Hyperbolic Differential Complexes

Pedro Lauridsen Ribeiro

pedro.ribeiro@ufabc.edu.br

Centro de Matemática, Computação e Cognição – Universidade Federal do ABC

29.V 2017

(work in progress with Michael Forger)

Script

1. Differential operators
2. Complexes of differential operators
3. The Cauchy problem
4. Hyperbolicity and gauge fixing
5. Coda

Differential operators between vector bundles

$\pi_j : \mathscr{E}_j \to \mathscr{M}$ = smooth (real or complex) vector bundles of rank D_j over the d-dim. smooth manifold $\mathscr{M}$, $j \in \mathbb{Z}$ (all our manifolds are assumed to be Hausdorff, paracompact, second countable and oriented). We denote by $\pi_j^{\mathbb{C}}$ the complexification of π_j.

$\Gamma(\pi_j) = \mathscr{C}^\infty(\mathscr{M})$-module of smooth sections of π_j.

A (linear) differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ is a (real or complex) linear map $P : \Gamma(\pi_1) \to \Gamma(\pi_2)$ such that:

- $k < 0$: $P = 0$;
- $k \geq 0$: for all $f \in \mathscr{C}^\infty(\mathscr{M})$, $\phi \in \Gamma(\pi_1)$, $[P, f]\phi = P(f\phi) - f(P\phi)$ defines a differential operator $[P, f]$ of type $\pi_1 \to \pi_2$ and order $k - 1$.

$\Rightarrow$ any differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ decreases supports: $\mathrm{supp}(P\phi) \subset \mathrm{supp}\phi$, for all $\phi \in \Gamma(\pi_1)$.

Differential operators between vector bundles

$\pi_j : \mathscr{E}_j \to \mathscr{M}$ = smooth (real or complex) vector bundles of rank D_j over the d-dim. smooth manifold $\mathscr{M}$, $j \in \mathbb{Z}$ (all our manifolds are assumed to be Hausdorff, paracompact, second countable and oriented). We denote by $\pi_j^{\mathbb{C}}$ the complexification of π_j.

$\Gamma(\pi_j) = \mathscr{C}^\infty(\mathscr{M})$-module of smooth sections of π_j.

A (linear) differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ is a (real or complex) linear map $P : \Gamma(\pi_1) \to \Gamma(\pi_2)$ such that:

- $k < 0$: $P = 0$;
- $k \geq 0$: for all $f \in \mathscr{C}^\infty(\mathscr{M})$, $\phi \in \Gamma(\pi_1)$, $[P, f]\phi = P(f\phi) - f(P\phi)$ defines a differential operator $[P, f]$ of type $\pi_1 \to \pi_2$ and order $k - 1$.

$\Rightarrow$ any differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ decreases supports: $\mathrm{supp}(P\phi) \subset \mathrm{supp}\phi$, for all $\phi \in \Gamma(\pi_1)$.

Differential operators between vector bundles

$\pi_j : \mathscr{E}_j \to \mathscr{M}$ = smooth (real or complex) vector bundles of rank D_j over the d-dim. smooth manifold $\mathscr{M}$, $j \in \mathbb{Z}$ (all our manifolds are assumed to be Hausdorff, paracompact, second countable and oriented). We denote by $\pi_j^{\mathbb{C}}$ the complexification of π_j.

$\Gamma(\pi_j) = \mathscr{C}^\infty(\mathscr{M})$-module of smooth sections of π_j.

A (linear) differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ is a (real or complex) linear map $P : \Gamma(\pi_1) \to \Gamma(\pi_2)$ such that:

- $k < 0$: $P = 0$;
- $k \geq 0$: for all $f \in \mathscr{C}^\infty(\mathscr{M})$, $\phi \in \Gamma(\pi_1)$, $[P, f]\phi = P(f\phi) - f(P\phi)$ defines a differential operator $[P, f]$ of type $\pi_1 \to \pi_2$ and order $k - 1$.

$\Rightarrow$ any differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ decreases supports: $\mathrm{supp}(P\phi) \subset \mathrm{supp}\phi$, for all $\phi \in \Gamma(\pi_1)$.

Universidade Federal do ABC

Differential operators between vector bundles

$\pi_j : \mathscr{E}_j \to \mathscr{M}$ = smooth (real or complex) vector bundles of rank D_j over the d-dim. smooth manifold $\mathscr{M}$, $j \in \mathbb{Z}$ (all our manifolds are assumed to be Hausdorff, paracompact, second countable and oriented). We denote by $\pi_j^{\mathbb{C}}$ the complexification of π_j.

$\Gamma(\pi_j) = \mathscr{C}^\infty(\mathscr{M})$-module of smooth sections of π_j.

A (linear) differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ is a (real or complex) linear map $P : \Gamma(\pi_1) \to \Gamma(\pi_2)$ such that:

- $k < 0$: $P = 0$;
- $k \geq 0$: for all $f \in \mathscr{C}^\infty(\mathscr{M})$, $\phi \in \Gamma(\pi_1)$, $[P, f]\phi = P(f\phi) - f(P\phi)$ defines a differential operator $[P, f]$ of type $\pi_1 \to \pi_2$ and order $k - 1$.

$\Rightarrow$ any differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ decreases supports: $\mathrm{supp}(P\phi) \subset \mathrm{supp}\phi$, for all $\phi \in \Gamma(\pi_1)$.

Differential operators between vector bundles

$\pi_j : \mathscr{E}_j \to \mathscr{M}$ = smooth (real or complex) vector bundles of rank D_j over the d-dim. smooth manifold $\mathscr{M}$, $j \in \mathbb{Z}$ (all our manifolds are assumed to be Hausdorff, paracompact, second countable and oriented). We denote by $\pi_j^{\mathbb{C}}$ the complexification of π_j.

$\Gamma(\pi_j) = \mathscr{C}^\infty(\mathscr{M})$-module of smooth sections of π_j.

A (linear) differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ is a (real or complex) linear map $P : \Gamma(\pi_1) \to \Gamma(\pi_2)$ such that:

- $k < 0$: $P = 0$;
- $k \geq 0$: for all $f \in \mathscr{C}^\infty(\mathscr{M})$, $\phi \in \Gamma(\pi_1)$, $[P, f]\phi = P(f\phi) - f(P\phi)$ defines a differential operator $[P, f]$ of type $\pi_1 \to \pi_2$ and order $k - 1$.

$\Rightarrow$ any differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ decreases supports: $\mathrm{supp}(P\phi) \subset \mathrm{supp}\phi$, for all $\phi \in \Gamma(\pi_1)$.

Differential operators between vector bundles

$\pi_j : \mathscr{E}_j \to \mathscr{M}$ = smooth (real or complex) vector bundles of rank D_j over the d-dim. smooth manifold $\mathscr{M}$, $j \in \mathbb{Z}$ (all our manifolds are assumed to be Hausdorff, paracompact, second countable and oriented). We denote by $\pi_j^{\mathbb{C}}$ the complexification of π_j.

$\Gamma(\pi_j) = \mathscr{C}^\infty(\mathscr{M})$-module of smooth sections of π_j.

A (linear) differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ is a (real or complex) linear map $P : \Gamma(\pi_1) \to \Gamma(\pi_2)$ such that:

- $k < 0$: $P = 0$;
- $k \geq 0$: for all $f \in \mathscr{C}^\infty(\mathscr{M})$, $\phi \in \Gamma(\pi_1)$, $[P, f]\phi = P(f\phi) - f(P\phi)$ defines a differential operator $[P, f]$ of type $\pi_1 \to \pi_2$ and order $k - 1$.

$\Rightarrow$ any differential operator of type $\pi_1 \to \pi_2$ and order $k \in \mathbb{Z}$ decreases supports: $\mathrm{supp}(P\phi) \subset \mathrm{supp}\phi$, for all $\phi \in \Gamma(\pi_1)$.

Conversely, Peetre's theorem asserts that any (real or complex) linear map $P:\Gamma(\pi_1)\to\Gamma(\pi_2)$ which decreases supports is (locally) a differential operator of type $\pi_1\to\pi_2$ and order k for some k. Moreover:

- $k\leq 0$: P is a $\mathscr{C}^\infty(\mathscr{M})$-linear map from $\Gamma(\pi_1)$ into $\Gamma(\pi_2)$ = smooth section a_k of $\pi'_1\otimes\pi_2$ ($\pi'_j:\mathscr{E}^*_j\to\mathscr{M}$ = dual bundle to π_j). $a_k=0$ if $k<0$;
- $k>0$: given a linear connection on π_1 and a linear connection on $\pi_{T\mathscr{M}}:T\mathscr{M}\to\mathscr{M}$, both denoted by ∇, we can use Leibniz's rule and extend them to a linear connection ∇ on $\oplus_{r\geq 0}\otimes^r\pi'_{T\mathscr{M}}\otimes\pi_1\Rightarrow$ there are $\mathscr{C}^\infty(\mathscr{M})$-linear maps $a_r=a_{P,\nabla,r}$ from $\Gamma(\otimes^r\phi'_{T\mathscr{M}}\otimes\pi_1)$ into $\Gamma(\pi_2)=$ smooth sections of $\otimes^r\pi_{T\mathscr{M}}\otimes\pi'_1\otimes\pi_2$, $r=0,1,\ldots,k$ such that for all $\phi\in\Gamma(\pi_1)$

$$P\phi=\sum_{r=0}^{k}a_r\nabla^r\phi\ ,\quad \nabla^0\phi=\phi\ ,\ \nabla^r\phi=\nabla(\nabla^{r-1}\phi)\ .$$

The a_r's may (and shall) be uniquely chosen to be symmetric with respect to their arguments in $T^*\mathscr{M}$, and extended to a smooth section of the complexified bundle $\otimes^r\pi^{\mathbb{C}}_{T\mathscr{M}}\otimes{\pi'}^{\mathbb{C}}_1\otimes\pi^{\mathbb{C}}_2$.

Conversely, Peetre's theorem asserts that any (real or complex) linear map $P : \Gamma(\pi_1) \to \Gamma(\pi_2)$ which decreases supports is (locally) a differential operator of type $\pi_1 \to \pi_2$ and order k for some k. Moreover:

- $k \leq 0$: P is a $\mathscr{C}^\infty(\mathscr{M})$-linear map from $\Gamma(\pi_1)$ into $\Gamma(\pi_2)$ = smooth section a_k of $\pi_1' \otimes \pi_2$ ($\pi_j' : \mathscr{E}_j^* \to \mathscr{M}$ = dual bundle to π_j). $a_k = 0$ if $k < 0$;
- $k > 0$: given a linear connection on π_1 and a linear connection on $\pi_{T\mathscr{M}} : T\mathscr{M} \to \mathscr{M}$, both denoted by ∇, we can use Leibniz's rule and extend them to a linear connection ∇ on $\oplus_{r\geq 0} \otimes^r \pi'_{T\mathscr{M}} \otimes \pi_1 \Rightarrow$ there are $\mathscr{C}^\infty(\mathscr{M})$-linear maps $a_r = a_{P,\nabla,r}$ from $\Gamma(\otimes^r \phi'_{T\mathscr{M}} \otimes \pi_1)$ into $\Gamma(\pi_2)$ = smooth sections of $\otimes^r \pi_{T\mathscr{M}} \otimes \pi_1' \otimes \pi_2$, $r = 0, 1, \ldots, k$ such that for all $\phi \in \Gamma(\pi_1)$

$$P\phi = \sum_{r=0}^{k} a_r \nabla^r \phi \, , \quad \nabla^0\phi = \phi \, , \, \nabla^r\phi = \nabla(\nabla^{r-1}\phi) \, .$$

The a_r's may (and shall) be uniquely chosen to be symmetric with respect to their arguments in $T^*\mathscr{M}$, and extended to a smooth section of the complexified bundle $\otimes^r \pi^{\mathbb{C}}_{T\mathscr{M}} \otimes \pi'^{\mathbb{C}}_1 \otimes \pi^{\mathbb{C}}_2$.

Universidade Federal do ABC

Conversely, Peetre's theorem asserts that any (real or complex) linear map $P:\Gamma(\pi_1)\to\Gamma(\pi_2)$ which decreases supports is (locally) a differential operator of type $\pi_1\to\pi_2$ and order k for some k. Moreover:

- $k\leq 0$: P is a $\mathscr{C}^\infty(\mathscr{M})$-linear map from $\Gamma(\pi_1)$ into $\Gamma(\pi_2)$ = smooth section a_k of $\pi'_1\otimes\pi_2$ ($\pi'_j:\mathscr{E}^*_j\to\mathscr{M}$ = dual bundle to π_j). $a_k=0$ if $k<0$;
- $k>0$: given a linear connection on π_1 and a linear connection on $\pi_{T\mathscr{M}}:T\mathscr{M}\to\mathscr{M}$, both denoted by ∇, we can use Leibniz's rule and extend them to a linear connection ∇ on $\oplus_{r\geq 0}\otimes^r\pi'_{T\mathscr{M}}\otimes\pi_1$ $\Rightarrow$ there are $\mathscr{C}^\infty(\mathscr{M})$-linear maps $a_r=a_{P,\nabla,r}$ from $\Gamma(\otimes^r\phi'_{T\mathscr{M}}\otimes\pi_1)$ into $\Gamma(\pi_2)$ = smooth sections of $\otimes^r\pi_{T\mathscr{M}}\otimes\pi'_1\otimes\pi_2$, $r=0,1,\ldots,k$ such that for all $\phi\in\Gamma(\pi_1)$

$$P\phi=\sum_{r=0}^{k}a_r\nabla^r\phi\ ,\quad \nabla^0\phi=\phi\ ,\ \nabla^r\phi=\nabla(\nabla^{r-1}\phi)\ .$$

The a_r's may (and shall) be uniquely chosen to be symmetric with respect to their arguments in $T^*\mathscr{M}$, and extended to a smooth section of the complexified bundle $\otimes^r\pi^{\mathbb{C}}_{T\mathscr{M}}\otimes\pi'^{\mathbb{C}}_1\otimes\pi^{\mathbb{C}}_2$.

Conversely, Peetre's theorem asserts that any (real or complex) linear map $P : \Gamma(\pi_1) \to \Gamma(\pi_2)$ which decreases supports is (locally) a differential operator of type $\pi_1 \to \pi_2$ and order k for some k. Moreover:

- $k \leq 0$: P is a $\mathscr{C}^\infty(\mathscr{M})$-linear map from $\Gamma(\pi_1)$ into $\Gamma(\pi_2)$ = smooth section a_k of $\pi_1' \otimes \pi_2$ ($\pi_j' : \mathscr{E}_j^* \to \mathscr{M}$ = dual bundle to π_j). $a_k = 0$ if $k < 0$;
- $k > 0$: given a linear connection on π_1 and a linear connection on $\pi_{T\mathscr{M}} : T\mathscr{M} \to \mathscr{M}$, both denoted by ∇, we can use Leibniz's rule and extend them to a linear connection ∇ on $\oplus_{r\geq 0} \otimes^r \pi_{T\mathscr{M}}' \otimes \pi_1 \Rightarrow$ there are $\mathscr{C}^\infty(\mathscr{M})$-linear maps $a_r = a_{P,\nabla,r}$ from $\Gamma(\otimes^r \phi_{T\mathscr{M}}' \otimes \pi_1)$ into $\Gamma(\pi_2)$ = smooth sections of $\otimes^r \pi_{T\mathscr{M}} \otimes \pi_1' \otimes \pi_2$, $r = 0, 1, \ldots, k$ such that for all $\phi \in \Gamma(\pi_1)$

$$P\phi = \sum_{r=0}^{k} a_r \nabla^r \phi \,, \quad \nabla^0 \phi = \phi \,, \nabla^r \phi = \nabla(\nabla^{r-1}\phi) \,.$$

The a_r's may (and shall) be uniquely chosen to be symmetric with respect to their arguments in $T^*\mathscr{M}$, and extended to a smooth section of the complexified bundle $\otimes^r \pi_{T\mathscr{M}}^{\mathbb{C}} \otimes {\pi'}_1^{\mathbb{C}} \otimes \pi_2^{\mathbb{C}}$.

Conversely, Peetre's theorem asserts that any (real or complex) linear map $P:\Gamma(\pi_1)\to\Gamma(\pi_2)$ which decreases supports is (locally) a differential operator of type $\pi_1\to\pi_2$ and order k for some k. Moreover:

- $k\leq 0$: P is a $\mathscr{C}^\infty(\mathscr{M})$-linear map from $\Gamma(\pi_1)$ into $\Gamma(\pi_2)$ = smooth section a_k of $\pi'_1\otimes\pi_2$ ($\pi'_j:\mathscr{E}^*_j\to\mathscr{M}$ = dual bundle to π_j). $a_k=0$ if $k<0$;
- $k>0$: given a linear connection on π_1 and a linear connection on $\pi_{T\mathscr{M}}:T\mathscr{M}\to\mathscr{M}$, both denoted by ∇, we can use Leibniz's rule and extend them to a linear connection ∇ on $\oplus_{r\geq 0}\otimes^r\pi'_{T\mathscr{M}}\otimes\pi_1$ $\Rightarrow$ there are $\mathscr{C}^\infty(\mathscr{M})$-linear maps $a_r=a_{P,\nabla,r}$ from $\Gamma(\otimes^r\phi'_{T\mathscr{M}}\otimes\pi_1)$ into $\Gamma(\pi_2)$ = smooth sections of $\otimes^r\pi_{T\mathscr{M}}\otimes\pi'_1\otimes\pi_2$, $r=0,1,\ldots,k$ such that for all $\phi\in\Gamma(\pi_1)$

$$P\phi=\sum_{r=0}^{k}a_r\nabla^r\phi\ ,\quad \nabla^0\phi=\phi\ ,\ \nabla^r\phi=\nabla(\nabla^{r-1}\phi)\ .$$

The a_r's may (and shall) be uniquely chosen to be symmetric with respect to their arguments in $T^*\mathscr{M}$, and extended to a smooth section of the complexified bundle $\otimes^r\pi^{\mathbb{C}}_{T\mathscr{M}}\otimes{\pi'}^{\mathbb{C}}_1\otimes\pi^{\mathbb{C}}_2$.

Conversely, Peetre's theorem asserts that any (real or complex) linear map $P:\Gamma(\pi_1)\to\Gamma(\pi_2)$ which decreases supports is (locally) a differential operator of type $\pi_1\to\pi_2$ and order k for some k. Moreover:

- $k\leq 0$: P is a $\mathscr{C}^\infty(\mathscr{M})$-linear map from $\Gamma(\pi_1)$ into $\Gamma(\pi_2)$ = smooth section a_k of $\pi'_1\otimes\pi_2$ ($\pi'_j:\mathscr{E}^*_j\to\mathscr{M}$ = dual bundle to π_j). $a_k=0$ if $k<0$;
- $k>0$: given a linear connection on π_1 and a linear connection on $\pi_{T\mathscr{M}}:T\mathscr{M}\to\mathscr{M}$, both denoted by ∇, we can use Leibniz's rule and extend them to a linear connection ∇ on $\oplus_{r\geq 0}\otimes^r\pi'_{T\mathscr{M}}\otimes\pi_1$ $\Rightarrow$ there are $\mathscr{C}^\infty(\mathscr{M})$-linear maps $a_r=a_{P,\nabla,r}$ from $\Gamma(\otimes^r\phi'_{T\mathscr{M}}\otimes\pi_1)$ into $\Gamma(\pi_2)=$ smooth sections of $\otimes^r\pi_{T\mathscr{M}}\otimes\pi'_1\otimes\pi_2$, $r=0,1,\ldots,k$ such that for all $\phi\in\Gamma(\pi_1)$

$$P\phi=\sum_{r=0}^{k}a_r\nabla^r\phi\ ,\quad \nabla^0\phi=\phi\ ,\ \nabla^r\phi=\nabla(\nabla^{r-1}\phi)\ .$$

The a_r's may (and shall) be uniquely chosen to be symmetric with respect to their arguments in $T^*\mathscr{M}$, and extended to a smooth section of the complexified bundle $\otimes^r\pi^{\mathbb{C}}_{T\mathscr{M}}\otimes{\pi'}^{\mathbb{C}}_1\otimes\pi^{\mathbb{C}}_2$.

Universidade Federal do ABC

$\Rightarrow a_k$ does not depend on the choice of ∇. Indeed, given $p \in \mathscr{M}$, $f \in \mathscr{C}^\infty(\mathscr{M})$ such that $f(p) = 0$, $\mathrm{d}f(p) = \xi$, we have that

$$a_k(p,\xi)\phi(p) \doteq a_k(\otimes^k \xi \otimes \phi(p)) = \frac{1}{k!} P(f^k \phi)(p)$$

$\Rightarrow (a_k)_p(\xi) \doteq a_k(p,\xi)$ is a linear map from $\pi_1^{-1}(p)$ into $\pi_2^{-1}(p)$ depending smoothly on $(p,\xi) \in T^*\mathscr{M}$, called the leading coefficient of P at (p,ξ).

(*Remark:* $a_k = i^{-k} \times$ the principal symbol of P).

Let $\pi_1 = \pi_2 = \pi : \mathscr{E} \to \mathscr{M}$, $(p,\xi) \in T^*\mathscr{M} \setminus 0$. We say that P is

Elliptic at (p,ξ) – $a_k(p,\xi)$ is non-singular $\Rightarrow (p,\xi) =$ non-characteristic covector for P;

Hyperbolic at (p,ξ) – (p,ξ) is non-characteristic for P and for all $\eta \in T^*_p\mathscr{M}$, the roots of the polynomial $q(\lambda) \doteq \det a_k(p, \lambda\xi + \eta)$ are all real.

$\Rightarrow a_k$ does not depend on the choice of ∇. Indeed, given $p \in \mathscr{M}$, $f \in \mathscr{C}^\infty(\mathscr{M})$ such that $f(p) = 0$, $\mathrm{d}f(p) = \xi$, we have that

$$a_k(p,\xi)\phi(p) \doteq a_k(\otimes^k \xi \otimes \phi(p)) = \frac{1}{k!} P(f^k \phi)(p)$$

$\Rightarrow (a_k)_p(\xi) \doteq a_k(p,\xi)$ is a linear map from $\pi_1^{-1}(p)$ into $\pi_2^{-1}(p)$ depending smoothly on $(p,\xi) \in T^*\mathscr{M}$, called the leading coefficient of P at (p,ξ).

(*Remark:* $a_k = i^{-k} \times$ the principal symbol of P).

Let $\pi_1 = \pi_2 = \pi : \mathscr{E} \to \mathscr{M}$, $(p,\xi) \in T^*\mathscr{M} \smallsetminus 0$. We say that P is

Elliptic at (p,ξ) – $a_k(p,\xi)$ is non-singular $\Rightarrow (p,\xi) =$ non-characteristic covector for P;

Hyperbolic at (p,ξ) – (p,ξ) is non-characteristic for P and for all $\eta \in T_p^*\mathscr{M}$, the roots of the polynomial $q(\lambda) \doteq \det a_k(p, \lambda\xi + \eta)$ are all real.

$\Rightarrow a_k$ does not depend on the choice of ∇. Indeed, given $p \in \mathscr{M}$, $f \in \mathscr{C}^\infty(\mathscr{M})$ such that $f(p) = 0$, $\mathrm{d}f(p) = \xi$, we have that

$$a_k(p,\xi)\phi(p) \doteq a_k(\otimes^k \xi \otimes \phi(p)) = \frac{1}{k!} P(f^k \phi)(p)$$

$\Rightarrow (a_k)_p(\xi) \doteq a_k(p,\xi)$ is a linear map from $\pi_1^{-1}(p)$ into $\pi_2^{-1}(p)$ depending smoothly on $(p,\xi) \in T^*\mathscr{M}$, called the leading coefficient of P at (p,ξ).

(*Remark:* $a_k = i^{-k} \times$ the principal symbol of P).

Let $\pi_1 = \pi_2 = \pi : \mathscr{E} \to \mathscr{M}$, $(p,\xi) \in T^*\mathscr{M} \smallsetminus 0$. We say that P is

Elliptic at (p,ξ) – $a_k(p,\xi)$ is non-singular $\Rightarrow (p,\xi) =$ non-characteristic covector for P;

Hyperbolic at (p,ξ) – (p,ξ) is non-characteristic for P and for all $\eta \in T_p^*\mathscr{M}$, the roots of the polynomial $q(\lambda) \doteq \det a_k(p, \lambda\xi + \eta)$ are all real.

$\Rightarrow a_k$ does not depend on the choice of ∇. Indeed, given $p \in \mathscr{M}$, $f \in \mathscr{C}^\infty(\mathscr{M})$ such that $f(p) = 0$, $\mathrm{d}f(p) = \xi$, we have that

$$a_k(p,\xi)\phi(p) \doteq a_k(\otimes^k \xi \otimes \phi(p)) = \frac{1}{k!} P(f^k \phi)(p)$$

$\Rightarrow (a_k)_p(\xi) \doteq a_k(p,\xi)$ is a linear map from $\pi_1^{-1}(p)$ into $\pi_2^{-1}(p)$ depending smoothly on $(p,\xi) \in T^*\mathscr{M}$, called the leading coefficient of P at (p,ξ).

(*Remark:* $a_k = i^{-k} \times$ the principal symbol of P).

Let $\pi_1 = \pi_2 = \pi : \mathscr{E} \to \mathscr{M}$, $(p,\xi) \in T^*\mathscr{M} \smallsetminus 0$. We say that P is

Elliptic at (p,ξ) – $a_k(p,\xi)$ is non-singular $\Rightarrow (p,\xi) =$ non-characteristic covector for P;

Hyperbolic at (p,ξ) – (p,ξ) is non-characteristic for P and for all $\eta \in T^*_p\mathscr{M}$, the roots of the polynomial $q(\lambda) \doteq \det a_k(p, \lambda\xi + \eta)$ are all real.

$\Rightarrow a_k$ does not depend on the choice of ∇. Indeed, given $p \in \mathscr{M}$, $f \in \mathscr{C}^\infty(\mathscr{M})$ such that $f(p) = 0$, $\mathrm{d}f(p) = \xi$, we have that

$$a_k(p,\xi)\phi(p) \doteq a_k(\otimes^k \xi \otimes \phi(p)) = \frac{1}{k!} P(f^k \phi)(p)$$

$\Rightarrow (a_k)_p(\xi) \doteq a_k(p,\xi)$ is a linear map from $\pi_1^{-1}(p)$ into $\pi_2^{-1}(p)$ depending smoothly on $(p,\xi) \in T^*\mathscr{M}$, called the leading coefficient of P at (p,ξ).

(*Remark:* $a_k = i^{-k} \times$ the principal symbol of P).

Let $\pi_1 = \pi_2 = \pi : \mathscr{E} \to \mathscr{M}$, $(p,\xi) \in T^*\mathscr{M} \smallsetminus 0$. We say that P is

Elliptic at (p,ξ) – $a_k(p,\xi)$ is non-singular $\Rightarrow (p,\xi) =$ non-characteristic covector for P;

Hyperbolic at (p,ξ) – (p,ξ) is non-characteristic for P and for all $\eta \in T^*_p\mathscr{M}$, the roots of the polynomial $q(\lambda) \doteq \det a_k(p, \lambda\xi + \eta)$ are all real.

$\Rightarrow a_k$ does not depend on the choice of ∇. Indeed, given $p \in \mathscr{M}$, $f \in \mathscr{C}^\infty(\mathscr{M})$ such that $f(p) = 0$, $\mathrm{d}f(p) = \xi$, we have that

$$a_k(p,\xi)\phi(p) \doteq a_k(\otimes^k \xi \otimes \phi(p)) = \frac{1}{k!} P(f^k \phi)(p)$$

$\Rightarrow (a_k)_p(\xi) \doteq a_k(p,\xi)$ is a linear map from $\pi_1^{-1}(p)$ into $\pi_2^{-1}(p)$ depending smoothly on $(p,\xi) \in T^*\mathscr{M}$, called the leading coefficient of P at (p,ξ).

(*Remark:* $a_k = i^{-k} \times$ the principal symbol of P).

Let $\pi_1 = \pi_2 = \pi : \mathscr{E} \to \mathscr{M}$, $(p,\xi) \in T^*\mathscr{M} \smallsetminus 0$. We say that P is

Elliptic at (p,ξ) – $a_k(p,\xi)$ is non-singular $\Rightarrow (p,\xi) =$ non-characteristic covector for P;

Hyperbolic at (p,ξ) – (p,ξ) is non-characteristic for P and for all $\eta \in T_p^*\mathscr{M}$, the roots of the polynomial $q(\lambda) \doteq \det a_k(p, \lambda\xi + \eta)$ are all real.

$\Rightarrow a_k$ does not depend on the choice of ∇. Indeed, given $p \in \mathscr{M}$, $f \in \mathscr{C}^\infty(\mathscr{M})$ such that $f(p) = 0$, $\mathrm{d}f(p) = \xi$, we have that

$$a_k(p,\xi)\phi(p) \doteq a_k(\otimes^k \xi \otimes \phi(p)) = \frac{1}{k!} P(f^k \phi)(p)$$

$\Rightarrow (a_k)_p(\xi) \doteq a_k(p,\xi)$ is a linear map from $\pi_1^{-1}(p)$ into $\pi_2^{-1}(p)$ depending smoothly on $(p,\xi) \in T^*\mathscr{M}$, called the leading coefficient of P at (p,ξ).

(*Remark:* $a_k = i^{-k} \times$ the principal symbol of P).

Let $\pi_1 = \pi_2 = \pi : \mathscr{E} \to \mathscr{M}$, $(p,\xi) \in T^*\mathscr{M} \smallsetminus 0$. We say that P is

Elliptic at (p,ξ) – $a_k(p,\xi)$ is non-singular $\Rightarrow (p,\xi) =$ non-characteristic covector for P;

Hyperbolic at (p,ξ) – (p,ξ) is non-characteristic for P and for all $\eta \in T^*_p\mathscr{M}$, the roots of the polynomial $q(\lambda) \doteq \det a_k(p, \lambda\xi + \eta)$ are all real.

Complexes of differential operators

A complex of differential operators (or simply a differential complex) is a sequence $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$, where P_j is a differential operator of type $\pi_j \to \pi_{j+1}$ and order $k_j \in \mathbb{Z}$:

$$\cdots \longrightarrow \Gamma(\pi_{j-1}) \xrightarrow{P_{j-1}} \Gamma(\pi_j) \xrightarrow{P_j} \Gamma(\pi_{j+1}) \longrightarrow \cdots$$

such that $P_j \circ P_{j-1} = 0$ for all j. At each $(p,\xi) \in \mathbb{C}T^*\mathscr{M} \smallsetminus 0$, there is an associated symbolic complex

$$\cdots \longrightarrow (\pi^{\mathbb{C}}_{j-1})^{-1}(p) \xrightarrow{\sigma_{P_{j-1}}(p,\xi)} (\pi^{\mathbb{C}}_{j})^{-1}(p) \xrightarrow{\sigma_{P_j}(p,\xi)} (\pi^{\mathbb{C}}_{j+1})^{-1}(p) \longrightarrow \cdots$$

where $\sigma_{P_j} \doteq a_{j,k_j}$ is the leading coefficient of P_j. We say that (p,ξ) is a non-characteristic covector for $\mathscr{P}$ if $\mathscr{P}$'s symbolic complex at (p,ξ) is exact. Otherwise, we say that (p,ξ) is characteristic. If every nonzero covector in $T^*\mathscr{M}$ is non-characteristic for $\mathscr{P}$, we say that $\mathscr{P}$ is an elliptic complex (Atiyah-Bott, Spencer).

Complexes of differential operators

A complex of differential operators (or simply a differential complex) is a sequence $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$, where P_j is a differential operator of type $\pi_j \to \pi_{j+1}$ and order $k_j \in \mathbb{Z}$:

$$\cdots \longrightarrow \Gamma(\pi_{j-1}) \xrightarrow{P_{j-1}} \Gamma(\pi_j) \xrightarrow{P_j} \Gamma(\pi_{j+1}) \longrightarrow \cdots$$

such that $P_j \circ P_{j-1} = 0$ for all j. At each $(p,\xi) \in \mathbb{C}T^*\mathscr{M} \smallsetminus 0$, there is an associated symbolic complex

$$\cdots \longrightarrow (\pi_{j-1}^{\mathbb{C}})^{-1}(p) \xrightarrow{\sigma_{P_{j-1}}(p,\xi)} (\pi_j^{\mathbb{C}})^{-1}(p) \xrightarrow{\sigma_{P_j}(p,\xi)} (\pi_{j+1}^{\mathbb{C}})^{-1}(p) \longrightarrow \cdots$$

where $\sigma_{P_j} \doteq a_{j,k_j}$ is the leading coefficient of P_j. We say that (p,ξ) is a non-characteristic covector for $\mathscr{P}$ if $\mathscr{P}$'s symbolic complex at (p,ξ) is exact. Otherwise, we say that (p,ξ) is characteristic. If every nonzero covector in $T^*\mathscr{M}$ is non-characteristic for $\mathscr{P}$, we say that $\mathscr{P}$ is an elliptic complex (Atiyah-Bott, Spencer).

Complexes of differential operators

A complex of differential operators (or simply a differential complex) is a sequence $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$, where P_j is a differential operator of type $\pi_j \to \pi_{j+1}$ and order $k_j \in \mathbb{Z}$:

$$\cdots \longrightarrow \Gamma(\pi_{j-1}) \xrightarrow{P_{j-1}} \Gamma(\pi_j) \xrightarrow{P_j} \Gamma(\pi_{j+1}) \longrightarrow \cdots$$

such that $P_j \circ P_{j-1} = 0$ for all j. At each $(p,\xi) \in \mathbb{C}T^*\mathscr{M} \smallsetminus 0$, there is an associated symbolic complex

$$\cdots \longrightarrow (\pi^{\mathbb{C}}_{j-1})^{-1}(p) \xrightarrow{\sigma_{P_{j-1}}(p,\xi)} (\pi^{\mathbb{C}}_{j})^{-1}(p) \xrightarrow{\sigma_{P_{j}}(p,\xi)} (\pi^{\mathbb{C}}_{j+1})^{-1}(p) \longrightarrow \cdots$$

where $\sigma_{P_j} \doteq a_{j,k_j}$ is the leading coefficient of P_j. We say that (p,ξ) is a non-characteristic covector for $\mathscr{P}$ if $\mathscr{P}$'s symbolic complex at (p,ξ) is exact. Otherwise, we say that (p,ξ) is characteristic. If every nonzero covector in $T^*\mathscr{M}$ is non-characteristic for $\mathscr{P}$, we say that $\mathscr{P}$ is an elliptic complex (Atiyah-Bott, Spencer).

Complexes of differential operators

A complex of differential operators (or simply a differential complex) is a sequence $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$, where P_j is a differential operator of type $\pi_j \to \pi_{j+1}$ and order $k_j \in \mathbb{Z}$:

$$\cdots \longrightarrow \Gamma(\pi_{j-1}) \xrightarrow{\quad P_{j-1} \quad} \Gamma(\pi_j) \xrightarrow{\quad P_j \quad} \Gamma(\pi_{j+1}) \longrightarrow \cdots$$

such that $P_j \circ P_{j-1} = 0$ for all j. At each $(p,\xi) \in \mathbb{C}T^*\mathscr{M} \smallsetminus 0$, there is an associated symbolic complex

$$\cdots \longrightarrow (\pi^{\mathbb{C}}_{j-1})^{-1}(p) \xrightarrow{\sigma_{P_{j-1}}(p,\xi)} (\pi^{\mathbb{C}}_{j})^{-1}(p) \xrightarrow{\sigma_{P_{j}}(p,\xi)} (\pi^{\mathbb{C}}_{j+1})^{-1}(p) \longrightarrow \cdots$$

where $\sigma_{P_j} \doteq a_{j,k_j}$ is the leading coefficient of P_j. We say that (p,ξ) is a non-characteristic covector for $\mathscr{P}$ if $\mathscr{P}$'s symbolic complex at (p,ξ) is exact. Otherwise, we say that (p,ξ) is characteristic. If every nonzero covector in $T^*\mathscr{M}$ is non-characteristic for $\mathscr{P}$, we say that $\mathscr{P}$ is an elliptic complex (Atiyah-Bott, Spencer).

Complexes of differential operators

A complex of differential operators (or simply a differential complex) is a sequence $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$, where P_j is a differential operator of type $\pi_j \to \pi_{j+1}$ and order $k_j \in \mathbb{Z}$:

$$\cdots \longrightarrow \Gamma(\pi_{j-1}) \xrightarrow{\quad P_{j-1} \quad} \Gamma(\pi_j) \xrightarrow{\quad P_j \quad} \Gamma(\pi_{j+1}) \longrightarrow \cdots$$

such that $P_j \circ P_{j-1} = 0$ for all j. At each $(p,\xi) \in \mathbb{C}T^*\mathscr{M} \smallsetminus 0$, there is an associated symbolic complex

$$\cdots \longrightarrow (\pi^{\mathbb{C}}_{j-1})^{-1}(p) \xrightarrow{\sigma_{P_{j-1}}(p,\xi)} (\pi^{\mathbb{C}}_{j})^{-1}(p) \xrightarrow{\sigma_{P_{j}}(p,\xi)} (\pi^{\mathbb{C}}_{j+1})^{-1}(p) \longrightarrow \cdots$$

where $\sigma_{P_j} \doteq a_{j,k_j}$ is the leading coefficient of P_j. We say that (p,ξ) is a non-characteristic covector for $\mathscr{P}$ if $\mathscr{P}$'s symbolic complex at (p,ξ) is exact. Otherwise, we say that (p,ξ) is characteristic. If every nonzero covector in $T^*\mathscr{M}$ is non-characteristic for $\mathscr{P}$, we say that $\mathscr{P}$ is an elliptic complex (Atiyah-Bott, Spencer).

Complexes of differential operators

A complex of differential operators (or simply a differential complex) is a sequence $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$, where P_j is a differential operator of type $\pi_j \to \pi_{j+1}$ and order $k_j \in \mathbb{Z}$:

$$\cdots \longrightarrow \Gamma(\pi_{j-1}) \xrightarrow{\quad P_{j-1} \quad} \Gamma(\pi_j) \xrightarrow{\quad P_j \quad} \Gamma(\pi_{j+1}) \longrightarrow \cdots$$

such that $P_j \circ P_{j-1} = 0$ for all j. At each $(p,\xi) \in \mathbb{C}T^*\mathscr{M} \smallsetminus 0$, there is an associated symbolic complex

$$\cdots \longrightarrow (\pi^{\mathbb{C}}_{j-1})^{-1}(p) \xrightarrow{\sigma_{P_{j-1}}(p,\xi)} (\pi^{\mathbb{C}}_{j})^{-1}(p) \xrightarrow{\sigma_{P_{j}}(p,\xi)} (\pi^{\mathbb{C}}_{j+1})^{-1}(p) \longrightarrow \cdots$$

where $\sigma_{P_j} \doteq a_{j,k_j}$ is the leading coefficient of P_j. We say that (p,ξ) is a non-characteristic covector for $\mathscr{P}$ if $\mathscr{P}$'s symbolic complex at (p,ξ) is exact. Otherwise, we say that (p,ξ) is characteristic. If every nonzero covector in $T^*\mathscr{M}$ is non-characteristic for $\mathscr{P}$, we say that $\mathscr{P}$ is an elliptic complex (Atiyah-Bott, Spencer).

Complexes of differential operators

A complex of differential operators (or simply a differential complex) is a sequence $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$, where P_j is a differential operator of type $\pi_j \to \pi_{j+1}$ and order $k_j \in \mathbb{Z}$:

$$\cdots \longrightarrow \Gamma(\pi_{j-1}) \xrightarrow{P_{j-1}} \Gamma(\pi_j) \xrightarrow{P_j} \Gamma(\pi_{j+1}) \longrightarrow \cdots$$

such that $P_j \circ P_{j-1} = 0$ for all j. At each $(p,\xi) \in \mathbb{C}T^*\mathscr{M} \smallsetminus 0$, there is an associated symbolic complex

$$\cdots \longrightarrow (\pi^{\mathbb{C}}_{j-1})^{-1}(p) \xrightarrow{\sigma_{P_{j-1}}(p,\xi)} (\pi^{\mathbb{C}}_{j})^{-1}(p) \xrightarrow{\sigma_{P_{j}}(p,\xi)} (\pi^{\mathbb{C}}_{j+1})^{-1}(p) \longrightarrow \cdots$$

where $\sigma_{P_j} \doteq a_{j,k_j}$ is the leading coefficient of P_j. We say that (p,ξ) is a non-characteristic covector for $\mathscr{P}$ if $\mathscr{P}$'s symbolic complex at (p,ξ) is exact. Otherwise, we say that (p,ξ) is characteristic. If every nonzero covector in $T^*\mathscr{M}$ is non-characteristic for $\mathscr{P}$, we say that $\mathscr{P}$ is an elliptic complex (Atiyah-Bott, Spencer).

Complexes of differential operators

A complex of differential operators (or simply a differential complex) is a sequence $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$, where P_j is a differential operator of type $\pi_j \to \pi_{j+1}$ and order $k_j \in \mathbb{Z}$:

$$\cdots \longrightarrow \Gamma(\pi_{j-1}) \xrightarrow{P_{j-1}} \Gamma(\pi_j) \xrightarrow{P_j} \Gamma(\pi_{j+1}) \longrightarrow \cdots$$

such that $P_j \circ P_{j-1} = 0$ for all j. At each $(p,\xi) \in \mathbb{C}T^*\mathscr{M} \smallsetminus 0$, there is an associated symbolic complex

$$\cdots \longrightarrow (\pi^{\mathbb{C}}_{j-1})^{-1}(p) \xrightarrow{\sigma_{P_{j-1}}(p,\xi)} (\pi^{\mathbb{C}}_{j})^{-1}(p) \xrightarrow{\sigma_{P_j}(p,\xi)} (\pi^{\mathbb{C}}_{j+1})^{-1}(p) \longrightarrow \cdots$$

where $\sigma_{P_j} \doteq a_{j,k_j}$ is the leading coefficient of P_j. We say that (p,ξ) is a non-characteristic covector for $\mathscr{P}$ if $\mathscr{P}$'s symbolic complex at (p,ξ) is exact. Otherwise, we say that (p,ξ) is characteristic. If every nonzero covector in $T^*\mathscr{M}$ is non-characteristic for $\mathscr{P}$, we say that $\mathscr{P}$ is an elliptic complex (Atiyah-Bott, Spencer).

Universidade Federal do ABC

Differential complexes generalize differential operators in the following sense: if P = differential operator of type $\pi_1 \to \pi_2$, then

$$0 \longrightarrow \Gamma(\pi_1) \xrightarrow{P} \Gamma(\pi_2) \longrightarrow 0$$

is clearly a differential complex (which we identify with P). Moreover, $(p, \xi) \in T^*\mathscr{M} \smallsetminus 0$ is a non-characteristic vector for $P \Leftrightarrow$ it is for the above differential complex.

More generally, we say that a differential operator P of type $\pi_1 \to \pi_2$ is underdetermined (resp. overdetermined) if there is a differential operator Q of type $\pi_0 \to \pi_1$ (resp. $\pi_2 \to \pi_3$) such that $PQ = 0$ (resp. $QP = 0$). Intuitively,

- Underdeterminacy = local symmetries $\phi \mapsto \phi + Q\phi_0 \Rightarrow$ uniqueness of solutions to $P\phi = \psi$ is spoiled;
- Overdeterminacy = constraints $\Rightarrow$ existence of solutions to $P\phi = \psi$ is spoiled, for a solution exists $\Rightarrow Q\psi = 0$;

In what follows, we shall only work with complexes of first-order differential operators $\mathscr{P}$ which are symbol surjective, i.e. $(\sigma_P)_j$ are fiberwise surjective for all j, and assume that $P_j = 0$ if $j < 0$ or $j > n$ for some $n > 1$.

Differential complexes generalize differential operators in the following sense: if P = differential operator of type $\pi_1 \to \pi_2$, then

$$0 \longrightarrow \Gamma(\pi_1) \xrightarrow{P} \Gamma(\pi_2) \longrightarrow 0$$

is clearly a differential complex (which we identify with P). Moreover, $(p, \xi) \in T^*\mathscr{M} \smallsetminus 0$ is a non-characteristic vector for $P \Leftrightarrow$ it is for the above differential complex.

More generally, we say that a differential operator P of type $\pi_1 \to \pi_2$ is underdetermined (resp. overdetermined) if there is a differential operator Q of type $\pi_0 \to \pi_1$ (resp. $\pi_2 \to \pi_3$) such that $PQ = 0$ (resp. $QP = 0$). Intuitively,

- Underdeterminacy = local symmetries $\phi \mapsto \phi + Q\phi_0 \Rightarrow$ uniqueness of solutions to $P\phi = \psi$ is spoiled;
- Overdeterminacy = constraints $\Rightarrow$ existence of solutions to $P\phi = \psi$ is spoiled, for a solution exists $\Rightarrow Q\psi = 0$;

In what follows, we shall only work with complexes of first-order differential operators $\mathscr{P}$ which are symbol surjective, i.e. $(\sigma_P)_j$ are fiberwise surjective for all j, and assume that $P_j = 0$ if $j < 0$ or $j > n$ for some $n > 1$.

Differential complexes generalize differential operators in the following sense: if P = differential operator of type $\pi_1 \to \pi_2$, then

$$0 \longrightarrow \Gamma(\pi_1) \xrightarrow{P} \Gamma(\pi_2) \longrightarrow 0$$

is clearly a differential complex (which we identify with P). Moreover, $(p,\xi) \in T^*\mathscr{M} \smallsetminus 0$ is a non-characteristic vector for $P \Leftrightarrow$ it is for the above differential complex.

More generally, we say that a differential operator P of type $\pi_1 \to \pi_2$ is underdetermined (resp. overdetermined) if there is a differential operator Q of type $\pi_0 \to \pi_1$ (resp. $\pi_2 \to \pi_3$) such that $PQ = 0$ (resp. $QP = 0$). Intuitively,

- Underdeterminacy = local symmetries $\phi \mapsto \phi + Q\phi_0 \Rightarrow$ uniqueness of solutions to $P\phi = \psi$ is spoiled;
- Overdeterminacy = constraints $\Rightarrow$ existence of solutions to $P\phi = \psi$ is spoiled, for a solution exists $\Rightarrow Q\psi = 0$;

In what follows, we shall only work with complexes of first-order differential operators $\mathscr{P}$ which are symbol surjective, i.e. $(\sigma_P)_j$ are fiberwise surjective for all j, and assume that $P_j = 0$ if $j < 0$ or $j > n$ for some $n > 1$.

Differential complexes generalize differential operators in the following sense: if P = differential operator of type $\pi_1 \to \pi_2$, then

$$0 \longrightarrow \Gamma(\pi_1) \xrightarrow{P} \Gamma(\pi_2) \longrightarrow 0$$

is clearly a differential complex (which we identify with P). Moreover, $(p,\xi) \in T^*\mathscr{M} \smallsetminus 0$ is a non-characteristic vector for $P \Leftrightarrow$ it is for the above differential complex.

More generally, we say that a differential operator P of type $\pi_1 \to \pi_2$ is underdetermined (resp. overdetermined) if there is a differential operator Q of type $\pi_0 \to \pi_1$ (resp. $\pi_2 \to \pi_3$) such that $PQ = 0$ (resp. $QP = 0$). Intuitively,

- Underdeterminacy = local symmetries $\phi \mapsto \phi + Q\phi_0 \Rightarrow$ uniqueness of solutions to $P\phi = \psi$ is spoiled;
- Overdeterminacy = constraints $\Rightarrow$ existence of solutions to $P\phi = \psi$ is spoiled, for a solution exists $\Rightarrow Q\psi = 0$;

In what follows, we shall only work with complexes of first-order differential operators $\mathscr{P}$ which are symbol surjective, i.e. $(\sigma_P)_j$ are fiberwise surjective for all j, and assume that $P_j = 0$ if $j < 0$ or $j > n$ for some $n > 1$.

Universidade Federal do ABC

Differential complexes generalize differential operators in the following sense: if P = differential operator of type $\pi_1 \to \pi_2$, then

$$0 \longrightarrow \Gamma(\pi_1) \xrightarrow{P} \Gamma(\pi_2) \longrightarrow 0$$

is clearly a differential complex (which we identify with P). Moreover, $(p,\xi) \in T^*\mathscr{M} \smallsetminus 0$ is a non-characteristic vector for $P \Leftrightarrow$ it is for the above differential complex.

More generally, we say that a differential operator P of type $\pi_1 \to \pi_2$ is underdetermined (resp. overdetermined) if there is a differential operator Q of type $\pi_0 \to \pi_1$ (resp. $\pi_2 \to \pi_3$) such that $PQ = 0$ (resp. $QP = 0$). Intuitively,

- Underdeterminacy = local symmetries $\phi \mapsto \phi + Q\phi_0 \Rightarrow$ uniqueness of solutions to $P\phi = \psi$ is spoiled;
- Overdeterminacy = constraints $\Rightarrow$ existence of solutions to $P\phi = \psi$ is spoiled, for a solution exists $\Rightarrow Q\psi = 0$;

In what follows, we shall only work with complexes of first-order differential operators $\mathscr{P}$ which are symbol surjective, i.e. $(\sigma_P)_j$ are fiberwise surjective for all j, and assume that $P_j = 0$ if $j < 0$ or $j > n$ for some $n > 1$.

Differential complexes generalize differential operators in the following sense: if P = differential operator of type $\pi_1 \to \pi_2$, then

$$0 \longrightarrow \Gamma(\pi_1) \xrightarrow{P} \Gamma(\pi_2) \longrightarrow 0$$

is clearly a differential complex (which we identify with P). Moreover, $(p,\xi) \in T^*\mathscr{M} \smallsetminus 0$ is a non-characteristic vector for $P \Leftrightarrow$ it is for the above differential complex.

More generally, we say that a differential operator P of type $\pi_1 \to \pi_2$ is underdetermined (resp. overdetermined) if there is a differential operator Q of type $\pi_0 \to \pi_1$ (resp. $\pi_2 \to \pi_3$) such that $PQ = 0$ (resp. $QP = 0$). Intuitively,

- Underdeterminacy = local symmetries $\phi \mapsto \phi + Q\phi_0 \Rightarrow$ uniqueness of solutions to $P\phi = \psi$ is spoiled;
- Overdeterminacy = constraints $\Rightarrow$ existence of solutions to $P\phi = \psi$ is spoiled, for a solution exists $\Rightarrow Q\psi = 0$;

In what follows, we shall only work with complexes of first-order differential operators $\mathscr{P}$ which are symbol surjective, i.e. $(\sigma_P)_j$ are fiberwise surjective for all j, and assume that $P_j = 0$ if $j < 0$ or $j > n$ for some $n > 1$.

Consider the graded vector bundle $\pi = \oplus_{j=0}^{n}\pi_j : \oplus_{j=1}^{n}\mathscr{E}_j \to \mathscr{M}$, so that $P = \oplus_{j=1}^{n}P_j : \Gamma(\pi) \to \Gamma(\pi)$ satisfies $P^2 = 0$. Notice that the leading coefficient of P at $p \in \mathscr{M}$

$$(\sigma_P)_p : T_p^*\mathscr{M} \to \mathrm{End}(\pi^{-1}(p))$$

is the leading coefficient of P at $p \in \mathscr{M}$ (hence, a graded linear map of degree 1) $\Rightarrow$ we may extend $(\sigma_P)_p$ to a graded linear map $(\sigma_P)_p : \otimes^j T_p^*\mathscr{M} \to \mathrm{End}(\pi^{-1}(p))$ of degree j for each $j > 0$ by setting

$$(\sigma_P)_p(\xi_1 \otimes \cdots \otimes \xi_j) = \sigma_P(p,\xi_1)\cdots\sigma_P(p,\xi_j)\ .$$

Since we clearly have $(\sigma_P)_p(\xi)^2 = 0$ for all $(p,\xi) \in T^*\mathscr{M}$ as well, this map descends further to a graded linear map of degree zero

$$(\sigma_P)_p : \wedge T_p^*\mathscr{M} \to \mathrm{End}(\pi^{-1}(p))\ ,$$

thus making $\pi^{-1}(p)$ a module over the exterior algebra $\wedge T_p^*\mathscr{M}$ and $\Gamma(\pi)$ a $\Gamma(\wedge\pi'_{T\mathscr{M}})$-module, with product

$$(\omega \wedge \phi)(p) \doteq (\sigma_P)_p(\omega(p))\phi(p)\ , \quad \omega \in \Gamma(\wedge\pi'_{T\mathscr{M}})\ , \phi \in \Gamma(\pi)\ .$$

Consider the graded vector bundle $\pi = \oplus_{j=0}^{n} \pi_j : \oplus_{j=1}^{n} \mathscr{E}_j \to \mathscr{M}$, so that $P = \oplus_{j=1}^{n} P_j : \Gamma(\pi) \to \Gamma(\pi)$ satisfies $P^2 = 0$. Notice that the leading coefficient of P at $p \in \mathscr{M}$

$$(\sigma_P)_p : T_p^* \mathscr{M} \to \mathrm{End}(\pi^{-1}(p))$$

is the leading coefficient of P at $p \in \mathscr{M}$ (hence, a graded linear map of degree 1) $\Rightarrow$ we may extend $(\sigma_P)_p$ to a graded linear map $(\sigma_P)_p : \otimes^j T_p^* \mathscr{M} \to \mathrm{End}(\pi^{-1}(p))$ of degree j for each $j > 0$ by setting

$$(\sigma_P)_p(\xi_1 \otimes \cdots \otimes \xi_j) = \sigma_P(p, \xi_1) \cdots \sigma_P(p, \xi_j) \ .$$

Since we clearly have $(\sigma_P)_p(\xi)^2 = 0$ for all $(p, \xi) \in T^* \mathscr{M}$ as well, this map descends further to a graded linear map of degree zero

$$(\sigma_P)_p : \wedge T_p^* \mathscr{M} \to \mathrm{End}(\pi^{-1}(p)) \ ,$$

thus making $\pi^{-1}(p)$ a module over the exterior algebra $\wedge T_p^* \mathscr{M}$ and $\Gamma(\pi)$ a $\Gamma(\wedge \pi'_{T\mathscr{M}})$-module, with product

$$(\omega \wedge \phi)(p) \doteq (\sigma_P)_p(\omega(p))\phi(p) \ , \quad \omega \in \Gamma(\wedge \pi'_{T\mathscr{M}}) \ , \phi \in \Gamma(\pi) \ .$$

Consider the graded vector bundle $\pi = \oplus_{j=0}^{n} \pi_j : \oplus_{j=1}^{n} \mathscr{E}_j \to \mathscr{M}$, so that $P = \oplus_{j=1}^{n} P_j : \Gamma(\pi) \to \Gamma(\pi)$ satisfies $P^2 = 0$. Notice that the leading coefficient of P at $p \in \mathscr{M}$

$$(\sigma_P)_p : T_p^*\mathscr{M} \to \mathrm{End}(\pi^{-1}(p))$$

is the leading coefficient of P at $p \in \mathscr{M}$ (hence, a graded linear map of degree 1) $\Rightarrow$ we may extend $(\sigma_P)_p$ to a graded linear map $(\sigma_P)_p : \otimes^j T_p^*\mathscr{M} \to \mathrm{End}(\pi^{-1}(p))$ of degree j for each $j > 0$ by setting

$$(\sigma_P)_p(\xi_1 \otimes \cdots \otimes \xi_j) = \sigma_P(p, \xi_1) \cdots \sigma_P(p, \xi_j) \ .$$

Since we clearly have $(\sigma_P)_p(\xi)^2 = 0$ for all $(p, \xi) \in T^*\mathscr{M}$ as well, this map descends further to a graded linear map of degree zero

$$(\sigma_P)_p : \wedge T_p^*\mathscr{M} \to \mathrm{End}(\pi^{-1}(p)) \ ,$$

thus making $\pi^{-1}(p)$ a module over the exterior algebra $\wedge T_p^*\mathscr{M}$ and $\Gamma(\pi)$ a $\Gamma(\wedge \pi'_{T\mathscr{M}})$-module, with product

$$(\omega \wedge \phi)(p) \doteq (\sigma_P)_p(\omega(p))\phi(p) \ , \quad \omega \in \Gamma(\wedge \pi'_{T\mathscr{M}}) \ , \phi \in \Gamma(\pi) \ .$$

Universidade Federal do ABC

Consider the graded vector bundle $\pi = \oplus_{j=0}^{n}\pi_j : \oplus_{j=1}^{n}\mathscr{E}_j \to \mathscr{M}$, so that $P = \oplus_{j=1}^{n} P_j : \Gamma(\pi) \to \Gamma(\pi)$ satisfies $P^2 = 0$. Notice that the leading coefficient of P at $p \in \mathscr{M}$

$$(\sigma_P)_p : T_p^*\mathscr{M} \to \mathrm{End}(\pi^{-1}(p))$$

is the leading coefficient of P at $p \in \mathscr{M}$ (hence, a graded linear map of degree 1) $\Rightarrow$ we may extend $(\sigma_P)_p$ to a graded linear map $(\sigma_P)_p : \otimes^j T_p^*\mathscr{M} \to \mathrm{End}(\pi^{-1}(p))$ of degree j for each $j > 0$ by setting

$$(\sigma_P)_p(\xi_1 \otimes \cdots \otimes \xi_j) = \sigma_P(p,\xi_1)\cdots\sigma_P(p,\xi_j)\ .$$

Since we clearly have $(\sigma_P)_p(\xi)^2 = 0$ for all $(p,\xi) \in T^*\mathscr{M}$ as well, this map descends further to a graded linear map of degree zero

$$(\sigma_P)_p : \wedge T_p^*\mathscr{M} \to \mathrm{End}(\pi^{-1}(p))\ ,$$

thus making $\pi^{-1}(p)$ a module over the exterior algebra $\wedge T_p^*\mathscr{M}$ and $\Gamma(\pi)$ a $\Gamma(\wedge\pi'_{T\mathscr{M}})$-module, with product

$$(\omega \wedge \phi)(p) \doteq (\sigma_P)_p(\omega(p))\phi(p)\ , \quad \omega \in \Gamma(\wedge\pi'_{T\mathscr{M}})\ ,\ \phi \in \Gamma(\pi)\ .$$

Consider the graded vector bundle $\pi = \oplus_{j=0}^{n} \pi_j : \oplus_{j=1}^{n} \mathscr{E}_j \to \mathscr{M}$, so that $P = \oplus_{j=1}^{n} P_j : \Gamma(\pi) \to \Gamma(\pi)$ satisfies $P^2 = 0$. Notice that the leading coefficient of P at $p \in \mathscr{M}$

$$(\sigma_P)_p : T_p^* \mathscr{M} \to \mathrm{End}(\pi^{-1}(p))$$

is the leading coefficient of P at $p \in \mathscr{M}$ (hence, a graded linear map of degree 1) $\Rightarrow$ we may extend $(\sigma_P)_p$ to a graded linear map $(\sigma_P)_p : \otimes^j T_p^* \mathscr{M} \to \mathrm{End}(\pi^{-1}(p))$ of degree j for each $j > 0$ by setting

$$(\sigma_P)_p(\xi_1 \otimes \cdots \otimes \xi_j) = \sigma_P(p, \xi_1) \cdots \sigma_P(p, \xi_j) \ .$$

Since we clearly have $(\sigma_P)_p(\xi)^2 = 0$ for all $(p, \xi) \in T^* \mathscr{M}$ as well, this map descends further to a graded linear map of degree zero

$$(\sigma_P)_p : \wedge T_p^* \mathscr{M} \to \mathrm{End}(\pi^{-1}(p)) \ ,$$

thus making $\pi^{-1}(p)$ a module over the exterior algebra $\wedge T_p^* \mathscr{M}$ and $\Gamma(\pi)$ a $\Gamma(\wedge \pi'_{T\mathscr{M}})$-module, with product

$$(\omega \wedge \phi)(p) \doteq (\sigma_P)_p(\omega(p))\phi(p) \ , \quad \omega \in \Gamma(\wedge \pi'_{T\mathscr{M}}) \ , \phi \in \Gamma(\pi) \ .$$

Moreover, $\Gamma(\pi)$ is even a differential $\Gamma(\wedge\pi'_{T\mathscr{M}})$-module with respect to P, for we have that for all $\omega\in\Gamma(\wedge^r\pi'_{T\mathscr{M}})$, $\phi\in\Gamma(\pi)$ (Guillemin)

$$P(\omega\wedge\phi)=\mathrm{d}\omega\wedge\phi+(-1)^r\omega\wedge P\phi\ ,$$

where d is the de Rham exterior differential. (Proof: for $r=0$, it follows from the definition of σ_P. The general case follows by induction on r)

Examples of (first-order) differential complexes

- $\pi_1=\pi_2=$ spinor bundle over the pseudo-Riemannian manifold $(\mathscr{M},g)$, $P=\not{D}$ (Dirac operator on $(\mathscr{M},g)$);
- de Rham complex: $\pi_j=$ bundle of j-forms on $\mathscr{M}$, $P_j=\mathrm{d}$;
- Dolbeault complex ($\mathscr{M}$ assumed complex) at degree r: $\pi_j=$ bundle of complex (r,j)-forms on $\mathscr{M}$, $P_j=\overline{\partial}$ (Cauchy-Riemann exterior differential).

Moreover, $\Gamma(\pi)$ is even a differential $\Gamma(\wedge\pi'_{T\mathscr{M}})$-module with respect to P, for we have that for all $\omega \in \Gamma(\wedge^r \pi'_{T\mathscr{M}})$, $\phi \in \Gamma(\pi)$ (Guillemin)

$$P(\omega \wedge \phi) = \mathrm{d}\omega \wedge \phi + (-1)^r \omega \wedge P\phi \ ,$$

where d is the de Rham exterior differential. (Proof: for $r = 0$, it follows from the definition of σ_P. The general case follows by induction on r)

Examples of (first-order) differential complexes

- $\pi_1 = \pi_2 =$ spinor bundle over the pseudo-Riemannian manifold $(\mathscr{M}, g)$, $P = \not{D}$ (Dirac operator on $(\mathscr{M}, g)$);
- de Rham complex: $\pi_j =$ bundle of j-forms on $\mathscr{M}$, $P_j = \mathrm{d}$;
- Dolbeault complex ($\mathscr{M}$ assumed complex) at degree r: $\pi_j =$ bundle of complex (r, j)-forms on $\mathscr{M}$, $P_j = \overline{\partial}$ (Cauchy-Riemann exterior differential).

Universidade Federal do ABC

Moreover, $\Gamma(\pi)$ is even a differential $\Gamma(\wedge\pi'_{T\mathscr{M}})$-module with respect to P, for we have that for all $\omega\in\Gamma(\wedge^r\pi'_{T\mathscr{M}})$, $\phi\in\Gamma(\pi)$ (Guillemin)

$$P(\omega\wedge\phi)=\mathrm{d}\omega\wedge\phi+(-1)^r\omega\wedge P\phi\ ,$$

where d is the de Rham exterior differential. (Proof: for $r=0$, it follows from the definition of σ_P. The general case follows by induction on r)

Examples of (first-order) differential complexes

- $\pi_1=\pi_2=$ spinor bundle over the pseudo-Riemannian manifold $(\mathscr{M},g)$, $P=\not{D}$ (Dirac operator on $(\mathscr{M},g)$);
- de Rham complex: $\pi_j=$ bundle of j-forms on $\mathscr{M}$, $P_j=\mathrm{d}$;
- Dolbeault complex ($\mathscr{M}$ assumed complex) at degree r: $\pi_j=$ bundle of complex (r,j)-forms on $\mathscr{M}$, $P_j=\overline{\partial}$ (Cauchy-Riemann exterior differential).

Moreover, $\Gamma(\pi)$ is even a differential $\Gamma(\wedge\pi'_{T\mathscr{M}})$-module with respect to P, for we have that for all $\omega \in \Gamma(\wedge^r\pi'_{T\mathscr{M}})$, $\phi \in \Gamma(\pi)$ (Guillemin)

$$P(\omega \wedge \phi) = \mathrm{d}\omega \wedge \phi + (-1)^r \omega \wedge P\phi \ ,$$

where d is the de Rham exterior differential. (Proof: for $r = 0$, it follows from the definition of σ_P. The general case follows by induction on r)

Examples of (first-order) differential complexes

- $\pi_1 = \pi_2 =$ spinor bundle over the pseudo-Riemannian manifold $(\mathscr{M}, g)$, $P = \not{D}$ (Dirac operator on $(\mathscr{M}, g)$);
- de Rham complex: $\pi_j =$ bundle of j-forms on $\mathscr{M}$, $P_j = \mathrm{d}$;
- Dolbeault complex ($\mathscr{M}$ assumed complex) at degree r: $\pi_j =$ bundle of complex (r, j)-forms on $\mathscr{M}$, $P_j = \overline{\partial}$ (Cauchy-Riemann exterior differential).

Moreover, $\Gamma(\pi)$ is even a differential $\Gamma(\wedge\pi'_{T\mathscr{M}})$-module with respect to P, for we have that for all $\omega \in \Gamma(\wedge^r \pi'_{T\mathscr{M}})$, $\phi \in \Gamma(\pi)$ (Guillemin)

$$P(\omega \wedge \phi) = \mathrm{d}\omega \wedge \phi + (-1)^r \omega \wedge P\phi \ ,$$

where d is the de Rham exterior differential. (Proof: for $r = 0$, it follows from the definition of σ_P. The general case follows by induction on r)

Examples of (first-order) differential complexes

- $\pi_1 = \pi_2 =$ spinor bundle over the pseudo-Riemannian manifold $(\mathscr{M}, g)$, $P = \not{D}$ (Dirac operator on $(\mathscr{M}, g)$);
- de Rham complex: $\pi_j =$ bundle of j-forms on $\mathscr{M}$, $P_j = \mathrm{d}$;
- Dolbeault complex ($\mathscr{M}$ assumed complex) at degree r: $\pi_j =$ bundle of complex (r, j)-forms on $\mathscr{M}$, $P_j = \overline{\partial}$ (Cauchy-Riemann exterior differential).

Moreover, $\Gamma(\pi)$ is even a differential $\Gamma(\wedge\pi'_{T\mathscr{M}})$-module with respect to P, for we have that for all $\omega \in \Gamma(\wedge^r\pi'_{T\mathscr{M}})$, $\phi \in \Gamma(\pi)$ (Guillemin)

$$P(\omega \wedge \phi) = \mathrm{d}\omega \wedge \phi + (-1)^r \omega \wedge P\phi \ ,$$

where d is the de Rham exterior differential. (Proof: for $r = 0$, it follows from the definition of σ_P. The general case follows by induction on r)

Examples of (first-order) differential complexes

- $\pi_1 = \pi_2 =$ spinor bundle over the pseudo-Riemannian manifold $(\mathscr{M}, g)$, $P = \not{D}$ (Dirac operator on $(\mathscr{M}, g)$);
- de Rham complex: $\pi_j =$ bundle of j-forms on $\mathscr{M}$, $P_j = \mathrm{d}$;
- Dolbeault complex ($\mathscr{M}$ assumed complex) at degree r: $\pi_j =$ bundle of complex (r, j)-forms on $\mathscr{M}$, $P_j = \overline{\partial}$ (Cauchy-Riemann exterior differential).

Universidade Federal do ABC

Moreover, $\Gamma(\pi)$ is even a differential $\Gamma(\wedge\pi'_{T\mathscr{M}})$-module with respect to P, for we have that for all $\omega\in\Gamma(\wedge^r\pi'_{T\mathscr{M}})$, $\phi\in\Gamma(\pi)$ (Guillemin)

$$P(\omega\wedge\phi)=\mathrm{d}\omega\wedge\phi+(-1)^r\omega\wedge P\phi\ ,$$

where d is the de Rham exterior differential. (Proof: for $r=0$, it follows from the definition of σ_P. The general case follows by induction on r)

Examples of (first-order) differential complexes

- $\pi_1=\pi_2=$ spinor bundle over the pseudo-Riemannian manifold $(\mathscr{M},g)$, $P=\not{D}$ (Dirac operator on $(\mathscr{M},g)$);
- de Rham complex: $\pi_j=$ bundle of j-forms on $\mathscr{M}$, $P_j=\mathrm{d}$;
- Dolbeault complex ($\mathscr{M}$ assumed complex) at degree r: $\pi_j=$ bundle of complex (r,j)-forms on $\mathscr{M}$, $P_j=\overline{\partial}$ (Cauchy-Riemann exterior differential).

The (non-characteristic) Cauchy problem

In this section we follow MacKichan 1975. Suppose that $t \in \mathscr{C}^\infty(\mathscr{M})$ is such that $\mathrm{d}t \in \Gamma(\pi'_{T\mathscr{M}})$ is a non-characteristic covector field for $\mathscr{P}$. Set $\Sigma = t^{-1}(0)$, and

- $\mathscr{J}_\Sigma$ = smallest d-closed ideal of $\Gamma(\wedge\pi'_{T\mathscr{M}})$ containing all smooth functions vanishing on Σ = ideal generated by $\{t, \mathrm{d}t\}$;
- $\mathscr{J}_\Sigma\Gamma(\pi) = \{\omega \wedge \pi \mid \omega \in \mathscr{J}_\Sigma\ , \phi \in \Gamma(\pi)\}$.

It is clear that the submodule $\mathscr{J}_\Sigma\Gamma(\pi)$ of $\Gamma(\pi)$ is P-closed (actually, it is the smallest P-closed submodule containing all smooth sections of π vanishing on Σ). We denote by $\mathscr{J}_\Sigma\mathscr{P}$ the complex obtained by restricting P to $\mathscr{J}_\Sigma\Gamma(\pi)$.

Lemma (MacKichan 1975, Lemma 2.1)

Let $\phi \in \mathscr{J}_\Sigma\Gamma(\pi) \Rightarrow$ there exist $\phi', \phi'' \in \Gamma(\pi)$ such that

$$\phi = \phi' + P\phi''\ , \quad \phi'|_\Sigma = 0\ , \ \phi''|_\Sigma = 0\ . \qquad \square$$

Moreover, if $f \in \mathscr{C}^\infty(\mathscr{M})$ is constant on the level sets of t, we see that $P(f\phi) = \mathrm{d}f \wedge \phi + fP\phi$, where $\mathrm{d}f$ must be of the form $a\mathrm{d}t$ for some $a \in \mathscr{C}^\infty(\mathscr{M})$ constant on the level sets of t. This means that P acts on $\Gamma(\pi)/\mathscr{J}_\Sigma\Gamma(\pi)$ only in directions tangential to the level sets of t.

The (non-characteristic) Cauchy problem

In this section we follow MacKichan 1975. Suppose that $t \in \mathscr{C}^\infty(\mathscr{M})$ is such that $\mathrm{d}t \in \Gamma(\pi'_{T\mathscr{M}})$ is a non-characteristic covector field for $\mathscr{P}$. Set $\Sigma = t^{-1}(0)$, and

- $\mathscr{I}_\Sigma$ = smallest d-closed ideal of $\Gamma(\wedge\pi'_{T\mathscr{M}})$ containing all smooth functions vanishing on Σ = ideal generated by $\{t, \mathrm{d}t\}$;
- $\mathscr{I}_\Sigma\Gamma(\pi) = \{\omega \wedge \pi \mid \omega \in \mathscr{I}_\Sigma\ , \phi \in \Gamma(\pi)\}$.

It is clear that the submodule $\mathscr{I}_\Sigma\Gamma(\pi)$ of $\Gamma(\pi)$ is P-closed (actually, it is the smallest P-closed submodule containing all smooth sections of π vanishing on Σ). We denote by $\mathscr{I}_\Sigma\mathscr{P}$ the complex obtained by restricting P to $\mathscr{I}_\Sigma\Gamma(\pi)$.

Lemma (MacKichan 1975, Lemma 2.1)

Let $\phi \in \mathscr{I}_\Sigma\Gamma(\pi) \Rightarrow$ there exist $\phi', \phi'' \in \Gamma(\pi)$ such that

$$\phi = \phi' + P\phi''\ , \quad \phi'|_\Sigma = 0\ , \phi''|_\Sigma = 0\ .$$ □

Moreover, if $f \in \mathscr{C}^\infty(\mathscr{M})$ is constant on the level sets of t, we see that $P(f\phi) = \mathrm{d}f \wedge \phi + fP\phi$, where $\mathrm{d}f$ must be of the form $a\mathrm{d}t$ for some $a \in \mathscr{C}^\infty(\mathscr{M})$ constant on the level sets of t. This means that P acts on $\Gamma(\pi)/\mathscr{I}_\Sigma\Gamma(\pi)$ only in directions tangential to the level sets of t.

The (non-characteristic) Cauchy problem

In this section we follow MacKichan 1975. Suppose that $t \in \mathscr{C}^\infty(\mathscr{M})$ is such that $\mathrm{d}t \in \Gamma(\pi'_{T\mathscr{M}})$ is a non-characteristic covector field for $\mathscr{P}$. Set $\Sigma = t^{-1}(0)$, and

- $\mathscr{I}_\Sigma =$ smallest d-closed ideal of $\Gamma(\wedge\pi'_{T\mathscr{M}})$ containing all smooth functions vanishing on $\Sigma =$ ideal generated by $\{t, \mathrm{d}t\}$;
- $\mathscr{I}_\Sigma\Gamma(\pi) = \{\omega \wedge \pi \mid \omega \in \mathscr{I}_\Sigma\ , \phi \in \Gamma(\pi)\}$.

It is clear that the submodule $\mathscr{I}_\Sigma\Gamma(\pi)$ of $\Gamma(\pi)$ is P-closed (actually, it is the smallest P-closed submodule containing all smooth sections of π vanishing on Σ). We denote by $\mathscr{I}_\Sigma\mathscr{P}$ the complex obtained by restricting P to $\mathscr{I}_\Sigma\Gamma(\pi)$.

Lemma (MacKichan 1975, Lemma 2.1)

Let $\phi \in \mathscr{I}_\Sigma\Gamma(\pi) \Rightarrow$ *there exist* $\phi', \phi'' \in \Gamma(\pi)$ *such that*

$$\phi = \phi' + P\phi''\ , \quad \phi'|_\Sigma = 0\ , \ \phi''|_\Sigma = 0\ . \qquad \square$$

Moreover, if $f \in \mathscr{C}^\infty(\mathscr{M})$ is constant on the level sets of t, we see that $P(f\phi) = \mathrm{d}f \wedge \phi + fP\phi$, where $\mathrm{d}f$ must be of the form $a\mathrm{d}t$ for some $a \in \mathscr{C}^\infty(\mathscr{M})$ constant on the level sets of t. This means that P acts on $\Gamma(\pi)/\mathscr{I}_\Sigma\Gamma(\pi)$ only in directions tangential to the level sets of t.

The (non-characteristic) Cauchy problem

In this section we follow MacKichan 1975. Suppose that $t \in \mathscr{C}^\infty(\mathscr{M})$ is such that $\mathrm{d}t \in \Gamma(\pi'_{T\mathscr{M}})$ is a non-characteristic covector field for $\mathscr{P}$. Set $\Sigma = t^{-1}(0)$, and

- $\mathscr{J}_\Sigma =$ smallest d-closed ideal of $\Gamma(\wedge\pi'_{T\mathscr{M}})$ containing all smooth functions vanishing on $\Sigma =$ ideal generated by $\{t, \mathrm{d}t\}$;
- $\mathscr{J}_\Sigma\Gamma(\pi) = \{\omega \wedge \pi \mid \omega \in \mathscr{J}_\Sigma\ , \phi \in \Gamma(\pi)\}$.

It is clear that the submodule $\mathscr{J}_\Sigma\Gamma(\pi)$ of $\Gamma(\pi)$ is P-closed (actually, it is the smallest P-closed submodule containing all smooth sections of π vanishing on Σ). We denote by $\mathscr{J}_\Sigma\mathscr{P}$ the complex obtained by restricting P to $\mathscr{J}_\Sigma\Gamma(\pi)$.

Lemma (MacKichan 1975, Lemma 2.1)

Let $\phi \in \mathscr{J}_\Sigma\Gamma(\pi) \Rightarrow$ there exist $\phi', \phi'' \in \Gamma(\pi)$ such that

$$\phi = \phi' + P\phi''\ , \quad \phi'|_\Sigma = 0\ ,\ \phi''|_\Sigma = 0\ . \qquad \square$$

Moreover, if $f \in \mathscr{C}^\infty(\mathscr{M})$ is constant on the level sets of t, we see that $P(f\phi) = \mathrm{d}f \wedge \phi + fP\phi$, where $\mathrm{d}f$ must be of the form $a\mathrm{d}t$ for some $a \in \mathscr{C}^\infty(\mathscr{M})$ constant on the level sets of t. This means that P acts on $\Gamma(\pi)/\mathscr{J}_\Sigma\Gamma(\pi)$ only in directions tangential to the level sets of t.

The (non-characteristic) Cauchy problem

In this section we follow MacKichan 1975. Suppose that $t \in \mathscr{C}^\infty(\mathscr{M})$ is such that $\mathrm{d}t \in \Gamma(\pi'_{T\mathscr{M}})$ is a non-characteristic covector field for $\mathscr{P}$. Set $\Sigma = t^{-1}(0)$, and

- $\mathscr{I}_\Sigma =$ smallest d-closed ideal of $\Gamma(\wedge\pi'_{T\mathscr{M}})$ containing all smooth functions vanishing on $\Sigma =$ ideal generated by $\{t, \mathrm{d}t\}$;
- $\mathscr{I}_\Sigma\Gamma(\pi) = \{\omega \wedge \pi \mid \omega \in \mathscr{I}_\Sigma \ , \phi \in \Gamma(\pi)\}$.

It is clear that the submodule $\mathscr{I}_\Sigma\Gamma(\pi)$ of $\Gamma(\pi)$ is P-closed (actually, it is the smallest P-closed submodule containing all smooth sections of π vanishing on Σ). We denote by $\mathscr{I}_\Sigma\mathscr{P}$ the complex obtained by restricting P to $\mathscr{I}_\Sigma\Gamma(\pi)$.

Lemma (MacKichan 1975, Lemma 2.1)

Let $\phi \in \mathscr{I}_\Sigma\Gamma(\pi) \Rightarrow$ there exist $\phi', \phi'' \in \Gamma(\pi)$ such that

$$\phi = \phi' + P\phi'' \ , \quad \phi'|_\Sigma = 0 \ , \ \phi''|_\Sigma = 0 \ . \qquad \square$$

Moreover, if $f \in \mathscr{C}^\infty(\mathscr{M})$ is constant on the level sets of t, we see that $P(f\phi) = \mathrm{d}f \wedge \phi + fP\phi$, where $\mathrm{d}f$ must be of the form $a\mathrm{d}t$ for some $a \in \mathscr{C}^\infty(\mathscr{M})$ constant on the level sets of t. This means that P acts on $\Gamma(\pi)/\mathscr{I}_\Sigma\Gamma(\pi)$ only in directions tangential to the level sets of t.

The (non-characteristic) Cauchy problem

In this section we follow MacKichan 1975. Suppose that $t \in \mathscr{C}^\infty(\mathscr{M})$ is such that $\mathrm{d}t \in \Gamma(\pi'_{T\mathscr{M}})$ is a non-characteristic covector field for $\mathscr{P}$. Set $\Sigma = t^{-1}(0)$, and

- $\mathscr{I}_\Sigma =$ smallest d-closed ideal of $\Gamma(\wedge \pi'_{T\mathscr{M}})$ containing all smooth functions vanishing on $\Sigma =$ ideal generated by $\{t, \mathrm{d}t\}$;
- $\mathscr{I}_\Sigma \Gamma(\pi) = \{\omega \wedge \pi \mid \omega \in \mathscr{I}_\Sigma \text{ , } \phi \in \Gamma(\pi)\}$.

It is clear that the submodule $\mathscr{I}_\Sigma\Gamma(\pi)$ of $\Gamma(\pi)$ is P-closed (actually, it is the smallest P-closed submodule containing all smooth sections of π vanishing on Σ).
We denote by $\mathscr{I}_\Sigma\mathscr{P}$ the complex obtained by restricting P to $\mathscr{I}_\Sigma\Gamma(\pi)$.

Lemma (MacKichan 1975, Lemma 2.1)

Let $\phi \in \mathscr{I}_\Sigma\Gamma(\pi) \Rightarrow$ there exist $\phi', \phi'' \in \Gamma(\pi)$ such that

$$\phi = \phi' + P\phi'' \ , \quad \phi'|_\Sigma = 0 \ , \ \phi''|_\Sigma = 0 \ . \qquad \square$$

Moreover, if $f \in \mathscr{C}^\infty(\mathscr{M})$ is constant on the level sets of t, we see that $P(f\phi) = \mathrm{d}f \wedge \phi + fP\phi$, where $\mathrm{d}f$ must be of the form $a\mathrm{d}t$ for some $a \in \mathscr{C}^\infty(\mathscr{M})$ constant on the level sets of t. This means that P acts on $\Gamma(\pi)/\mathscr{I}_\Sigma\Gamma(\pi)$ only in directions tangential to the level sets of t.

Universidade Federal do ABC

THE (NON-CHARACTERISTIC) CAUCHY PROBLEM

In this section we follow MacKichan 1975. Suppose that $t \in \mathscr{C}^\infty(\mathscr{M})$ is such that $\mathrm{d}t \in \Gamma(\pi'_{T\mathscr{M}})$ is a non-characteristic covector field for $\mathscr{P}$. Set $\Sigma = t^{-1}(0)$, and

- $\mathscr{I}_\Sigma =$ smallest d-closed ideal of $\Gamma(\wedge\pi'_{T\mathscr{M}})$ containing all smooth functions vanishing on $\Sigma =$ ideal generated by $\{t, \mathrm{d}t\}$;
- $\mathscr{I}_\Sigma\Gamma(\pi) = \{\omega \wedge \pi \mid \omega \in \mathscr{I}_\Sigma \ , \phi \in \Gamma(\pi)\}$.

It is clear that the submodule $\mathscr{I}_\Sigma\Gamma(\pi)$ of $\Gamma(\pi)$ is P-closed (actually, it is the smallest P-closed submodule containing all smooth sections of π vanishing on Σ). We denote by $\mathscr{I}_\Sigma\mathscr{P}$ the complex obtained by restricting P to $\mathscr{I}_\Sigma\Gamma(\pi)$.

LEMMA (MACKICHAN 1975, LEMMA 2.1)

Let $\phi \in \mathscr{I}_\Sigma\Gamma(\pi) \Rightarrow$ there exist $\phi', \phi'' \in \Gamma(\pi)$ such that

$$\phi = \phi' + P\phi'' \ , \quad \phi'|_\Sigma = 0 \ , \ \phi''|_\Sigma = 0 \ . \qquad \square$$

Moreover, if $f \in \mathscr{C}^\infty(\mathscr{M})$ is constant on the level sets of t, we see that $P(f\phi) = \mathrm{d}f \wedge \phi + fP\phi$, where $\mathrm{d}f$ must be of the form $a\mathrm{d}t$ for some $a \in \mathscr{C}^\infty(\mathscr{M})$ constant on the level sets of t. This means that P acts on $\Gamma(\pi)/\mathscr{I}_\Sigma\Gamma(\pi)$ only in directions tangential to the level sets of t.

The (non-characteristic) Cauchy problem

In this section we follow MacKichan 1975. Suppose that $t \in \mathscr{C}^\infty(\mathscr{M})$ is such that $\mathrm{d}t \in \Gamma(\pi'_{T\mathscr{M}})$ is a non-characteristic covector field for $\mathscr{P}$. Set $\Sigma = t^{-1}(0)$, and

- $\mathscr{I}_\Sigma =$ smallest d-closed ideal of $\Gamma(\wedge\pi'_{T\mathscr{M}})$ containing all smooth functions vanishing on $\Sigma =$ ideal generated by $\{t, \mathrm{d}t\}$;
- $\mathscr{I}_\Sigma\Gamma(\pi) = \{\omega \wedge \pi \mid \omega \in \mathscr{I}_\Sigma \ , \phi \in \Gamma(\pi)\}$.

It is clear that the submodule $\mathscr{I}_\Sigma\Gamma(\pi)$ of $\Gamma(\pi)$ is P-closed (actually, it is the smallest P-closed submodule containing all smooth sections of π vanishing on Σ). We denote by $\mathscr{I}_\Sigma\mathscr{P}$ the complex obtained by restricting P to $\mathscr{I}_\Sigma\Gamma(\pi)$.

Lemma (MacKichan 1975, Lemma 2.1)

Let $\phi \in \mathscr{I}_\Sigma\Gamma(\pi) \Rightarrow$ *there exist* $\phi', \phi'' \in \Gamma(\pi)$ *such that*

$$\phi = \phi' + P\phi'' \ , \quad \phi'|_\Sigma = 0 \ , \ \phi''|_\Sigma = 0 \ . \qquad \square$$

Moreover, if $f \in \mathscr{C}^\infty(\mathscr{M})$ is constant on the level sets of t, we see that $P(f\phi) = \mathrm{d}f \wedge \phi + fP\phi$, where $\mathrm{d}f$ must be of the form $a\mathrm{d}t$ for some $a \in \mathscr{C}^\infty(\mathscr{M})$ constant on the level sets of t. This means that P acts on $\Gamma(\pi)/\mathscr{I}_\Sigma\Gamma(\pi)$ only in directions tangential to the level sets of t.

The (non-characteristic) Cauchy problem

In this section we follow MacKichan 1975. Suppose that $t \in \mathscr{C}^\infty(\mathscr{M})$ is such that $\mathrm{d}t \in \Gamma(\pi'_{T\mathscr{M}})$ is a non-characteristic covector field for $\mathscr{P}$. Set $\Sigma = t^{-1}(0)$, and

- $\mathscr{J}_\Sigma =$ smallest d-closed ideal of $\Gamma(\wedge\pi'_{T\mathscr{M}})$ containing all smooth functions vanishing on $\Sigma =$ ideal generated by $\{t, \mathrm{d}t\}$;
- $\mathscr{J}_\Sigma\Gamma(\pi) = \{\omega \wedge \pi \mid \omega \in \mathscr{J}_\Sigma \ , \phi \in \Gamma(\pi)\}$.

It is clear that the submodule $\mathscr{J}_\Sigma\Gamma(\pi)$ of $\Gamma(\pi)$ is P-closed (actually, it is the smallest P-closed submodule containing all smooth sections of π vanishing on Σ). We denote by $\mathscr{J}_\Sigma\mathscr{P}$ the complex obtained by restricting P to $\mathscr{J}_\Sigma\Gamma(\pi)$.

Lemma (MacKichan 1975, Lemma 2.1)

Let $\phi \in \mathscr{J}_\Sigma\Gamma(\pi) \Rightarrow$ there exist $\phi', \phi'' \in \Gamma(\pi)$ such that

$$\phi = \phi' + P\phi'' \ , \quad \phi'|_\Sigma = 0 \ , \ \phi''|_\Sigma = 0 \ . \qquad \square$$

Moreover, if $f \in \mathscr{C}^\infty(\mathscr{M})$ is constant on the level sets of t, we see that $P(f\phi) = \mathrm{d}f \wedge \phi + fP\phi$, where $\mathrm{d}f$ must be of the form $a\mathrm{d}t$ for some $a \in \mathscr{C}^\infty(\mathscr{M})$ constant on the level sets of t. This means that P acts on $\Gamma(\pi)/\mathscr{J}_\Sigma\Gamma(\pi)$ only in directions tangential to the level sets of t.

THE (NON-CHARACTERISTIC) CAUCHY PROBLEM

In this section we follow MacKichan 1975. Suppose that $t \in \mathscr{C}^\infty(\mathscr{M})$ is such that $\mathrm{d}t \in \Gamma(\pi'_{T\mathscr{M}})$ is a non-characteristic covector field for $\mathscr{P}$. Set $\Sigma = t^{-1}(0)$, and

- $\mathscr{I}_\Sigma$ = smallest d-closed ideal of $\Gamma(\wedge \pi'_{T\mathscr{M}})$ containing all smooth functions vanishing on Σ = ideal generated by $\{t, \mathrm{d}t\}$;
- $\mathscr{I}_\Sigma \Gamma(\pi) = \{\omega \wedge \pi \mid \omega \in \mathscr{I}_\Sigma \ , \phi \in \Gamma(\pi)\}$.

It is clear that the submodule $\mathscr{I}_\Sigma \Gamma(\pi)$ of $\Gamma(\pi)$ is P-closed (actually, it is the smallest P-closed submodule containing all smooth sections of π vanishing on Σ). We denote by $\mathscr{I}_\Sigma \mathscr{P}$ the complex obtained by restricting P to $\mathscr{I}_\Sigma \Gamma(\pi)$.

LEMMA (MACKICHAN 1975, LEMMA 2.1)

Let $\phi \in \mathscr{I}_\Sigma \Gamma(\pi) \Rightarrow$ there exist $\phi', \phi'' \in \Gamma(\pi)$ such that

$$\phi = \phi' + P\phi'' \ , \quad \phi'|_\Sigma = 0 \ , \ \phi''|_\Sigma = 0 \ . \qquad \square$$

Moreover, if $f \in \mathscr{C}^\infty(\mathscr{M})$ is constant on the level sets of t, we see that $P(f\phi) = \mathrm{d}f \wedge \phi + fP\phi$, where $\mathrm{d}f$ must be of the form $a\mathrm{d}t$ for some $a \in \mathscr{C}^\infty(\mathscr{M})$ constant on the level sets of t. This means that P acts on $\Gamma(\pi)/\mathscr{I}_\Sigma\Gamma(\pi)$ only in directions tangential to the level sets of t.

The (non-characteristic) Cauchy problem

In this section we follow MacKichan 1975. Suppose that $t \in \mathscr{C}^\infty(\mathscr{M})$ is such that $\mathrm{d}t \in \Gamma(\pi'_{T\mathscr{M}})$ is a non-characteristic covector field for $\mathscr{P}$. Set $\Sigma = t^{-1}(0)$, and

- $\mathscr{J}_\Sigma =$ smallest d-closed ideal of $\Gamma(\wedge\pi'_{T\mathscr{M}})$ containing all smooth functions vanishing on $\Sigma =$ ideal generated by $\{t, \mathrm{d}t\}$;
- $\mathscr{J}_\Sigma\Gamma(\pi) = \{\omega \wedge \pi \mid \omega \in \mathscr{J}_\Sigma\ , \phi \in \Gamma(\pi)\}$.

It is clear that the submodule $\mathscr{J}_\Sigma\Gamma(\pi)$ of $\Gamma(\pi)$ is P-closed (actually, it is the smallest P-closed submodule containing all smooth sections of π vanishing on Σ). We denote by $\mathscr{J}_\Sigma\mathscr{P}$ the complex obtained by restricting P to $\mathscr{J}_\Sigma\Gamma(\pi)$.

Lemma (MacKichan 1975, Lemma 2.1)

Let $\phi \in \mathscr{J}_\Sigma\Gamma(\pi) \Rightarrow$ there exist $\phi', \phi'' \in \Gamma(\pi)$ such that

$$\phi = \phi' + P\phi''\ , \quad \phi'|_\Sigma = 0\ , \ \phi''|_\Sigma = 0\ . \qquad \square$$

Moreover, if $f \in \mathscr{C}^\infty(\mathscr{M})$ is constant on the level sets of t, we see that $P(f\phi) = \mathrm{d}f \wedge \phi + fP\phi$, where $\mathrm{d}f$ must be of the form $a\mathrm{d}t$ for some $a \in \mathscr{C}^\infty(\mathscr{M})$ constant on the level sets of t. This means that P acts on $\Gamma(\pi)/\mathscr{J}_\Sigma\Gamma(\pi)$ only in directions tangential to the level sets ot t.

We have a short exact sequence of differential complexes

$$0 \to \mathscr{J}_\Sigma \mathscr{P} \to \mathscr{P} \to \mathscr{P}_\Sigma \to 0 \,,$$

detailed as follows (first vertical map = inclusion, second vertical map = quotient):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathscr{J}_\Sigma \Gamma(\pi)_{j-1} \arrow[r, "P_{j-1}"] \arrow[d] & \mathscr{J}_\Sigma \Gamma(\pi)_j \arrow[r, "P_j"] \arrow[d] & \mathscr{J}_\Sigma \Gamma(\pi)_{j+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \Gamma(\pi_{j-1}) \arrow[r, "P_{j-1}"] \arrow[d] & \Gamma(\pi_j) \arrow[r, "P_j"] \arrow[d] & \Gamma(\pi_{j+1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \frac{\Gamma(\pi_{j-1})}{\mathscr{J}_\Sigma \Gamma(\pi)_{j-1}} \arrow[r, "P_{j-1}"] \arrow[d] & \frac{\Gamma(\pi_j)}{\mathscr{J}_\Sigma \Gamma(\pi)_j} \arrow[r, "P_j"] \arrow[d] & \frac{\Gamma(\pi_{j+1})}{\mathscr{J}_\Sigma \Gamma(\pi)_{j+1}} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We have a short exact sequence of differential complexes

$$0 \to \mathscr{J}_\Sigma \mathscr{P} \to \mathscr{P} \to \mathscr{P}_\Sigma \to 0 \,,$$

detailed as follows (first vertical map = inclusion, second vertical map = quotient):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathscr{J}_\Sigma\Gamma(\pi)_{j-1} \arrow[r, "P_{j-1}"] \arrow[d] & \mathscr{J}_\Sigma\Gamma(\pi)_{j} \arrow[r, "P_{j}"] \arrow[d] & \mathscr{J}_\Sigma\Gamma(\pi)_{j+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \Gamma(\pi_{j-1}) \arrow[r, "P_{j-1}"] \arrow[d] & \Gamma(\pi_{j}) \arrow[r, "P_{j}"] \arrow[d] & \Gamma(\pi_{j+1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \frac{\Gamma(\pi_{j-1})}{\mathscr{J}_\Sigma\Gamma(\pi)_{j-1}} \arrow[r, "P_{j-1}"] \arrow[d] & \frac{\Gamma(\pi_{j})}{\mathscr{J}_\Sigma\Gamma(\pi)_{j}} \arrow[r, "P_{j}"] \arrow[d] & \frac{\Gamma(\pi_{j+1})}{\mathscr{J}_\Sigma\Gamma(\pi)_{j+1}} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We have a short exact sequence of differential complexes

$$0 \to \mathscr{I}_\Sigma \mathscr{P} \to \mathscr{P} \to \mathscr{P}_\Sigma \to 0 \,,$$

detailed as follows (first vertical map = inclusion, second vertical map = quotient):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathscr{I}_\Sigma \Gamma(\pi)_{j-1} \arrow[r, "P_{j-1}"] \arrow[d] & \mathscr{I}_\Sigma \Gamma(\pi)_j \arrow[r, "P_j"] \arrow[d] & \mathscr{I}_\Sigma \Gamma(\pi)_{j+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \Gamma(\pi_{j-1}) \arrow[r, "P_{j-1}"] \arrow[d] & \Gamma(\pi_j) \arrow[r, "P_j"] \arrow[d] & \Gamma(\pi_{j+1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \frac{\Gamma(\pi_{j-1})}{\mathscr{I}_\Sigma \Gamma(\pi)_{j-1}} \arrow[r, "P_{j-1}"] \arrow[d] & \frac{\Gamma(\pi_j)}{\mathscr{I}_\Sigma \Gamma(\pi)_j} \arrow[r, "P_j"] \arrow[d] & \frac{\Gamma(\pi_{j+1})}{\mathscr{I}_\Sigma \Gamma(\pi)_{j+1}} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Using an auxiliary pseudo-Riemannian metric g on $\mathscr{M}$ such that $g^{-1}(\mathrm{d}t, \mathrm{d}t) \neq 0$, we can find a splitting of this exact sequence, so that the last row $\mathscr{P}_\Sigma$ can be seen as a differential complex over $\Sigma =$ "spatial part" of $\mathscr{P} \Rightarrow$ tangential complex associated to the pair $(\mathscr{P}, \Sigma)$ (the precise choice of t is immaterial).

Now we can state the Cauchy problem for $\mathscr{P}$ (at level j) with initial data on Σ:

Given $\psi \in \Gamma(\pi_{j+1})$, $\phi_0 \in \Gamma(\pi_j|_\Sigma)$, find $\phi \in \Gamma(\pi_j)$ such that

$$P_j\phi = \psi \ , \quad \phi|_\Sigma = \phi_0 \ . \tag{1}$$

It is clear that a necessary condition for existence of solutions of (1) is that $P_{j+1}\psi = 0$, so that P_{j+1} represents a constraint on admissible sources. Moreover, if $\tilde{\phi}_0 \in \Gamma(\pi_j)$ extends ϕ_0, then $\phi - \tilde{\phi}_0$ vanishes on Σ, thus

$$P(\phi - \tilde{\phi}_0) = \psi - P\tilde{\phi}_0$$

belongs to $\mathscr{J}_\Sigma\Gamma(\pi)_{j+1}$, regardless of the choice of $\tilde{\phi}_0$.

Using an auxiliary pseudo-Riemannian metric g on $\mathscr{M}$ such that $g^{-1}(\mathrm{d}t, \mathrm{d}t) \neq 0$, we can find a splitting of this exact sequence, so that the last row $\mathscr{P}_\Sigma$ can be seen as a differential complex over $\Sigma =$ "spatial part" of $\mathscr{P} \Rightarrow$ tangential complex associated to the pair $(\mathscr{P}, \Sigma)$ (the precise choice of t is immaterial).

Now we can state the Cauchy problem for $\mathscr{P}$ (at level j) with initial data on Σ:

Given $\psi \in \Gamma(\pi_{j+1})$, $\phi_0 \in \Gamma(\pi_j|_\Sigma)$, find $\phi \in \Gamma(\pi_j)$ such that

$$P_j \phi = \psi \,, \quad \phi|_\Sigma = \phi_0 \,. \tag{1}$$

It is clear that a necessary condition for existence of solutions of (1) is that $P_{j+1}\psi = 0$, so that P_{j+1} represents a constraint on admissible sources. Moreover, if $\tilde{\phi}_0 \in \Gamma(\pi_j)$ extends ϕ_0, then $\phi - \tilde{\phi}_0$ vanishes on Σ, thus

$$P(\phi - \tilde{\phi}_0) = \psi - P\tilde{\phi}_0$$

belongs to $\mathscr{J}_\Sigma \Gamma(\pi)_{j+1}$, regardless of the choice of $\tilde{\phi}_0$.

Using an auxiliary pseudo-Riemannian metric g on $\mathscr{M}$ such that $g^{-1}(\mathrm{d}t, \mathrm{d}t) \neq 0$, we can find a splitting of this exact sequence, so that the last row $\mathscr{P}_\Sigma$ can be seen as a differential complex over $\Sigma =$ "spatial part" of $\mathscr{P} \Rightarrow$ tangential complex associated to the pair $(\mathscr{P}, \Sigma)$ (the precise choice of t is immaterial).

Now we can state the Cauchy problem for $\mathscr{P}$ (at level j) with initial data on Σ:

Given $\psi \in \Gamma(\pi_{j+1})$, $\phi_0 \in \Gamma(\pi_j|_\Sigma)$, find $\phi \in \Gamma(\pi_j)$ such that

$$P_j \phi = \psi \ , \quad \phi|_\Sigma = \phi_0 \ . \tag{1}$$

It is clear that a necessary condition for existence of solutions of (1) is that $P_{j+1}\psi = 0$, so that P_{j+1} represents a constraint on admissible sources. Moreover, if $\tilde{\phi}_0 \in \Gamma(\pi_j)$ extends ϕ_0, then $\phi - \tilde{\phi}_0$ vanishes on Σ, thus

$$P(\phi - \tilde{\phi}_0) = \psi - P\tilde{\phi}_0$$

belongs to $\mathscr{I}_\Sigma \Gamma(\pi)_{j+1}$, regardless of the choice of $\tilde{\phi}_0$.

Using an auxiliary pseudo-Riemannian metric g on $\mathscr{M}$ such that $g^{-1}(\mathrm{d}t, \mathrm{d}t) \neq 0$, we can find a splitting of this exact sequence, so that the last row $\mathscr{P}_\Sigma$ can be seen as a differential complex over $\Sigma =$ "spatial part" of $\mathscr{P} \Rightarrow$ tangential complex associated to the pair $(\mathscr{P}, \Sigma)$ (the precise choice of t is immaterial).

Now we can state the Cauchy problem for $\mathscr{P}$ (at level j) with initial data on Σ:

Given $\psi \in \Gamma(\pi_{j+1})$, $\phi_0 \in \Gamma(\pi_j|_\Sigma)$, find $\phi \in \Gamma(\pi_j)$ such that

$$P_j\phi = \psi \ , \quad \phi|_\Sigma = \phi_0 \ . \tag{1}$$

It is clear that a necessary condition for existence of solutions of (1) is that $P_{j+1}\psi = 0$, so that P_{j+1} represents a constraint on admissible sources. Moreover, if $\tilde{\phi}_0 \in \Gamma(\pi_j)$ extends ϕ_0, then $\phi - \tilde{\phi}_0$ vanishes on Σ, thus

$$P(\phi - \tilde{\phi}_0) = \psi - P\tilde{\phi}_0$$

belongs to $\mathscr{J}_\Sigma\Gamma(\pi)_{j+1}$, regardless of the choice of $\tilde{\phi}_0$.

Using an auxiliary pseudo-Riemannian metric g on $\mathscr{M}$ such that $g^{-1}(\mathrm{d}t, \mathrm{d}t) \neq 0$, we can find a splitting of this exact sequence, so that the last row $\mathscr{P}_\Sigma$ can be seen as a differential complex over $\Sigma =$ "spatial part" of $\mathscr{P} \Rightarrow$ tangential complex associated to the pair $(\mathscr{P}, \Sigma)$ (the precise choice of t is immaterial).

Now we can state the Cauchy problem for $\mathscr{P}$ (at level j) with initial data on Σ:

Given $\psi \in \Gamma(\pi_{j+1})$, $\phi_0 \in \Gamma(\pi_j|_\Sigma)$, find $\phi \in \Gamma(\pi_j)$ such that

$$P_j\phi = \psi \ , \quad \phi|_\Sigma = \phi_0 \ . \tag{1}$$

It is clear that a necessary condition for existence of solutions of (1) is that $P_{j+1}\psi = 0$, so that P_{j+1} represents a constraint on admissible sources. Moreover, if $\tilde{\phi}_0 \in \Gamma(\pi_j)$ extends ϕ_0, then $\phi - \tilde{\phi}_0$ vanishes on Σ, thus

$$P(\phi - \tilde{\phi}_0) = \psi - P\tilde{\phi}_0$$

belongs to $\mathscr{I}_\Sigma\Gamma(\pi)_{j+1}$, regardless of the choice of $\tilde{\phi}_0$.

Using an auxiliary pseudo-Riemannian metric g on $\mathscr{M}$ such that $g^{-1}(\mathrm{d}t, \mathrm{d}t) \neq 0$, we can find a splitting of this exact sequence, so that the last row $\mathscr{P}_\Sigma$ can be seen as a differential complex over $\Sigma =$ "spatial part" of $\mathscr{P} \Rightarrow$ tangential complex associated to the pair $(\mathscr{P}, \Sigma)$ (the precise choice of t is immaterial).

Now we can state the Cauchy problem for $\mathscr{P}$ (at level j) with initial data on Σ:

Given $\psi \in \Gamma(\pi_{j+1})$, $\phi_0 \in \Gamma(\pi_j|_\Sigma)$, find $\phi \in \Gamma(\pi_j)$ such that

$$P_j \phi = \psi \ , \quad \phi|_\Sigma = \phi_0 \ . \tag{1}$$

It is clear that a necessary condition for existence of solutions of (1) is that $P_{j+1}\psi = 0$, so that P_{j+1} represents a constraint on admissible sources. Moreover, if $\tilde{\phi}_0 \in \Gamma(\pi_j)$ extends ϕ_0, then $\phi - \tilde{\phi}_0$ vanishes on Σ, thus

$$P(\phi - \tilde{\phi}_0) = \psi - P\tilde{\phi}_0$$

belongs to $\mathscr{I}_\Sigma \Gamma(\pi)_{j+1}$, regardless of the choice of $\tilde{\phi}_0$.

Using an auxiliary pseudo-Riemannian metric g on $\mathscr{M}$ such that $g^{-1}(\mathrm{d}t, \mathrm{d}t) \neq 0$, we can find a splitting of this exact sequence, so that the last row $\mathscr{P}_\Sigma$ can be seen as a differential complex over $\Sigma =$ "spatial part" of $\mathscr{P} \Rightarrow$ tangential complex associated to the pair $(\mathscr{P}, \Sigma)$ (the precise choice of t is immaterial).

Now we can state the Cauchy problem for $\mathscr{P}$ (at level j) with initial data on Σ:

Given $\psi \in \Gamma(\pi_{j+1})$, $\phi_0 \in \Gamma(\pi_j|_\Sigma)$, find $\phi \in \Gamma(\pi_j)$ such that

$$P_j \phi = \psi \ , \quad \phi|_\Sigma = \phi_0 \ . \tag{1}$$

It is clear that a necessary condition for existence of solutions of (1) is that $P_{j+1}\psi = 0$, so that P_{j+1} represents a constraint on admissible sources. Moreover, if $\tilde{\phi}_0 \in \Gamma(\pi_j)$ extends ϕ_0, then $\phi - \tilde{\phi}_0$ vanishes on Σ, thus

$$P(\phi - \tilde{\phi}_0) = \psi - P\tilde{\phi}_0$$

belongs to $\mathscr{J}_\Sigma \Gamma(\pi)_{j+1}$, regardless of the choice of $\tilde{\phi}_0$.

Hence, our first reformulation of (1) is

Find $\phi \in \Gamma(\pi_j)$ such that $P_j\phi = \psi$ and $\phi|_\Sigma = \phi_0$, where

$$P_{j+1}\psi = 0 \ , \ \psi - P\tilde{\phi}_0 \in \mathscr{I}_\Sigma\Gamma(\pi) \tag{2}$$

for all $\tilde{\phi}_0 \in \Gamma(\pi_j)$ extending ϕ_0.

Actually, (2) is equivalent to the (apparently) special case

Find $\phi' \in \Gamma(\pi_j)$ such that $P_j\phi' = \psi'$ and $\phi'|_\Sigma = 0$, where

$$\psi' \in \mathscr{I}_\Sigma\Gamma(\pi)_{j+1} \ , \ P_{j+1}\psi' = 0 \ , \tag{3}$$

since a solution ϕ' of (3) with $\psi' = \psi - P_j\tilde{\phi}_0$ yields the solution $\phi = \phi' + \tilde{\phi}_0$ of (2). Finally, we can reformulate (3) as

Find $\phi \in \mathscr{I}_\Sigma\Gamma(\pi)_j$ such that $P_j\phi = \psi$, where

$$\psi \in \mathscr{I}_\Sigma\Gamma(\phi)_{j+1} \ , \ P_{j+1}\psi = 0 \ . \tag{4}$$

Hence, our first reformulation of (1) is

Find $\phi \in \Gamma(\pi_j)$ such that $P_j\phi = \psi$ and $\phi|_\Sigma = \phi_0$, where

$$P_{j+1}\psi = 0 \ , \ \psi - P\tilde{\phi}_0 \in \mathscr{I}_\Sigma\Gamma(\pi) \tag{2}$$

for all $\tilde{\phi}_0 \in \Gamma(\pi_j)$ extending ϕ_0.

Actually, (2) is equivalent to the (apparently) special case

Find $\phi' \in \Gamma(\pi_j)$ such that $P_j\phi' = \psi'$ and $\phi'|_\Sigma = 0$, where

$$\psi' \in \mathscr{I}_\Sigma\Gamma(\pi)_{j+1} \ , \ P_{j+1}\psi' = 0 \ , \tag{3}$$

since a solution ϕ' of (3) with $\psi' = \psi - P_j\tilde{\phi}_0$ yields the solution $\phi = \phi' + \tilde{\phi}_0$ of (2). Finally, we can reformulate (3) as

Find $\phi \in \mathscr{I}_\Sigma\Gamma(\pi)_j$ such that $P_j\phi = \psi$, where

$$\psi \in \mathscr{I}_\Sigma\Gamma(\phi)_{j+1} \ , \ P_{j+1}\psi = 0 \ . \tag{4}$$

Hence, our first reformulation of (1) is

Find $\phi \in \Gamma(\pi_j)$ such that $P_j\phi = \psi$ and $\phi|_\Sigma = \phi_0$, where

$$P_{j+1}\psi = 0 \ , \ \psi - P\tilde{\phi}_0 \in \mathscr{J}_\Sigma\Gamma(\pi) \tag{2}$$

for all $\tilde{\phi}_0 \in \Gamma(\pi_j)$ extending ϕ_0.

Actually, (2) is equivalent to the (apparently) special case

Find $\phi' \in \Gamma(\pi_j)$ such that $P_j\phi' = \psi'$ and $\phi'|_\Sigma = 0$, where

$$\psi' \in \mathscr{J}_\Sigma\Gamma(\pi)_{j+1} \ , \ P_{j+1}\psi' = 0 \ , \tag{3}$$

since a solution ϕ' of (3) with $\psi' = \psi - P_j\tilde{\phi}_0$ yields the solution $\phi = \phi' + \tilde{\phi}_0$ of (2). Finally, we can reformulate (3) as

Find $\phi \in \mathscr{J}_\Sigma\Gamma(\pi)_j$ such that $P_j\phi = \psi$, where

$$\psi \in \mathscr{J}_\Sigma\Gamma(\phi)_{j+1} \ , \ P_{j+1}\psi = 0 \ . \tag{4}$$

Hence, our first reformulation of (1) is

Find $\phi \in \Gamma(\pi_j)$ such that $P_j\phi = \psi$ and $\phi|_\Sigma = \phi_0$, where

$$P_{j+1}\psi = 0 \ , \ \psi - P\tilde{\phi}_0 \in \mathscr{J}_\Sigma\Gamma(\pi) \tag{2}$$

for all $\tilde{\phi}_0 \in \Gamma(\pi_j)$ extending ϕ_0.

Actually, (2) is equivalent to the (apparently) special case

Find $\phi' \in \Gamma(\pi_j)$ such that $P_j\phi' = \psi'$ and $\phi'|_\Sigma = 0$, where

$$\psi' \in \mathscr{J}_\Sigma\Gamma(\pi)_{j+1} \ , \ P_{j+1}\psi' = 0 \ , \tag{3}$$

since a solution ϕ' of (3) with $\psi' = \psi - P_j\tilde{\phi}_0$ yields the solution $\phi = \phi' + \tilde{\phi}_0$ of (2). Finally, we can reformulate (3) as

Find $\phi \in \mathscr{J}_\Sigma\Gamma(\pi)_j$ such that $P_j\phi = \psi$, where

$$\psi \in \mathscr{J}_\Sigma\Gamma(\phi)_{j+1} \ , \ P_{j+1}\psi = 0 \ . \tag{4}$$

Clearly, (3) is a special case of (4). Conversely, if ϕ solves (4), then by the above Lemma there are $\phi' \in \mathscr{J}_\Sigma\Gamma(\pi)_j$, $\phi'' \in \mathscr{J}_\Sigma\Gamma(\pi)_{j-1}$ such that $\phi = \phi' + P_{j-1}\phi''$ and ϕ' is a solution of (3) $\Rightarrow$ ϕ solves (3) "up to a gauge transformation". This is as close to uniqueness as we can get. In fact:

Theorem (MacKichan 1975, Theorem 3.1)

The Cauchy problem (4) *at level j is solvable $\Leftrightarrow$ the $(j+1)$-th cohomology class of ψ in the complex $\mathscr{J}_\Sigma\mathscr{P}$ is zero. Moreover, the cohomology class of the solution ϕ is unique $\Leftrightarrow$ the j-th cohomology group of $\mathscr{J}_\Sigma\mathscr{P}$ is zero.* □

In particular, the well-posedness of the Cauchy problem at all levels can be stated as follows:

Solvability of (4) + uniqueness of cohomology class of solutions $\Leftrightarrow$ exactness of the complex $\mathscr{J}_\Sigma\mathscr{P}$ $\Leftrightarrow$ the cohomology groups of $\mathscr{P}$ and $\mathscr{P}_\Sigma$ are isomorphic!

Clearly, (3) is a special case of (4). Conversely, if ϕ solves (4), then by the above Lemma there are $\phi' \in \mathscr{I}_\Sigma\Gamma(\pi)_j$, $\phi'' \in \mathscr{I}_\Sigma\Gamma(\pi)_{j-1}$ such that $\phi = \phi' + P_{j-1}\phi''$ and ϕ' is a solution of (3) $\Rightarrow$ ϕ solves (3) "up to a gauge transformation". This is as close to uniqueness as we can get. In fact:

Theorem (MacKichan 1975, Theorem 3.1)

The Cauchy problem (4) at level j is solvable $\Leftrightarrow$ the $(j+1)$-th cohomology class of ψ in the complex $\mathscr{I}_\Sigma\mathscr{P}$ is zero. Moreover, the cohomology class of the solution ϕ is unique $\Leftrightarrow$ the j-th cohomology group of $\mathscr{I}_\Sigma\mathscr{P}$ is zero. □

In particular, the well-posedness of the Cauchy problem at all levels can be stated as follows:

Solvability of (4) + uniqueness of cohomology class of solutions $\Leftrightarrow$ exactness of the complex $\mathscr{I}_\Sigma\mathscr{P}$ $\Leftrightarrow$ the cohomology groups of $\mathscr{P}$ and $\mathscr{P}_\Sigma$ are isomorphic!

Clearly, (3) is a special case of (4). Conversely, if ϕ solves (4), then by the above Lemma there are $\phi' \in \mathscr{J}_\Sigma\Gamma(\pi)_j$, $\phi'' \in \mathscr{J}_\Sigma\Gamma(\pi)_{j-1}$ such that $\phi = \phi' + P_{j-1}\phi''$ and ϕ' is a solution of (3) $\Rightarrow$ ϕ solves (3) "up to a gauge transformation". This is as close to uniqueness as we can get. In fact:

Theorem (MacKichan 1975, Theorem 3.1)

The Cauchy problem (4) at level j is solvable $\Leftrightarrow$ the $(j+1)$-th cohomology class of ψ in the complex $\mathscr{J}_\Sigma\mathscr{P}$ is zero. Moreover, the cohomology class of the solution ϕ is unique $\Leftrightarrow$ the j-th cohomology group of $\mathscr{J}_\Sigma\mathscr{P}$ is zero. □

In particular, the well-posedness of the Cauchy problem at all levels can be stated as follows:

Solvability of (4) + uniqueness of cohomology class of solutions $\Leftrightarrow$ exactness of the complex $\mathscr{J}_\Sigma\mathscr{P}$ $\Leftrightarrow$ the cohomology groups of $\mathscr{P}$ and $\mathscr{P}_\Sigma$ are isomorphic!

Clearly, (3) is a special case of (4). Conversely, if ϕ solves (4), then by the above Lemma there are $\phi' \in \mathscr{J}_\Sigma \Gamma(\pi)_j$, $\phi'' \in \mathscr{J}_\Sigma \Gamma(\pi)_{j-1}$ such that $\phi = \phi' + P_{j-1}\phi''$ and ϕ' is a solution of (3) $\Rightarrow$ ϕ solves (3) "up to a gauge transformation". This is as close to uniqueness as we can get. In fact:

Theorem (MacKichan 1975, Theorem 3.1)

The Cauchy problem (4) *at level j is solvable $\Leftrightarrow$ the $(j+1)$-th cohomology class of ψ in the complex $\mathscr{J}_\Sigma \mathscr{P}$ is zero. Moreover, the cohomology class of the solution ϕ is unique $\Leftrightarrow$ the j-th cohomology group of $\mathscr{J}_\Sigma \mathscr{P}$ is zero.* □

In particular, the well-posedness of the Cauchy problem at all levels can be stated as follows:

Solvability of (4) + uniqueness of cohomology class of solutions $\Leftrightarrow$ exactness of the complex $\mathscr{J}_\Sigma \mathscr{P}$ $\Leftrightarrow$ the cohomology groups of $\mathscr{P}$ and $\mathscr{P}_\Sigma$ are isomorphic!

Clearly, (3) is a special case of (4). Conversely, if ϕ solves (4), then by the above Lemma there are $\phi' \in \mathscr{J}_\Sigma\Gamma(\pi)_j$, $\phi'' \in \mathscr{J}_\Sigma\Gamma(\pi)_{j-1}$ such that $\phi = \phi' + P_{j-1}\phi''$ and ϕ' is a solution of (3) $\Rightarrow$ ϕ solves (3) "up to a gauge transformation". This is as close to uniqueness as we can get. In fact:

Theorem (MacKichan 1975, Theorem 3.1)

The Cauchy problem (4) *at level j is solvable $\Leftrightarrow$ the $(j+1)$-th cohomology class of ψ in the complex $\mathscr{J}_\Sigma\mathscr{P}$ is zero. Moreover, the cohomology class of the solution ϕ is unique $\Leftrightarrow$ the j-th cohomology group of $\mathscr{J}_\Sigma\mathscr{P}$ is zero.* □

In particular, the well-posedness of the Cauchy problem at all levels can be stated as follows:

Solvability of (4) + uniqueness of cohomology class of solutions $\Leftrightarrow$ exactness of the complex $\mathscr{J}_\Sigma\mathscr{P}$ $\Leftrightarrow$ the cohomology groups of $\mathscr{P}$ and $\mathscr{P}_\Sigma$ are isomorphic!

Clearly, (3) is a special case of (4). Conversely, if ϕ solves (4), then by the above Lemma there are $\phi' \in \mathscr{J}_\Sigma\Gamma(\pi)_j$, $\phi'' \in \mathscr{J}_\Sigma\Gamma(\pi)_{j-1}$ such that $\phi = \phi' + P_{j-1}\phi''$ and ϕ' is a solution of (3) $\Rightarrow$ ϕ solves (3) "up to a gauge transformation". This is as close to uniqueness as we can get. In fact:

Theorem (MacKichan 1975, Theorem 3.1)

The Cauchy problem (4) *at level* j *is solvable* $\Leftrightarrow$ *the* $(j+1)$*-th cohomology class of* ψ *in the complex* $\mathscr{J}_\Sigma\mathscr{P}$ *is zero. Moreover, the cohomology class of the solution* ϕ *is unique* $\Leftrightarrow$ *the* j*-th cohomology group of* $\mathscr{J}_\Sigma\mathscr{P}$ *is zero.* $\square$

In particular, the well-posedness of the Cauchy problem at all levels can be stated as follows:

Solvability of (4) + uniqueness of cohomology class of solutions $\Leftrightarrow$ exactness of the complex $\mathscr{J}_\Sigma\mathscr{P}$ $\Leftrightarrow$ the cohomology groups of $\mathscr{P}$ and $\mathscr{P}_\Sigma$ are isomorphic!

Remarks:

- The second equivalence follows from the long exact sequence in cohomology associated to $0 \to \mathscr{J}_\Sigma \mathscr{P} \to \mathscr{P} \to \mathscr{P}_\Sigma \to 0$;
- It is enough to assume only local existence of the "time function" t, for local existence and uniqueness of solutions in the sense of cohomology lead to global well-posedness in the same sense thanks to paracompactness of $\mathscr{M}$ and a Mayer-Vietoris-type argument (Andreotti-Denson Hill-Łojasiewicz-MacKichan 1976).

Differential complexes can be seen as "resolutions" of (possibly underdetermined or overdetermined) linear differential operators (Cartan-Kuranishi, Spencer). The quotes are due to the fact that (nontrivial) solutions of (4) are represented by nontrivial cohomology classes. Hence, exact differential complexes have only "pure gauge" solutions!

Remarks:

- The second equivalence follows from the long exact sequence in cohomology associated to $0 \to \mathscr{J}_\Sigma \mathscr{P} \to \mathscr{P} \to \mathscr{P}_\Sigma \to 0$;
- It is enough to assume only local existence of the "time function" t, for local existence and uniqueness of solutions in the sense of cohomology lead to global well-posedness in the same sense thanks to paracompactness of $\mathscr{M}$ and a Mayer-Vietoris-type argument (Andreotti-Denson Hill-Łojasiewicz-MacKichan 1976).

Differential complexes can be seen as "resolutions" of (possibly underdetermined or overdetermined) linear differential operators (Cartan-Kuranishi, Spencer). The quotes are due to the fact that (nontrivial) solutions of (4) are represented by nontrivial cohomology classes. Hence, exact differential complexes have only "pure gauge" solutions!

Remarks:

- The second equivalence follows from the long exact sequence in cohomology associated to $0 \to \mathscr{J}_\Sigma \mathscr{P} \to \mathscr{P} \to \mathscr{P}_\Sigma \to 0$;
- It is enough to assume only local existence of the "time function" t, for local existence and uniqueness of solutions in the sense of cohomology lead to global well-posedness in the same sense thanks to paracompactness of $\mathscr{M}$ and a Mayer-Vietoris-type argument (Andreotti-Denson Hill-Łojasiewicz-MacKichan 1976).

Differential complexes can be seen as "resolutions" of (possibly underdetermined or overdetermined) linear differential operators (Cartan-Kuranishi, Spencer). The quotes are due to the fact that (nontrivial) solutions of (4) are represented by nontrivial cohomology classes. Hence, exact differential complexes have only "pure gauge" solutions!

Universidade Federal do ABC

Remarks:

- The second equivalence follows from the long exact sequence in cohomology associated to $0 \to \mathscr{J}_\Sigma \mathscr{P} \to \mathscr{P} \to \mathscr{P}_\Sigma \to 0$;
- It is enough to assume only local existence of the "time function" t, for local existence and uniqueness of solutions in the sense of cohomology lead to global well-posedness in the same sense thanks to paracompactness of $\mathscr{M}$ and a Mayer-Vietoris-type argument (Andreotti-Denson Hill-Łojasiewicz-MacKichan 1976).

Differential complexes can be seen as "resolutions" of (possibly underdetermined or overdetermined) linear differential operators (Cartan-Kuranishi, Spencer). The quotes are due to the fact that (nontrivial) solutions of (4) are represented by nontrivial cohomology classes. Hence, exact differential complexes have only "pure gauge" solutions!

Remarks:

- The second equivalence follows from the long exact sequence in cohomology associated to $0 \to \mathscr{J}_\Sigma \mathscr{P} \to \mathscr{P} \to \mathscr{P}_\Sigma \to 0$;
- It is enough to assume only local existence of the "time function" t, for local existence and uniqueness of solutions in the sense of cohomology lead to global well-posedness in the same sense thanks to paracompactness of $\mathscr{M}$ and a Mayer-Vietoris-type argument (Andreotti-Denson Hill-Łojasiewicz-MacKichan 1976).

Differential complexes can be seen as "resolutions" of (possibly underdetermined or overdetermined) linear differential operators (Cartan-Kuranishi, Spencer). The quotes are due to the fact that (nontrivial) solutions of (4) are represented by nontrivial cohomology classes. Hence, exact differential complexes have only "pure gauge" solutions!

Hyperbolicity and gauge fixing

In order to discuss hyperbolicity for differential complexes, we need a few more tools. The (formal) transpose of a differential operator P of order k and type $\pi_1 \to \pi_2$ is the differential operator P' of order k and type $\pi_2^{\circledast} \to \pi_1^{\circledast}$ ($\pi_j^{\circledast} : \mathscr{E}_j^{\circledast} = \mathscr{E}_j^* \otimes \wedge^d T^*\mathscr{M} \to \mathscr{M}$ = twisted dual of π_j) defined by "integration by parts":

$$\int_{\mathscr{M}} (P'\alpha)(\phi) \doteq \int_{\mathscr{M}} \alpha(P\phi) \ , \quad \alpha \in \Sigma(\pi_2^{\circledast}) \ , \ \phi \in \Gamma_c(\pi_1) \ ,$$

where $\Gamma_c(\pi_j)$ = space of smooth sections of π_j with compact support. If $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$ is a differential complex, the transposed complex is just the complex $\mathscr{P}' = (P_j')_{j\in\mathbb{Z}}$.

Hyperbolicity and gauge fixing

In order to discuss hyperbolicity for differential complexes, we need a few more tools. The (formal) transpose of a differential operator P of order k and type $\pi_1 \to \pi_2$ is the differential operator P' of order k and type $\pi_2^{\circledast} \to \pi_1^{\circledast}$ ($\pi_j^{\circledast} : \mathscr{E}_j^{\circledast} = \mathscr{E}_j^* \otimes \wedge^d T^*\mathscr{M} \to \mathscr{M}$ = twisted dual of π_j) defined by "integration by parts":

$$\int_{\mathscr{M}} (P'\alpha)(\phi) \doteq \int_{\mathscr{M}} \alpha(P\phi) \ , \quad \alpha \in \Sigma(\pi_2^{\circledast}) \ , \ \phi \in \Gamma_c(\pi_1) \ ,$$

where $\Gamma_c(\pi_j)$ = space of smooth sections of π_j with compact support. If $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$ is a differential complex, the transposed complex is just the complex $\mathscr{P}' = (P_j')_{j\in\mathbb{Z}}$.

Hyperbolicity and gauge fixing

In order to discuss hyperbolicity for differential complexes, we need a few more tools. The (formal) transpose of a differential operator P of order k and type $\pi_1 \to \pi_2$ is the differential operator P' of order k and type $\pi_2^{\circledast} \to \pi_1^{\circledast}$ ($\pi_j^{\circledast} : \mathscr{E}_j^{\circledast} = \mathscr{E}_j^* \otimes \wedge^d T^*\mathscr{M} \to \mathscr{M}$ = twisted dual of π_j) defined by "integration by parts":

$$\int_{\mathscr{M}} (P'\alpha)(\phi) \doteq \int_{\mathscr{M}} \alpha(P\phi) \ , \quad \alpha \in \Sigma(\pi_2^{\circledast}) \ , \phi \in \Gamma_c(\pi_1) \ ,$$

where $\Gamma_c(\pi_j)$ = space of smooth sections of π_j with compact support. If $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$ is a differential complex, the transposed complex is just the complex $\mathscr{P}' = (P_j')_{j\in\mathbb{Z}}$.

Hyperbolicity and gauge fixing

In order to discuss hyperbolicity for differential complexes, we need a few more tools. The (formal) transpose of a differential operator P of order k and type $\pi_1 \to \pi_2$ is the differential operator P' of order k and type $\pi_2^{\circledast} \to \pi_1^{\circledast}$ ($\pi_j^{\circledast} : \mathscr{E}_j^{\circledast} = \mathscr{E}_j^* \otimes \wedge^d T^*\mathscr{M} \to \mathscr{M}$ = twisted dual of π_j) defined by "integration by parts":

$$\int_{\mathscr{M}} (P'\alpha)(\phi) \doteq \int_{\mathscr{M}} \alpha(P\phi) \ , \quad \alpha \in \Sigma(\pi_2^{\circledast}) \ , \phi \in \Gamma_c(\pi_1) \ ,$$

where $\Gamma_c(\pi_j)$ = space of smooth sections of π_j with compact support. If $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$ is a differential complex, the transposed complex is just the complex $\mathscr{P}' = (P'_j)_{j\in\mathbb{Z}}$.

Hyperbolicity and gauge fixing

In order to discuss hyperbolicity for differential complexes, we need a few more tools. The (formal) transpose of a differential operator P of order k and type $\pi_1 \to \pi_2$ is the differential operator P' of order k and type $\pi_2^{\circledast} \to \pi_1^{\circledast}$ ($\pi_j^{\circledast} : \mathscr{E}_j^{\circledast} = \mathscr{E}_j^* \otimes \wedge^d T^*\mathscr{M} \to \mathscr{M}$ = twisted dual of π_j) defined by "integration by parts":

$$\int_{\mathscr{M}} (P'\alpha)(\phi) \doteq \int_{\mathscr{M}} \alpha(P\phi) \ , \quad \alpha \in \Sigma(\pi_2^{\circledast}) \ , \phi \in \Gamma_c(\pi_1) \ ,$$

where $\Gamma_c(\pi_j)$ = space of smooth sections of π_j with compact support. If $\mathscr{P} = (P_j)_{j\in\mathbb{Z}}$ is a differential complex, the transposed complex is just the complex $\mathscr{P}' = (P'_j)_{j\in\mathbb{Z}}$.

Suppose now that $\pi_{T\mathscr{M}}$ is endowed with a pseudo-Riemannian metric g and π_j is endowed with a pseudo-Riemannian (π_j real) or pseudo-Hermitian (π_j complex) metric e_j, $j \in \mathbb{Z}$, all naturally lifted to the associated duals and tensor bundles. These allow us to define bundle isomorphisms $* : \pi_j^{\circledast} \to \pi_j$. The (formal) adjoint complex to $\mathscr{P}$ (w.r.t. $g, (e_j)_{j\in\mathbb{Z}}$) is the complex $\mathscr{P}^* = (P_j^*)_{j\in\mathbb{Z}}$, where

$$P_j^* = *P_j' *^{-1} : \pi_{j+1} \to \pi_j \ .$$

It is not difficult to show that $\sigma_{P_j^*}(p,\xi) = \sigma_{P_j}(p,\xi)^*$, the adjoint of the linear map $\sigma_{P_j}(p,\xi) : \pi_j^{-1}(p) \to \pi_{j+1}^{-1}(p)$ with respect to the pair $e_j(p), e_{j+1}(p)$ (particularly, it does not depend on g).

Adjunction has the effect of exchanging the roles of overdeterminedness and underdeterminedness. An useful byproduct is that if $P_{j-1} \neq 0$, then P_{j-1}^* yields a natural, Lorenz-type gauge-fixing condition for P_j:

$$P_{j-1}^* \phi = 0 \ .$$

Universidade Federal do ABC

Suppose now that $\pi_{T\mathscr{M}}$ is endowed with a pseudo-Riemannian metric g and π_j is endowed with a pseudo-Riemannian (π_j real) or pseudo-Hermitian (π_j complex) metric e_j, $j \in \mathbb{Z}$, all naturally lifted to the associated duals and tensor bundles. These allow us to define bundle isomorphisms $* : \pi_j^{\circledast} \to \pi_j$. The (formal) adjoint complex to $\mathscr{P}$ (w.r.t. $g, (e_j)_{j\in\mathbb{Z}}$) is the complex $\mathscr{P}^* = (P_j^*)_{j\in\mathbb{Z}}$, where

$$P_j^* = *P_j' *^{-1} : \pi_{j+1} \to \pi_j \ .$$

It is not difficult to show that $\sigma_{P_j^*}(p,\xi) = \sigma_{P_j}(p,\xi)^*$, the adjoint of the linear map $\sigma_{P_j}(p,\xi) : \pi_j^{-1}(p) \to \pi_{j+1}^{-1}(p)$ with respect to the pair $e_j(p), e_{j+1}(p)$ (particularly, it does not depend on g).

Adjunction has the effect of exchanging the roles of overdeterminedness and underdeterminedness. An useful byproduct is that if $P_{j-1} \neq 0$, then P_{j-1}^* yields a natural, Lorenz-type gauge-fixing condition for P_j:

$$P_{j-1}^* \phi = 0 \ .$$

Suppose now that $\pi_{T\mathscr{M}}$ is endowed with a pseudo-Riemannian metric g and π_j is endowed with a pseudo-Riemannian (π_j real) or pseudo-Hermitian (π_j complex) metric e_j, $j \in \mathbb{Z}$, all naturally lifted to the associated duals and tensor bundles. These allow us to define bundle isomorphisms $* : \pi_j^{\circledast} \to \pi_j$. The (formal) adjoint complex to $\mathscr{P}$ (w.r.t. $g, (e_j)_{j\in\mathbb{Z}}$) is the complex $\mathscr{P}^* = (P_j^*)_{j\in\mathbb{Z}}$, where

$$P_j^* = *P_j' *^{-1} : \pi_{j+1} \to \pi_j \ .$$

It is not difficult to show that $\sigma_{P_j^*}(p,\xi) = \sigma_{P_j}(p,\xi)^*$, the adjoint of the linear map $\sigma_{P_j}(p,\xi) : \pi_j^{-1}(p) \to \pi_{j+1}^{-1}(p)$ with respect to the pair $e_j(p), e_{j+1}(p)$ (particularly, it does not depend on g).

Adjunction has the effect of exchanging the roles of overdeterminedness and underdeterminedness. An useful byproduct is that if $P_{j-1} \neq 0$, then P_{j-1}^* yields a natural, Lorenz-type gauge-fixing condition for P_j:

$$P_{j-1}^* \phi = 0 \ .$$

Suppose now that $\pi_{T\mathscr{M}}$ is endowed with a pseudo-Riemannian metric g and π_j is endowed with a pseudo-Riemannian (π_j real) or pseudo-Hermitian (π_j complex) metric e_j, $j \in \mathbb{Z}$, all naturally lifted to the associated duals and tensor bundles. These allow us to define bundle isomorphisms $* : \pi_j^{\circledast} \to \pi_j$. The (formal) adjoint complex to $\mathscr{P}$ (w.r.t. $g, (e_j)_{j\in\mathbb{Z}}$) is the complex $\mathscr{P}^* = (P_j^*)_{j\in\mathbb{Z}}$, where

$$P_j^* = *P_j' *^{-1} : \pi_{j+1} \to \pi_j \ .$$

It is not difficult to show that $\sigma_{P_j^*}(p,\xi) = \sigma_{P_j}(p,\xi)^*$, the adjoint of the linear map $\sigma_{P_j}(p,\xi) : \pi_j^{-1}(p) \to \pi_{j+1}^{-1}(p)$ with respect to the pair $e_j(p), e_{j+1}(p)$ (particularly, it does not depend on g).

Adjunction has the effect of exchanging the roles of overdeterminedness and underdeterminedness. An useful byproduct is that if $P_{j-1} \neq 0$, then P_{j-1}^* yields a natural, Lorenz-type gauge-fixing condition for P_j:

$$P_{j-1}^* \phi = 0 \ .$$

A reasonable notion of hyperbolicity for (first-order, symbol surjective) differential complexes ought to include the following examples:

(I) The Dirac operator $\not{D}$ associated to a Lorentzian manifold $(\mathscr{M}, g)$;

(II) The Maxwell system $\mathrm{d}^* : \Gamma(\wedge^2 \pi'_{T\mathscr{M}}) \to \Gamma(\wedge^1 \pi'_{T\mathscr{M}})$ when $T\mathscr{M}$ is endowed with a Lorentz metric g, subject to the constraint $\mathrm{d}\omega = 0$.

In example (i), the spinor bundle π_1 is endowed with a pseudo-Hermitian metric e_1 of signature zero. It is easy to show (Forger-PLR-Vidal 2017, to appear) that the g being Lorentz is equivalent to $\not{D}$ being symmetric hyperbolic (SH):

- $\gamma_p(\xi) \doteq \sigma_{\not{D}}(p, \xi) : \pi_1^{-1}(p) \to \pi_1^{-1}(p)$ is pseudo-Hermitian:

$$e_1(\psi_1, \gamma_p(\xi)\psi_2) = e_1(\gamma_p(\xi)\psi_1, \psi_2) \; ; \tag{5}$$

- At each $p \in \mathscr{M}$ there is $\tau \in T_p^*\mathscr{M}$ (e.g. timelike w.r.t. g) s.t.

$$e_1^\tau \doteq e_1(\cdot, \gamma_p(\tau)\cdot) \text{ is positive definite.} \tag{6}$$

A reasonable notion of hyperbolicity for (first-order, symbol surjective) differential complexes ought to include the following examples:

(I) The Dirac operator $\not{D}$ associated to a Lorentzian manifold $(\mathscr{M}, g)$;

(II) The Maxwell system $\mathrm{d}^* : \Gamma(\wedge^2 \pi'_{T\mathscr{M}}) \to \Gamma(\wedge^1 \pi'_{T\mathscr{M}})$ when $T\mathscr{M}$ is endowed with a Lorentz metric g, subject to the constraint $\mathrm{d}\omega = 0$.

In example (i), the spinor bundle π_1 is endowed with a pseudo-Hermitian metric e_1 of signature zero. It is easy to show (Forger-PLR-Vidal 2017, to appear) that the g being Lorentz is equivalent to $\not{D}$ being symmetric hyperbolic (SH):

- $\gamma_p(\xi) \doteq \sigma_{\not{D}}(p, \xi) : \pi_1^{-1}(p) \to \pi_1^{-1}(p)$ is pseudo-Hermitian:

$$e_1(\psi_1, \gamma_p(\xi)\psi_2) = e_1(\gamma_p(\xi)\psi_1, \psi_2) \; ; \tag{5}$$

- At each $p \in \mathscr{M}$ there is $\tau \in T_p^*\mathscr{M}$ (e.g. timelike w.r.t. g) s.t.

$$e_1^\tau \doteq e_1(\cdot, \gamma_p(\tau)\cdot) \text{ is positive definite.} \tag{6}$$

A reasonable notion of hyperbolicity for (first-order, symbol surjective) differential complexes ought to include the following examples:

(I) The Dirac operator $\not{D}$ associated to a Lorentzian manifold $(\mathscr{M}, g)$;

(II) The Maxwell system $\mathrm{d}^* : \Gamma(\wedge^2 \pi'_{T\mathscr{M}}) \to \Gamma(\wedge^1 \pi'_{T\mathscr{M}})$ when $T\mathscr{M}$ is endowed with a Lorentz metric g, subject to the constraint $\mathrm{d}\omega = 0$.

In example (i), the spinor bundle π_1 is endowed with a pseudo-Hermitian metric e_1 of signature zero. It is easy to show (Forger-PLR-Vidal 2017, to appear) that the g being Lorentz is equivalent to $\not{D}$ being symmetric hyperbolic (SH):

- $\gamma_p(\xi) \doteq \sigma_{\not{D}}(p, \xi) : \pi_1^{-1}(p) \to \pi_1^{-1}(p)$ is pseudo-Hermitian:

$$e_1(\psi_1, \gamma_p(\xi)\psi_2) = e_1(\gamma_p(\xi)\psi_1, \psi_2) \; ; \tag{5}$$

- At each $p \in \mathscr{M}$ there is $\tau \in T_p^*\mathscr{M}$ (e.g. timelike w.r.t. g) s.t.

$$e_1^\tau \doteq e_1(\cdot, \gamma_p(\tau)\cdot) \text{ is positive definite.} \tag{6}$$

A reasonable notion of hyperbolicity for (first-order, symbol surjective) differential complexes ought to include the following examples:

(I) The Dirac operator $\not{D}$ associated to a Lorentzian manifold $(\mathscr{M}, g)$;

(II) The Maxwell system $\mathrm{d}^* : \Gamma(\wedge^2 \pi'_{T\mathscr{M}}) \to \Gamma(\wedge^1 \pi'_{T\mathscr{M}})$ when $T\mathscr{M}$ is endowed with a Lorentz metric g, subject to the constraint $\mathrm{d}\omega = 0$.

In example (i), the spinor bundle π_1 is endowed with a pseudo-Hermitian metric e_1 of signature zero. It is easy to show (Forger-PLR-Vidal 2017, to appear) that the g being Lorentz is equivalent to $\not{D}$ being symmetric hyperbolic (SH):

- $\gamma_p(\xi) \doteq \sigma_{\not{D}}(p, \xi) : \pi_1^{-1}(p) \to \pi_1^{-1}(p)$ is pseudo-Hermitian:

$$e_1(\psi_1, \gamma_p(\xi)\psi_2) = e_1(\gamma_p(\xi)\psi_1, \psi_2) \; ; \tag{5}$$

- At each $p \in \mathscr{M}$ there is $\tau \in T_p^*\mathscr{M}$ (e.g. timelike w.r.t. g) s.t.

$$e_1^\tau \doteq e_1(\cdot, \gamma_p(\tau)\cdot) \text{ is positive definite.} \tag{6}$$

A reasonable notion of hyperbolicity for (first-order, symbol surjective) differential complexes ought to include the following examples:

(I) The Dirac operator $\not{D}$ associated to a Lorentzian manifold $(\mathscr{M}, g)$;

(II) The Maxwell system $\mathrm{d}^* : \Gamma(\wedge^2\pi'_{T\mathscr{M}}) \to \Gamma(\wedge^1\pi'_{T\mathscr{M}})$ when $T\mathscr{M}$ is endowed with a Lorentz metric g, subject to the constraint $\mathrm{d}\omega = 0$.

In example (i), the spinor bundle π_1 is endowed with a pseudo-Hermitian metric e_1 of signature zero. It is easy to show (Forger-PLR-Vidal 2017, to appear) that the g being Lorentz is equivalent to $\not{D}$ being symmetric hyperbolic (SH):

- $\gamma_p(\xi) \doteq \sigma_{\not{D}}(p, \xi) : \pi_1^{-1}(p) \to \pi_1^{-1}(p)$ is pseudo-Hermitian:

$$e_1(\psi_1, \gamma_p(\xi)\psi_2) = e_1(\gamma_p(\xi)\psi_1, \psi_2) \; ; \tag{5}$$

- At each $p \in \mathscr{M}$ there is $\tau \in T_p^*\mathscr{M}$ (e.g. timelike w.r.t. g) s.t.

$$e_1^\tau \doteq e_1(\cdot, \gamma_p(\tau)\cdot) \text{ is positive definite.} \tag{6}$$

A reasonable notion of hyperbolicity for (first-order, symbol surjective) differential complexes ought to include the following examples:

(i) The Dirac operator $\not{D}$ associated to a Lorentzian manifold $(\mathscr{M}, g)$;

(ii) The Maxwell system $\mathrm{d}^* : \Gamma(\wedge^2 \pi'_{T\mathscr{M}}) \to \Gamma(\wedge^1 \pi'_{T\mathscr{M}})$ when $T\mathscr{M}$ is endowed with a Lorentz metric g, subject to the constraint $\mathrm{d}\omega = 0$.

In example (i), the spinor bundle π_1 is endowed with a pseudo-Hermitian metric e_1 of signature zero. It is easy to show (Forger-PLR-Vidal 2017, to appear) that the g being Lorentz is equivalent to $\not{D}$ being symmetric hyperbolic (SH):

- $\gamma_p(\xi) \doteq \sigma_{\not{D}}(p, \xi) : \pi_1^{-1}(p) \to \pi_1^{-1}(p)$ is pseudo-Hermitian:

$$e_1(\psi_1, \gamma_p(\xi)\psi_2) = e_1(\gamma_p(\xi)\psi_1, \psi_2) \; ; \tag{5}$$

- At each $p \in \mathscr{M}$ there is $\tau \in T_p^*\mathscr{M}$ (e.g. timelike w.r.t. g) s.t.

$$e_1^\tau \doteq e_1(\cdot, \gamma_p(\tau)\cdot) \text{ is positive definite.} \tag{6}$$

As for example (ii), we cannot expect to have a condition similar to (6), we need to restrict to a sub-bundle of $\wedge\pi'_{T\mathscr{M}}$ – such a restriction is sometimes called an hyperbolization of P (Geroch). In order to do so, one needs to reformulate Maxwell's equations in terms of $\mathrm{d}+\mathrm{d}^*$ instead of d^* (the Dirac-Hodge operator). It has a structure similar to that of $\not{D}$, but it is not symmetric hyperbolic, since for all $\omega \in \wedge T^*_p\mathscr{M}$, $\xi \in T^*_p\mathscr{M}$, $p \in \mathscr{M}$

$$g(\omega, \xi \wedge \omega - i_{g^\sharp(\xi)}\omega) = 0 \ .$$

It turns out, however, that Maxwell's equations can be expressed in a way more closely resembling Dirac's equation.

From now on, we assume $\mathscr{M}$ to be 4-dim. and g with signature convention $(-+++)$. Let $\tau = \mathrm{d}t$ with $g^{-1}(\tau,\tau) = -1$, and define the electric and magnetic parts of $\omega \in \Gamma(\wedge^2\pi'_{T\mathscr{M}})$ respectively as ($*$ = Hodge star operator associated to g)

$$E = i_{g^\sharp(\tau)}\omega \ , \ B = -i_{g^\sharp(\tau)}\omega \ ,$$

so that

$$\omega = \tau \wedge E + *(\tau \wedge B) \ , \ *\omega = \tau \wedge B - *(\tau \wedge E) \ .$$

As for example (ii), we cannot expect to have a condition similar to (6), we need to restrict to a sub-bundle of $\wedge\pi'_{T\mathscr{M}}$ – such a restriction is sometimes called an hyperbolization of P (Geroch). In order to do so, one needs to reformulate Maxwell's equations in terms of $\mathrm{d} + \mathrm{d}^*$ instead of d^* (the Dirac-Hodge operator). It has a structure similar to that of $\not{D}$, but it is not symmetric hyperbolic, since for all $\omega \in \wedge T^*_p\mathscr{M}$, $\xi \in T^*_p\mathscr{M}$, $p \in \mathscr{M}$

$$g(\omega, \xi \wedge \omega - i_{g^\sharp(\xi)}\omega) = 0 \ .$$

It turns out, however, that Maxwell's equations can be expressed in a way more closely resembling Dirac's equation.

From now on, we assume $\mathscr{M}$ to be 4-dim. and g with signature convention $(- + ++)$. Let $\tau = \mathrm{d}t$ with $g^{-1}(\tau, \tau) = -1$, and define the electric and magnetic parts of $\omega \in \Gamma(\wedge^2\pi'_{T\mathscr{M}})$ respectively as ($*$ = Hodge star operator associated to g)

$$E = i_{g^\sharp(\tau)}\omega \ , \ B = -i_{g^\sharp(\tau)}\omega \ ,$$

so that

$$\omega = \tau \wedge E + *(\tau \wedge B) \ , \ *\omega = \tau \wedge B - *(\tau \wedge E) \ .$$

As for example (ii), we cannot expect to have a condition similar to (6), we need to restrict to a sub-bundle of $\wedge\pi'_{T\mathscr{M}}$ – such a restriction is sometimes called an hyperbolization of P (Geroch). In order to do so, one needs to reformulate Maxwell's equations in terms of $\mathrm{d}+\mathrm{d}^*$ instead of d^* (the Dirac-Hodge operator). It has a structure similar to that of $\not{D}$, but it is not symmetric hyperbolic, since for all $\omega\in\wedge T^*_p\mathscr{M}$, $\xi\in T^*_p\mathscr{M}$, $p\in\mathscr{M}$

$$g(\omega,\xi\wedge\omega-i_{g^\sharp(\xi)}\omega)=0\ .$$

It turns out, however, that Maxwell's equations can be expressed in a way more closely resembling Dirac's equation.

From now on, we assume $\mathscr{M}$ to be 4-dim. and g with signature convention $(-+++)$. Let $\tau=\mathrm{d}t$ with $g^{-1}(\tau,\tau)=-1$, and define the electric and magnetic parts of $\omega\in\Gamma(\wedge^2\pi'_{T\mathscr{M}})$ respectively as ($*$ = Hodge star operator associated to g)

$$E=i_{g^\sharp(\tau)}\omega\ ,\ B=-i_{g^\sharp(\tau)}\omega\ ,$$

so that

$$\omega=\tau\wedge E+*(\tau\wedge B)\ ,\ *\omega=\tau\wedge B-*(\tau\wedge E)\ .$$

As for example (ii), we cannot expect to have a condition similar to (6), we need to restrict to a sub-bundle of $\wedge\pi'_{T\mathscr{M}}$ – such a restriction is sometimes called an hyperbolization of P (Geroch). In order to do so, one needs to reformulate Maxwell's equations in terms of $\mathrm{d}+\mathrm{d}^*$ instead of d^* (the Dirac-Hodge operator). It has a structure similar to that of $\not{D}$, but it is not symmetric hyperbolic, since for all $\omega\in\wedge T_p^*\mathscr{M}$, $\xi\in T_p^*\mathscr{M}$, $p\in\mathscr{M}$

$$g(\omega,\xi\wedge\omega-i_{g^\sharp(\xi)}\omega)=0\ .$$

It turns out, however, that Maxwell's equations can be expressed in a way more closely resembling Dirac's equation.

From now on, we assume $\mathscr{M}$ to be 4-dim. and g with signature convention $(-+++)$. Let $\tau=\mathrm{d}t$ with $g^{-1}(\tau,\tau)=-1$, and define the electric and magnetic parts of $\omega\in\Gamma(\wedge^2\pi'_{T\mathscr{M}})$ respectively as ($*=$ Hodge star operator associated to g)

$$E=i_{g^\sharp(\tau)}\omega\ ,\ B=-i_{g^\sharp(\tau)}\omega\ ,$$

so that

$$\omega=\tau\wedge E+*(\tau\wedge B)\ ,\ *\omega=\tau\wedge B-*(\tau\wedge E)\ .$$

As for example (ii), we cannot expect to have a condition similar to (6), we need to restrict to a sub-bundle of $\wedge\pi'_{T\mathscr{M}}$ – such a restriction is sometimes called an hyperbolization of P (Geroch). In order to do so, one needs to reformulate Maxwell's equations in terms of $\mathrm{d}+\mathrm{d}^*$ instead of d^* (the Dirac-Hodge operator). It has a structure similar to that of $\not{D}$, but it is not symmetric hyperbolic, since for all $\omega\in\wedge T^*_p\mathscr{M}$, $\xi\in T^*_p\mathscr{M}$, $p\in\mathscr{M}$

$$g(\omega,\xi\wedge\omega-i_{g^\sharp(\xi)}\omega)=0\ .$$

It turns out, however, that Maxwell's equations can be expressed in a way more closely resembling Dirac's equation.

From now on, we assume $\mathscr{M}$ to be 4-dim. and g with signature convention $(-+++)$. Let $\tau=\mathrm{d}t$ with $g^{-1}(\tau,\tau)=-1$, and define the electric and magnetic parts of $\omega\in\Gamma(\wedge^2\pi'_{T\mathscr{M}})$ respectively as ($*=$ Hodge star operator associated to g)

$$E=i_{g^\sharp(\tau)}\omega\ ,\ B=-i_{g^\sharp(\tau)}\omega\ ,$$

so that

$$\omega=\tau\wedge E+*(\tau\wedge B)\ ,\ *\omega=\tau\wedge B-*(\tau\wedge E)\ .$$

Maxwell's equations (with electric current j_e and magnetic current j_m)

$$\mathrm{d}^*\omega = 4\pi j_e \ , \ \mathrm{d}^*(*\omega) = 4\pi j_m$$

can then be written as a (complex) Dirac-type equation

$$\mathrm{d}^*(\tau \wedge \psi + i * (\tau \wedge \psi)) = 4\pi(j_e + i j_m) \ ,$$

where $\psi = E + iB$. This complex form of Maxwell's equations was proposed independently by Majorana (1932) and Oppenheimer (1931) (see also Esposito (1998)).

The restriction of $\mathrm{d} + \mathrm{d}^*$ to subbundles of $\wedge(\pi'_{T\mathscr{M}})^{\mathbb{C}}$ of the form

$$\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}} - i\tau \wedge (\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}}) \ , \ k > 0 \ ,$$

endowed with the complexification of g, is symmetric hyperbolic!

Remarks:

- The above observation can be extended to $k > 1$, corresponding to Abelian k-form gauge fields.
- Likewise, back to example (i), if we rewrite $\not{D}$ as a differential complex, the corresponding operator P becomes symmetric hyperbolic if we restrict to the "diagonal" subbundle of $\pi_1 \oplus \pi_1$.

Maxwell's equations (with electric current j_e and magnetic current j_m)

$$\mathrm{d}^*\omega = 4\pi j_e \ , \ \mathrm{d}^*(*\omega) = 4\pi j_m$$

can then be written as a (complex) Dirac-type equation

$$\mathrm{d}^*(\tau \wedge \psi + i * (\tau \wedge \psi)) = 4\pi(j_e + i j_m) \ ,$$

where $\psi = E + iB$. This complex form of Maxwell's equations was proposed independently by Majorana (1932) and Oppenheimer (1931) (see also Esposito (1998)).

The restriction of $\mathrm{d} + \mathrm{d}^*$ to subbundles of $\wedge(\pi'_{T\mathscr{M}})^{\mathbb{C}}$ of the form

$$\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}} - i\tau \wedge (\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}}) \ , \ k > 0 \ ,$$

endowed with the complexification of g, is symmetric hyperbolic!

Remarks:

- The above observation can be extended to $k > 1$, corresponding to Abelian k-form gauge fields.
- Likewise, back to example (i), if we rewrite $\not{D}$ as a differential complex, the corresponding operator P becomes symmetric hyperbolic if we restrict to the "diagonal" subbundle of $\pi_1 \oplus \pi_1$.

Maxwell's equations (with electric current j_e and magnetic current j_m)

$$\mathrm{d}^*\omega = 4\pi j_e \ , \ \mathrm{d}^*(*\omega) = 4\pi j_m$$

can then be written as a (complex) Dirac-type equation

$$\mathrm{d}^*(\tau \wedge \psi + i * (\tau \wedge \psi)) = 4\pi(j_e + i j_m) \ ,$$

where $\psi = E + iB$. This complex form of Maxwell's equations was proposed independently by Majorana (1932) and Oppenheimer (1931) (see also Esposito (1998)).

The restriction of $\mathrm{d} + \mathrm{d}^*$ to subbundles of $\wedge(\pi'_{T\mathscr{M}})^{\mathbb{C}}$ of the form

$$\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}} - i\tau \wedge (\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}}) \ , \ k > 0 \ ,$$

endowed with the complexification of g, is symmetric hyperbolic!

Remarks:

- The above observation can be extended to $k > 1$, corresponding to Abelian k-form gauge fields.
- Likewise, back to example (i), if we rewrite $\not{D}$ as a differential complex, the corresponding operator P becomes symmetric hyperbolic if we restrict to the "diagonal" subbundle of $\pi_1 \oplus \pi_1$.

Maxwell's equations (with electric current j_e and magnetic current j_m)

$$\mathrm{d}^*\omega = 4\pi j_e \ , \ \mathrm{d}^*(*\omega) = 4\pi j_m$$

can then be written as a (complex) Dirac-type equation

$$\mathrm{d}^*(\tau \wedge \psi + i * (\tau \wedge \psi)) = 4\pi(j_e + ij_m) \ ,$$

where $\psi = E + iB$. This complex form of Maxwell's equations was proposed independently by Majorana (1932) and Oppenheimer (1931) (see also Esposito (1998)).

The restriction of $\mathrm{d} + \mathrm{d}^*$ to subbundles of $\wedge(\pi'_{T\mathscr{M}})^{\mathbb{C}}$ of the form

$$\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}} - i\tau \wedge (\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}}) \ , \ k > 0 \ ,$$

endowed with the complexification of g, is symmetric hyperbolic!

Remarks:

- The above observation can be extended to $k > 1$, corresponding to Abelian k-form gauge fields.
- Likewise, back to example (i), if we rewrite $\not{D}$ as a differential complex, the corresponding operator P becomes symmetric hyperbolic if we restrict to the "diagonal" subbundle of $\pi_1 \oplus \pi_1$.

Maxwell's equations (with electric current j_e and magnetic current j_m)

$$\mathrm{d}^*\omega = 4\pi j_e \ , \ \mathrm{d}^*(*\omega) = 4\pi j_m$$

can then be written as a (complex) Dirac-type equation

$$\mathrm{d}^*(\tau \wedge \psi + i * (\tau \wedge \psi)) = 4\pi(j_e + ij_m) \ ,$$

where $\psi = E + iB$. This complex form of Maxwell's equations was proposed independently by Majorana (1932) and Oppenheimer (1931) (see also Esposito (1998)).

The restriction of $\mathrm{d} + \mathrm{d}^*$ to subbundles of $\wedge(\pi'_{T\mathscr{M}})^{\mathbb{C}}$ of the form

$$\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}} - i\tau \wedge (\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}}) \ , \ k > 0 \ ,$$

endowed with the complexification of g, is symmetric hyperbolic!

Remarks:

- The above observation can be extended to $k > 1$, corresponding to Abelian k-form gauge fields.
- Likewise, back to example (i), if we rewrite $\not{D}$ as a differential complex, the corresponding operator P becomes symmetric hyperbolic if we restrict to the "diagonal" subbundle of $\pi_1 \oplus \pi_1$.

Universidade Federal do ABC

Maxwell's equations (with electric current j_e and magnetic current j_m)

$$\mathrm{d}^*\omega = 4\pi j_e \ , \ \mathrm{d}^*(*\omega) = 4\pi j_m$$

can then be written as a (complex) Dirac-type equation

$$\mathrm{d}^*(\tau \wedge \psi + i * (\tau \wedge \psi)) = 4\pi(j_e + i j_m) \ ,$$

where $\psi = E + iB$. This complex form of Maxwell's equations was proposed independently by Majorana (1932) and Oppenheimer (1931) (see also Esposito (1998)).

The restriction of $\mathrm{d} + \mathrm{d}^*$ to subbundles of $\wedge(\pi'_{T\mathscr{M}})^{\mathbb{C}}$ of the form

$$\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}} - i\tau \wedge (\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}}) \ , \ k > 0 \ ,$$

endowed with the complexification of g, is symmetric hyperbolic!

Remarks:

- The above observation can be extended to $k > 1$, corresponding to Abelian k-form gauge fields.
- Likewise, back to example (i), if we rewrite $\not{D}$ as a differential complex, the corresponding operator P becomes symmetric hyperbolic if we restrict to the "diagonal" subbundle of $\pi_1 \oplus \pi_1$.

Maxwell's equations (with electric current j_e and magnetic current j_m)

$$\mathrm{d}^*\omega = 4\pi j_e \ , \ \mathrm{d}^*(*\omega) = 4\pi j_m$$

can then be written as a (complex) Dirac-type equation

$$\mathrm{d}^*(\tau \wedge \psi + i * (\tau \wedge \psi)) = 4\pi(j_e + i j_m) \ ,$$

where $\psi = E + iB$. This complex form of Maxwell's equations was proposed independently by Majorana (1932) and Oppenheimer (1931) (see also Esposito (1998)).

The restriction of $\mathrm{d} + \mathrm{d}^*$ to subbundles of $\wedge(\pi'_{T\mathscr{M}})^{\mathbb{C}}$ of the form

$$\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}} - i\tau \wedge (\wedge^k(\pi'_{T\mathscr{M}})^{\mathbb{C}}) \ , \ k > 0 \ ,$$

endowed with the complexification of g, is symmetric hyperbolic!

Remarks:

- The above observation can be extended to $k > 1$, corresponding to Abelian k-form gauge fields.
- Likewise, back to example (i), if we rewrite $\not{D}$ as a differential complex, the corresponding operator P becomes symmetric hyperbolic if we restrict to the "diagonal" subbundle of $\pi_1 \oplus \pi_1$.

Coda

- Differential complexes provide an unified way to deal with underdetermined and overdetermined linear PDE systems. Adjunction with respect to a suitable metric yields natural gauge-fixing conditions in the underdetermined case and "gauges" constraints (à la Faddeev-Jackiw) in the overdetermined case.
- For nonlinear PDE systems, the presence of local gauge symmetries lead to differential complexes "modulo the ideal generated by the equations of motion" starting from the linearized system, and the operators in the complex modelling such symmetries usually carry a Lie bracket (i.e. a Lie-Rinehart pair structure) "modulo the ideal generated by the equations of motion", which is the algebraic incarnation of a Lie algebroid structure.
- The ambiguity entailed by the quotient "modulo the equations of motion" accounts for the possibility of "open gauge algebras". "Closed gauge algebras" should correspond to a suitable splitting in the category of Lie-Rinehart pairs.

Coda

- Differential complexes provide an unified way to deal with underdetermined and overdetermined linear PDE systems. Adjunction with respect to a suitable metric yields natural gauge-fixing conditions in the underdetermined case and "gauges" constraints (à la Faddeev-Jackiw) in the overdetermined case.
- For nonlinear PDE systems, the presence of local gauge symmetries lead to differential complexes "modulo the ideal generated by the equations of motion" starting from the linearized system, and the operators in the complex modelling such symmetries usually carry a Lie bracket (i.e. a Lie-Rinehart pair structure) "modulo the ideal generated by the equations of motion", which is the algebraic incarnation of a Lie algebroid structure.
- The ambiguity entailed by the quotient "modulo the equations of motion" accounts for the possibility of "open gauge algebras". "Closed gauge algebras" should correspond to a suitable splitting in the category of Lie-Rinehart pairs.

Coda

- Differential complexes provide an unified way to deal with underdetermined and overdetermined linear PDE systems. Adjunction with respect to a suitable metric yields natural gauge-fixing conditions in the underdetermined case and "gauges" constraints (à la Faddeev-Jackiw) in the overdetermined case.
- For nonlinear PDE systems, the presence of local gauge symmetries lead to differential complexes "modulo the ideal generated by the equations of motion" starting from the linearized system, and the operators in the complex modelling such symmetries usually carry a Lie bracket (i.e. a Lie-Rinehart pair structure) "modulo the ideal generated by the equations of motion", which is the algebraic incarnation of a Lie algebroid structure.
- The ambiguity entailed by the quotient "modulo the equations of motion" accounts for the possibility of "open gauge algebras". "Closed gauge algebras" should correspond to a suitable splitting in the category of Lie-Rinehart pairs.

- MacKichan 1975 used hyperbolicity together with a condition called "δ-estimate" as a replacement for symmetric hyperbolicity, derived from the Spencer complexes associated to the symbolic sequence of a differential complex, and proved that the Maxwell system satisfies it. However, this does not seem to be the case for the Dirac operator, due to the indefiniteness of the bundle metric with respect to which symmetric hyperbolicity of $\not{D}$ holds.
- Yang 1987 proposed a notion of involutive hyperbolic systems, which extend the notion of strictly hyperbolic PDE's and also include Maxwell's equations, but this notion does not include more general symmetric hyperbolic systems.
- Therefore, how to encode the symmetric hyperbolicity of $\not{D}$ in its associated complex? Can such a characterization (if it exists) be extended to more general complexes, so as to cover cases such as the Maxwell equations, Abelian k-form gauge fields and linearized Einstein gravity?

Thanks a lot for your attention!

Universidade Federal do ABC

- MacKichan 1975 used hyperbolicity together with a condition called "δ-estimate" as a replacement for symmetric hyperbolicity, derived from the Spencer complexes associated to the symbolic sequence of a differential complex, and proved that the Maxwell system satisfies it. However, this does not seem to be the case for the Dirac operator, due to the indefiniteness of the bundle metric with respect to which symmetric hyperbolicity of $\not{D}$ holds.
- Yang 1987 proposed a notion of involutive hyperbolic systems, which extend the notion of strictly hyperbolic PDE's and also include Maxwell's equations, but this notion does not include more general symmetric hyperbolic systems.
- Therefore, how to encode the symmetric hyperbolicity of $\not{D}$ in its associated complex? Can such a characterization (if it exists) be extended to more general complexes, so as to cover cases such as the Maxwell equations, Abelian k-form gauge fields and linearized Einstein gravity?

Thanks a lot for your attention!

Universidade Federal do ABC

- MacKichan 1975 used hyperbolicity together with a condition called "δ-estimate" as a replacement for symmetric hyperbolicity, derived from the Spencer complexes associated to the symbolic sequence of a differential complex, and proved that the Maxwell system satisfies it. However, this does not seem to be the case for the Dirac operator, due to the indefiniteness of the bundle metric with respect to which symmetric hyperbolicity of $\not{D}$ holds.
- Yang 1987 proposed a notion of involutive hyperbolic systems, which extend the notion of strictly hyperbolic PDE's and also include Maxwell's equations, but this notion does not include more general symmetric hyperbolic systems.
- Therefore, how to encode the symmetric hyperbolicity of $\not{D}$ in its associated complex? Can such a characterization (if it exists) be extended to more general complexes, so as to cover cases such as the Maxwell equations, Abelian k-form gauge fields and linearized Einstein gravity?

Thanks a lot for your attention!

- MacKichan 1975 used hyperbolicity together with a condition called "δ-estimate" as a replacement for symmetric hyperbolicity, derived from the Spencer complexes associated to the symbolic sequence of a differential complex, and proved that the Maxwell system satisfies it. However, this does not seem to be the case for the Dirac operator, due to the indefiniteness of the bundle metric with respect to which symmetric hyperbolicity of $\not{D}$ holds.
- Yang 1987 proposed a notion of involutive hyperbolic systems, which extend the notion of strictly hyperbolic PDE's and also include Maxwell's equations, but this notion does not include more general symmetric hyperbolic systems.
- Therefore, how to encode the symmetric hyperbolicity of $\not{D}$ in its associated complex? Can such a characterization (if it exists) be extended to more general complexes, so as to cover cases such as the Maxwell equations, Abelian k-form gauge fields and linearized Einstein gravity?

Thanks a lot for your attention!