, \mathcal{O}_2) \pm (\mathcal{O}_1, s\mathcal{O}_2). \tag{16}$$

- S is not shift-invariant anymore, but the violation of shift symmetry is s-exact

$$(\frac{\delta}{\delta \bar{\mathcal{A}}} - \frac{\delta}{\delta A}) S = s(\delta\psi), \tag{17}$$

- Therefore, the gauge-fixed YM is also background-independent.

Realization of BRST symmetry at quantum level

- The BV-BRST formalism can be cast to the locally covariant and algebraic approach to pQFT [Hollands, '07], [Fredenhagen, Rejzner, '11] [today talk by K.Rejzner].
- In the presence of the background connection $\bar{\mathcal{A}}$, QFT is constructed to be covariant under *background gauge transformations* [Zahn, '12].
- In particular our gauge-fixing behaves covariantly under such transformations.
- At the quantum level, the BRST symmetry (when preserved) is implemented by the renormalized BRST charge $Q_{\bar{\mathcal{A}}}$

$$[Q_{\bar{\mathcal{A}}}, \mathcal{O}_{\bar{\mathcal{A}}}] = i\hbar(\hat{q}\mathcal{O})_{\bar{\mathcal{A}}}, \tag{18}$$

$$[Q_{\bar{\mathcal{A}}}, T_{\bar{\mathcal{A}}}(\mathcal{O}_1 \otimes \mathcal{O}_2)] = i\hbar T_{\bar{\mathcal{A}}}\big(\hat{q}\mathcal{O}_1 \otimes \mathcal{O}_2 + \mathcal{O}_1 \otimes \hat{q}\mathcal{O}_2\big) + \hbar^2\big((\mathcal{O}_1,\mathcal{O}_2)_\hbar\big)_{\bar{\mathcal{A}}}. \tag{19}$$

New algebraic structures at quantum level [Rejzner '13], [M. TT '17]

- *quantum BRST operator*

$$\hat{q}\mathcal{O} := \hat{s}\mathcal{O} + \hat{A}_1(\mathcal{O}), \qquad \text{with } \hat{q}^2 = 0. \tag{20}$$

- *quantum anti-bracket* $(\mathcal{O}_1,\mathcal{O}_2)_\hbar := (\mathcal{O}_1,\mathcal{O}_2) + \hat{A}_2(\mathcal{O}_1 \otimes \mathcal{O}_2)$ with

$$\hat{q}(\mathcal{O}_1,\mathcal{O}_2)_\hbar = (\hat{q}\mathcal{O}_1,\mathcal{O}_2)_\hbar \pm (\mathcal{O}_1,\hat{q}\mathcal{O}_2)_\hbar. \tag{21}$$

$\hat{A}_1(\mathcal{O}), \hat{A}_2(\mathcal{O}_1 \otimes \mathcal{O}_2)$ are of order $O(\hbar)$ (quantum corrections).

Background-independent observables

- For each $\bar{\mathcal{A}}$, the following algebra admits a Hilbet space representation:

$$\mathcal{F}_{\bar{\mathcal{A}}} \equiv \{\text{algebra of physical, gauge-invariant observables}\} = \frac{\ker\,[Q_{\bar{\mathcal{A}}}, -]}{\mathrm{im}\,[Q_{\bar{\mathcal{A}}}, -]}. \quad (22)$$

Geometrical definition of background-independence

- Let $\mathcal{S}$ be the manifold of on-shell background field configurations, i.e. solutions to the non-linear equation $\bar{\nabla}^\mu \bar{F}_{\mu\nu} = 0$.
- We consider the algebra bundle $\mathcal{F} = \bigsqcup_{\bar{\mathcal{A}}} \mathcal{F}_{\bar{\mathcal{A}}} \to \mathcal{S}$.
- Interacting fields $\mathcal{O}_{\bar{\mathcal{A}}}$ are viewed as smooth sections of $\mathcal{F}$.
- $\delta\bar{A} \in T_{\bar{\mathcal{A}}}\mathcal{S}$ is a $\mathfrak{g}$-valued 1-form which is a solutions to linearized field equation.
- One then constructs a connection $\nabla^\hbar_{\delta\bar{A}}$ on $\mathcal{F} \to \mathcal{S}$ which must be
  1. well-defined on $[Q_{\bar{\mathcal{A}}}, -]$-cohomology, i.e.

$$\nabla^\hbar_{\delta\bar{A}} \circ [Q_{\bar{\mathcal{A}}}, -] - [Q_{\bar{\mathcal{A}}}, -] \circ \nabla^\hbar_{\delta\bar{A}} = 0. \quad (23)$$

  2. flat up to a $[Q_{\bar{\mathcal{A}}}, -]$-exact term:

$$([\nabla^\hbar_{\delta\bar{A}}, \nabla^\hbar_{\delta\bar{A}'}] - \nabla^\hbar_{[\delta\bar{A}, \delta\bar{A}']})\mathcal{O}_{\bar{A}} \in \mathrm{im}\,[Q_{\bar{\mathcal{A}}}, -]. \quad (24)$$

- Background-independent observables are flat sections of $\nabla^\hbar_{\delta\bar{A}}$ mod $[Q_{\bar{\mathcal{A}}}, -]$-exact

$$\nabla^\hbar_{\delta\bar{A}}\mathcal{O}_{\bar{\mathcal{A}}} \in \mathrm{im}\,[Q_{\bar{\mathcal{A}}}, -]. \quad (25)$$

Possible obstructions of background independence

- Recall that the gauge-fixed S is not shift-invariant $\langle \delta\bar{A}, (\frac{\delta}{\delta\bar{\mathcal{A}}} - \frac{\delta}{\delta A})S\rangle = s(\delta_{\delta\bar{A}}\psi)$

$$\delta_{\delta\bar{A}}\psi = \langle \delta\bar{A}^{\mu}_{I}, [A_{\mu}, \bar{C}]^{I}\rangle, \qquad \text{with ghost number } -1. \tag{26}$$

- Define $\nabla^{\hbar}_{\delta\bar{A}} := \delta^{R}_{\delta\bar{A}} - \langle \delta\bar{A}, \frac{\delta}{\delta\bar{A}}\rangle$

Necessary conditions for background independence of observables

- If $\hat{A}_1(\delta_{\delta\bar{A}}\psi) = 0$, then we can write

$$\nabla^{\hbar}_{\delta\bar{A}}\mathcal{O}_{\bar{\mathcal{A}}} = (D_{\delta\bar{A}}\mathcal{O})_{\bar{\mathcal{A}}} + [Q_{\bar{\mathcal{A}}}, \mathcal{O}'_{\bar{\mathcal{A}}}], \qquad \text{for some } \mathcal{O}'. \tag{27}$$

$$\text{where } D_{\delta\bar{A}}\mathcal{O} := \langle \delta\bar{A}, (\frac{\delta}{\delta\bar{\mathcal{A}}} - \frac{\delta}{\delta\bar{A}})\mathcal{O}\rangle - (\mathcal{O}, \delta_{\delta\bar{A}}\psi)_{\hbar}. \tag{28}$$

- If $\hat{A}_2(Q \otimes \delta_{\delta\bar{A}}\psi) = 0$, then

$$D_{\delta\bar{A}} \circ \hat{q} - \hat{q} \circ D_{\delta\bar{A}} = 0, \tag{29}$$

  thus $\nabla^{\hbar}_{\delta\bar{A}}$ is well-defined on $[Q_{\bar{\mathcal{A}}}, -]$-cohomology.
- If $\hat{A}_2(\delta_{\delta\bar{A}}\psi \otimes \delta_{\delta\bar{A}'}\psi) = 0$, then

$$[D_{\delta\bar{A}}, D_{\delta\bar{A}'}] - D_{[\delta\bar{A}, \delta\bar{A}']} = 0, \tag{30}$$

  thus $\nabla^{\hbar}_{\delta\bar{A}}$ is flat up to im $[Q_{\bar{\mathcal{A}}}, \mathcal{O}'_{\bar{\mathcal{A}}}]$.

Absence of obstructions for pure YM

(1) The anomaly of retarded variation (e.g. chiral gauge or gravitational anomaly),

(2) The anomaly of Ward identity $A(e_{\otimes}^{S^{\mathrm{int}}})$ (gauge anomaly).

- The first two anomalies can be shown to be absent in 4 dimensions for pure YM.
- Other anomalies appear because of $\langle \delta \bar{A}, (\frac{\delta}{\delta \bar{\mathcal{A}}} - \frac{\delta}{\delta A}) S \rangle = s(\delta_{\delta \bar{A}} \psi)$

$$\delta_{\delta \bar{A}} \psi = \langle \delta \bar{A}, \frac{\delta}{\delta \bar{\mathcal{A}}} \Psi \rangle = \langle \delta \bar{A}_I^\mu, [A_\mu, \bar{C}]^I \rangle, \qquad \text{with ghost number } -1. \tag{31}$$

(3) $\hat{A}_1(\delta_{\delta \bar{A}} \psi) = \langle \delta \bar{A}_I^\mu, a_\mu^I \rangle$, where $a_\mu^I(x)$ has ghost nr. 0, dim 3 and satisfies $s a_\mu^I = 0$. Thus, it only transforms covariantly under background gauge transformations. Hence, it is only made out of background configurations, i.e. it is a c-number.

(4) $\hat{A}_2(Q \otimes \delta_{\delta \bar{A}} \psi) = \langle \delta \bar{A}_I^\mu, a_{\mu\nu}^I \rangle$. $a_{\mu\nu}^I(x)$ has ghost nr. $+1$, and $s a_{\mu\nu}^I = 0$. Since $H_1(s)$ is trivial, it only contains backgrounds but there is no background with gh. nr. 1.

(5) $\hat{A}_2(\delta_{\delta \bar{A}} \psi \otimes \delta_{\delta \bar{A}'} \psi) = \langle \delta \bar{A}_I^\mu \delta \bar{A}_J'^\nu, a_{\mu\nu}^{IJ} \rangle$. $a_{\mu\nu}^{IJ}(x)$ has ghost nr. 0 and $s a_{\mu\nu}^{IJ} = 0$. Similarly, there is no candidate for this term.

Potential obstructions in non-renormalizable theories

- In power-counting non-renormalizable theories, e.g. YM in higher dimensions, there is no dimension constraint for $a^I_\mu(x)$.
- An example for an obstruction term in a non-renormalizable setting would be

$$a^I_\mu = \bar{F}^I_{\mu\nu} \bar{\nabla}^\nu (F^J_{\rho\sigma} F^{\rho\sigma}_J). \tag{32}$$

Perturbative quantum gravity in 4 dim

- One splits the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, into a background $\bar{g}$ and a perturbation h.
- The construction of $\nabla^\hbar_{\delta\bar{g}}$ can be carried out similar to the YM case.
- The potential anomaly comes from $\delta_{\delta\bar{g}}\psi = \langle \delta\bar{g}^{\mu\nu}, \frac{\delta}{\delta\bar{g}^{\mu\nu}}\psi\rangle$.
- $\hat{A}_1(\delta_{\delta\bar{g}}\psi) = \langle \delta\bar{g}^{\mu\nu}, a_{\mu\nu}\rangle$, where $a_{\mu\nu}(x)$ has ghost nr.=0, and $s a_{\mu\nu} = 0$.
- Let $\mathcal{O}[g]$ be any local and gauge-invariant scalar observable [Brunetti, Fredenhagen, Rejzner '13], [Khavkine '15]
- A potential anomaly candidate would be

$$a_{\mu\nu} = \bar{g}_{\mu\nu}\mathcal{O}. \tag{33}$$

- There are indeed infinitely many of them!