Stability of thermal states in perturbative algebraic quantum field theory

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1. Stability in Quantum Statistical Mechanics

2. An interacting KMS state for pAQFT

3. Stability Properties of $\omega^{\beta,V}$

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If $\delta(A)$ is the generator of $\alpha_t(A)$, then $\delta_P(A) = \delta(A) + i[P, A]$ generates the perturbed dynamics

$$\alpha_t^P(A) \doteq \alpha_t(A) + \sum_{n\geq 1} i^n \int_{tS_n} dT \, [\alpha_{t_n}(P), [\ldots, [\alpha_{t_1}(P), \alpha_t(A)]]] \, .$$

This equation is obtained by an intertwining relation

$$\alpha_t^P(A) = U_P(t)\alpha_t(A)U_P(t)^*,$$

where $U_P(t) \in \mathcal{A}$ is a one-parameter family of unitaries which satisfy the **co-cycle relation**

$$U_P(t+s) = U_P(t)\alpha_t\left(U_P(s)\right).$$

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Remark

$U_P(t)$ can be expanded as a series expansion. In the C^*-algebraic context all the series are norm convergent.

An interacting KMS state in Quantum Statistical Mechanics

Theorem (Araki)

Let ω^β be a pure (β, α)-KMS state. Considering the perturbed dynamics α^P for a self-adjoint perturbation $P \in \mathcal{A}$ and taking an analytic extension, then

$$\omega^{\beta,P}(A) \doteq \frac{\omega^\beta(AU_P(i\beta))}{\omega^\beta(U_P(i\beta))}$$

is an extremal (β, α^P)-KMS state.

It can be shown that the Araki state can be written in term of the connected functions:

$$\omega^{\beta,P}(A) = \omega^\beta(A) + \sum_{n\geq 1}(-1)^n \int_{\beta S_n} dU\, \omega^{\beta,c}\left(A \otimes \bigotimes_{k=1}^{n} \alpha_{iu_k}(P)\right)$$

Stability Properties of the Interacting KMS state

[Haag, Kastler, Trych-Pohlmeyer - Robinson - Bratteli, Kishimoto, Robinson]

Stability/Return to Equilibrium

$$\lim_{t\to\infty} \omega^\beta(\alpha_t^P(A)) = \omega^{\beta,P}(A) \quad \text{and} \quad \lim_{t\to\infty} \omega^{\beta,P}(\alpha_t(A)) = \omega^\beta(A)$$

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Theorem

If ω^β satisfies the Strong Clustering Condition

$$\lim_{t\to\pm\infty} \omega^\beta\left(A\alpha_t(B)\right) = \omega^\beta(A)\omega^\beta(B)$$

then the stability property holds.

Sketch of the proof for the first order stability

The proof is realized by equating the two series term-by-term. At first order:

$$\omega^\beta(\alpha_t^P(A)) = \omega^\beta(A) + i\int_0^T dt\,\omega^\beta\left([\alpha_T(P), \alpha_t(A)]\right)$$

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Manipulating the first term

$$i\int_0^T dt\,\omega^\beta\left([\alpha_T(P),\alpha_t(A)]\right) = i\int_0^T dt\,\omega^\beta\left([\alpha_t(P),A]\right) =$$

$$i\int_0^T dt\,\Big(\omega^\beta(A\alpha_{t+i\beta}(P)) - \omega^\beta(A\alpha_t(P))\Big) \overset{*}{=} \int_0^\beta du\,\Big(\omega^\beta(A\alpha_{T+iu}(P)) - \omega^\beta(A\alpha_{iu}(P))\Big)$$

Where in $*$ we used the Cauchy theorem.

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Using the strongly clustering condition

$$\lim_{T\to\infty}\int_0^\beta du\, \omega^\beta\left(A\alpha_{T+iu}(P)\right) = \int_0^\beta du\, \omega^\beta(A)\omega^\beta(P).$$

The Massive Klein-Gordon Theory

Klein-Gordon massive self-interacting model on Minkowski $\mathbb{M}$ $(-,+,+,+)$

$$P\varphi + \lambda V^{(1)}(\varphi) = 0 \qquad P \doteq -\Box + m^2, \quad V(\varphi) = \int \varphi^n(x) g(x) d^4x, \; g \in C_0^\infty(\mathbb{M})$$

- Observables are realized as functionals over the **off-shell** configurations $\phi \in \mathcal{E} \doteq C^\infty(\mathbb{M}, \mathbb{R})$:

$$\mathscr{F}_{\mu c} \doteq \{F : \mathcal{E} \to \mathbb{C} \,|\, \text{smooth, compactly supported and microcausal}\}$$

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- F is smooth and compactly supported if all its functional derivatives are well defined as compactly supported distributions.

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$$\mathscr{F}_{\mu c} \doteq \{F : \mathcal{E} \to \mathbb{C} \,|\, \text{smooth, compactly supported and microcausal}\}$$

- F is Microcausal if it is smooth, compactly supported and satisfies

$$\mathrm{WF}(F^{(n)}) \cap \left(\mathbb{M}^n \times \left(\overline{V_+^n} \cup \overline{V_-^n}\right)\right) = \emptyset$$

where $(\overline{V_\pm})_x \subset T_x^*\mathbb{M}$ is the closed past/future lightcone

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- Linear Functional $F_f(\phi) \doteq \int f \phi d\mu$ for $f \in C_0^\infty(\mathbb{M}, \mathbb{C})$
- Local Functional $\mathscr{F}_{\mathrm{loc}} \subset \mathscr{F}_{\mu c}$ such that for all $n \in \mathbb{N}$

$$\mathrm{spt}\left(F^{(n)}(\phi)\right) \subseteq diag^n \quad diag^n \doteq \{(x_1, \ldots, x_n) \in \mathbb{M}^n \,|\, x_1 = x_2 = \ldots = x_n\}$$

Formal Deformation Quantization

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Definition (Hadamard bi-distribution (Radzikowski '96))

An Hadamard bi-distribution ω is the 2-point function of an Hadamard state:

- *Microcausal Wavefront Set* ▸ def
- $\omega(x, y) - \omega(y, x) = \frac{i}{2}\Delta(x, y)$
- *Solution of the equations of motion* $(\mod C^\infty)$

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$$(F \star_{\omega^\beta} G)(\phi) \doteq e^{\hbar \left\langle \omega^\beta , \frac{\delta^2}{\delta\phi\delta\phi'} \right\rangle} F(\phi) G(\phi')|_{\phi'=\phi} \quad F, G \in \mathscr{F}_{\mu c}$$

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For linear functionals $F_f(\phi) = \int \phi(x) f(x) dx$, $F_g(\phi') = \int \phi'(x) g(x) dx$

$$F_f \star_{\omega^\beta} F_g = F_f \cdot F_g + \frac{i}{2} \hbar \, \omega^\beta(f, g)$$
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Algebras obtained with different Hadamard bi-distributions are *-isomorphic

Time-Order Product

Let $T : \mathscr{F}_{\mathrm{loc}}^{\otimes^n} \to \mathscr{F}_{\mu c}$ such that (causal factorisation) [1]

$T(A, B) = T^{-1}(A) \star T^{-1}(B) \quad \text{if } A \succeq B \qquad T(A, B) = T^{-1}(B) \star T^{-1}(A) \quad \text{if } B \succeq A$

Construction of interacting field as time-ordered exponential of $F \in \mathscr{F}_{\mathrm{loc}}$ (formal power series in the coupling constant) [Renormalization ambiguities]

$$S(F) \doteq \sum_n \frac{(i\lambda)^n}{\hbar^n} T\left(T^{-1}(F)^{\otimes^n}\right)$$

[1] [Brunetti, Dütsch, Fredenhagen, Hollands, Rejzner, Wald]

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Interacting observables are represented in the free algebra via the Bogoliubov Map:

$$R_V(F) \doteq \frac{d}{d\lambda} S(V) \star S(V+\lambda F)|_{\lambda=0}$$

We call $\mathscr{F}_I \subset \mathscr{F}_{\mu c}$ the algebra generated by elements of the form $R_V(F)$

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Linear functionals are off-shell solutions of the interacting equation of motion:

$$R_V\left(F_{Pf}\right) + \lambda R_V(V^{(1)}) = F_{Pf}$$

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An interacting KMS State for pAQFT: Support properties

[Fredenhagen, Lindner '14]: time-slice axiom:

$$\mathbb{M} = \mathbb{M}_- \cup \Sigma_\varepsilon \cup \mathbb{M}_+ \quad \Sigma_\varepsilon \doteq (-\varepsilon, \varepsilon) \times \mathbb{R}^3$$

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Time-slice Axiom [Fredenhagen, Chilian]

For every $A \in \mathscr{F}_I$ there exists $B \in \mathscr{F}_I(\Sigma_\epsilon)$ such that, for every state ω

$$\omega(A) = \omega(B).$$

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Remark

By the causal factorisation property of the S-matrix, the algebras build considering two support functions g, g' are unitarily equivalent

- *if* $\mathrm{spt}(g - g') \cap J^+(\mathcal{O}) = \emptyset$, *then* $R_{V_g}(F) = R_{V_{g'}}(F)$ *for all* F
- *if* $\mathrm{spt}(g - g') \cap J^+(\mathcal{O}) \cap J^-(\mathcal{O}) = \emptyset$, *then there exists a unitary co-cycle* $W(g, g')$ *such that*

$$R_{V_g}(F) = W(g, g') \star R_{V_{g'}}(F) \star W(g, g')^{-1} \quad W(g, g') \equiv S_{V_g}(V_g - V_{g'})$$

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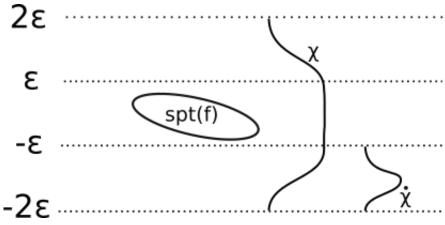
Changing the cutoff for the potential $V_g(\phi) = \int \phi^n g \, d\mu$:

$$V_g(\phi) \to V_{\chi hg}(\phi) \Rightarrow \text{ adiabatic limit } g \to 1$$

$$\begin{cases} h \in C_0^\infty(\mathbb{R}^3) \\ h(x) \equiv 1 \text{ if } x \in \mathcal{O} \subset \Sigma_\varepsilon \end{cases} \qquad \begin{cases} \mathrm{spt}(\chi) \subset \Sigma_{2\varepsilon} \\ \chi(t) = 1 \text{ if } t \in (-\varepsilon, \varepsilon) \end{cases}$$



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This allows us to construct the interacting dynamics:

$$\alpha_t^V \left(R_V(F)\right) = R_V\left(\alpha_t F\right) \quad \forall\, F \in \mathscr{F}_{\mu c}$$

An interacting KMS State for pAQFT: Interacting dynamics

In [Fredenhagen, Lindner '14] an interacting time-evolution was defined using a co-cycle (Araki construction):

$$\alpha_t^V(A) = U_h(t)\alpha_t(A)U_h(t)^{-1}$$

$$U_h(t) \doteq 1 + \sum_{n=1}^{\infty} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \, \alpha_{t_n}(K_h) \star \dots \star \alpha_{t_1}(K_h)$$

where the generator is obtained as

$$K_h \doteq R_V(\dot{V}), \quad \dot{V} \doteq \int \phi^n(x)\dot{\chi}(t)h(\mathbf{x})d^4x \quad \mathrm{spt}(\dot{\chi}) \subset (-2\varepsilon, -\varepsilon)$$

Definition of the interacting KMS state in the adiabatic limit:

$$\omega^{\beta,V}(A) = \lim_{h\to 1} \frac{\omega^\beta(A \star U_h(i\beta))}{\omega^\beta(U_h(i\beta))}, \quad A \equiv R_V(F_1) \star \dots \star R_V(F_n) \in \mathscr{F}_I$$

Clustering Condition for α_t - Spatial Compact Support

Proposition

Given $A, B \in \mathscr{F}_I(\mathcal{O})$ and an interacting potential $V_{\chi,h}$, it holds that

$$\lim_{t\to\infty} \omega^\beta \left(A \star \alpha_t(B)\right) = \omega^\beta(A)\omega^\beta(B)$$

in the sense of formal power series in the coupling constant.

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Sketch of the proof

- A, B are sums of $\star$-products of the form $R_V(F_1) \star \ldots \star R_V(F_n)$, $F_i \in \mathscr{F}_{\mathrm{loc}}$, hence their support is compact

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- $\omega^\beta(A \star \alpha_t(B)) - \omega^\beta(A)\omega^\beta(B) = \sum_{n\geq 1} \frac{1}{n!} \langle A^{(n)}, \omega_2^{\beta^n}(\alpha_t(B))^{(n)} \rangle$

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- $\omega^\beta(A \star \alpha_t(B)) - \omega^\beta(A)\omega^\beta(B) = \sum_{n\geq 1} \frac{1}{n!} \langle A^{(n)}, \omega_2^{\beta^n}(\alpha_t(B))^{(n)} \rangle$
- For sufficiently large t, there are no null geodesics connecting intersecting both the supports of $A^{(n)}$ and $(\alpha_t(B))^{(n)}$, hence we find a compact set in which $\omega_2^{\beta^n}$ is a smooth function

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in the sense of formal power series in the coupling constant.

Sketch of the proof

- A, B are sums of $\star$-products of the form $R_V(F_1) \star \ldots \star R_V(F_n)$, $F_i \in \mathscr{F}_{\mathrm{loc}}$, hence their support is compact
- $\omega^\beta(A \star \alpha_t(B)) - \omega^\beta(A)\omega^\beta(B) = \sum_{n\geq 1} \frac{1}{n!} \langle A^{(n)}, \omega_2^{\beta^n}(\alpha_t(B))^{(n)}\rangle$
- For sufficiently large t, there are no null geodesics connecting intersecting both the supports of $A^{(n)}$ and $(\alpha_t(B))^{(n)}$, hence we find a compact set in which $\omega_2^{\beta^n}$ is a smooth function
- $\left|D^{(\alpha)}\omega_2^\beta(x; y^0 + t, \mathbf{y})\right| \leq \frac{C_\alpha}{t^{3/2}}$ for a multi-index α

Return to equilibrium for $\omega^{\beta,V}$

Proposition

The clustering condition for ω^β implies the return to equilibrium in the sense of formal power series for $\omega^{\beta,V}$:

$$\lim_{T\to\infty} \omega^{\beta,V}(\alpha_T(A)) = \lim_{T\to\infty} \frac{\omega^\beta\left(\alpha_T(A) \star U_h(i\beta)\right)}{\omega^\beta\left(U_h(i\beta)\right)} = \omega^\beta(A)$$

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- Application of the clustering condition

First order stability

Theorem

Let ω^β be the pure free KMS state with respect to the evolution α_t. Then the state is stable under a spatially compact perturbation $V_{\chi,h}$, namely

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up to the first order in perturbation theory.

The proof is analogous to the C^*-algebraic case

Clustering Condition for α_t^V

Proposition

The clustering condition for α_t^V

$$\lim_{t\to\infty}\left[\omega^\beta\left(A\star\alpha_t^V(B)\right)-\omega^\beta(A)\omega^\beta\left(\alpha_t^V(B)\right)\right]=0\quad\forall A,B\in\mathscr{F}_I(\mathcal{O})$$

holds in the sense of formal power series in the coupling constant whenever the interacting potential $V_{\chi,h}$ has spatial compact support.

Clustering Condition for α_t^V - Sketch of the proof

We check that $\omega^{\beta,c}\left(A \otimes \alpha_t^V(B)\right) = \omega^{\beta}\left(A \star \alpha_t^V(B)\right) - \omega^{\beta}(A)\omega^{\beta}\left(\alpha_t^V(B)\right)$ vanishes for large or negative times t. Expanding α_t^V

$$\omega^{\beta,c}\left(A \otimes \alpha_t^V(B)\right) = \sum_{n\geq 0} i^n \int_{tS_n} dT\, \omega^{\beta,c}\left(A \otimes [\alpha_{t_1}(K_h), \ldots, [\alpha_{t_n}(K_h), \alpha_t(B)]]\right).$$

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$n=0$ is proved thanks to the clustering condition for α_t.
For generic n we notice that the integrand can be expanded as a sum over connected graphs where the lines are ω_2^{β}:

$$[A,B]_\star=m\left(e^{D_{12}}-e^{D_{21}}\right)A\otimes B$$

for $D_{ij}\doteq\langle\omega_2^{\beta},\frac{\delta^2}{\delta\phi_i\delta\phi_j}\rangle$ and $m(A_1\otimes\ldots\otimes A_n)=A_1\cdot\ldots\cdot A_n$.

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The integral can be performed and vanishes in the limit thanks to the decay properties of ω_2^β in the limit $t\to\infty$.

Stability in pAQFT - Compact spatial support

Theorem

The free KMS state ω^β is invariant under spatially compactly supported perturbations $V_{\chi,h}$, namely

$$\lim_{T\to\infty} \omega^\beta \left(\alpha_T^V(A)\right) = \omega^{\beta,V}(A) \quad \forall A \in \mathscr{F}_I(\mathcal{O})$$

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The proof is achieved with a formal computation using:

- Inverse co-cycle $\alpha_{-T}(U_h(T)) = U_h(T)^{-1}$
- Co-cycle relation $U(t+s) = U(t)\alpha_t\left(U(s)\right)$
- Clustering property for α_t^V

Stability in pAQFT - Adiabatic Limit

We studied the stability of the state

$$\omega_T^{V,+}(A) := \lim_{h\to 1} \frac{1}{T}\int_0^T \omega^\beta(\alpha_\tau^V(A))d\tau$$

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Theorem (Adiabatic Limit)

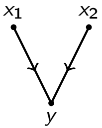
Suppose that $\left.\frac{\delta^2 V_{\chi,h}}{\delta\phi\delta\phi}\right|_{\phi=0} \neq 0$. *If the adiabatic limit is considered, the clustering condition fails at first order in perturbation theory also when the ergodic mean is considered, i.e.*

$$\lim_{T\to\infty}\lim_{h\to 1}\left(\frac{1}{T}\int_0^T dt\, \omega^\beta(A\star\alpha_t(K_h)) - \omega^\beta(A)\omega^\beta(K_h)\right) \neq 0$$

for $A = R_V(F_f)\star R_V(F_g)$ *where* F_f *and* F_g *are linear functionals.*

Stability in the adiabatic limit - Sketch of the proof

At the first order a contribution of this form appears:



which corresponds to a contribution of the form

$$\omega^{\beta}\left(A \star_{\omega^{\beta}} \alpha_t(K)\right) - \omega^{\beta}(A)\,\omega^{\beta}(K) = \lambda \int \omega_2^{\beta}(f, y)\omega_2^{\beta}(g, y)\dot{\chi}(y^0 + t)h(\mathbf{y})dy^0 d\mathbf{y} + O(\lambda^2)$$

Taking the adiabatic limit

$$\lim_{h \to 1} \int \omega_2^{\beta}(f, y)\omega_2^{\beta}(g, y)\dot{\chi}(y^0 + t)h(\mathbf{y})d\mathbf{y} + O(\lambda^2) \equiv \langle F_t, f \otimes g \rangle$$

The 2-point function for the free KMS state is

$$\omega_2^\beta(x,y) = \frac{1}{(2\pi)^3} \int d\mathbf{k} \left(b_+(k) \frac{e^{i\omega_{\mathbf{k}}(x^0 - y^0)}}{2\omega_{\mathbf{k}}} + b_-(k) \frac{e^{-i\omega_{\mathbf{k}}(x^0 - y^0)}}{2\omega_{\mathbf{k}}} \right) e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}$$

where $b_+(k) = (1 - e^{-\beta\omega_{\mathbf{k}}})^{-1}$, $b_-(k) = e^{-\beta\omega_{\mathbf{k}}} b_+(k)$ and $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$.

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$$F_t(x_1,x_2) = \frac{1}{(2\pi)^6}\int d\mathbf{y}dy^0\dot{\chi}(y^0+t)$$
$$\prod_{j=1}^{2}\int d\mathbf{k}_j\left(b_+(\mathbf{k}_j)\frac{e^{i\omega_{\mathbf{k}_j}(x_j^0-y^0)}}{2\omega_{\mathbf{k}_j}} + b_-(\mathbf{k}_j)\frac{e^{-i\omega_{\mathbf{k}_j}(x_j^0-y^0)}}{2\omega_{\mathbf{k}_j}}\right)e^{-i\mathbf{k}_j(\mathbf{x}_j-\mathbf{y})}$$

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The integral in $d\mathbf{y}$ forces $\mathbf{k}_1+\mathbf{k}_2=0$, hence in the product of the modes there is a contribution which remains unaffected by the t translation, which is

$$w(x_1,x_2) \equiv \frac{1}{(2\pi)^3}\int \frac{d\mathbf{k}}{2\omega_k^2}b_+(\mathbf{k})b_-(\mathbf{k})\cos\left(\omega_\mathbf{k}(x_1^0-x_2^0)\right)e^{i\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)}.$$

The other contributions vanish by Riemann-Lebesgue lemma.

Some infrared divergences...

The state $\omega_T^{V,+}$ is ill-defined due to some infra-red divergences.

Proposition

The contribution

$$Q_T^{(n)}(A) = \frac{1}{T}\int_0^T dt_{n+1} \cdots \int_0^{t_2} dt_1 \omega^\beta([\alpha_{t_1}(K), \ldots, [\alpha_{t_n}(K), \alpha_{t_{n+1}}(A)]] \ldots])$$

to the ergodic mean $\omega_T^{V,+}(A)$ does not converge for $n \geq 3$ in the sense of perturbation theory for large T, if the adiabatic limit is taken in advance.

A NESS for Quantum Field Theory

Definition (NESS ▸ Def)

$$\omega^+(A) := \lim_{T\to\infty} \lim_{h\to 1} \frac{1}{T}\int_0^T \omega^{\beta,V}(\alpha_\tau(A))d\tau$$

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Theorem

- *The functional ω^+, defined in the sense of formal power series, is a state for the free algebra $\mathscr{F}_{\mu c}$. Furthermore, ω^+ is invariant under the free evolution α_t.*
- *ω^+ does not satisfy the KMS condition with respect to α_t.*

Conclusions

What was done...

- Stability/return to equilibrium property for compactly-supported perturbation
- Failure of the stability/return to equilibrium in the adiabatic limit
- Definition of a non-equilibrium steady state for the massive free Klein-Gordon theory

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... and what is still to do

- Study of the properties of ω^+ (e.g. is it unique?)
- Comparison with the result of [Bros, Buchholz]
- Definition of Entropy Production and its relationship with the relative entropy ([Jakšić, Pillet])

What's Next? - Spoiler alert!

Definition

We define the relative entropy of $\omega^{\beta,V}$ *w.r.t.* ω^{β} *as*

$$S_h(\omega^{\beta,V} \,|\, \omega^{\beta}) \doteq -\beta\omega^{\beta}\left(K_h\right) + \log\omega^{\beta}\left(U_h(i\beta)\right).$$

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Explicit expression

$$S_h\left(\omega^{\beta,V}|\omega^{\beta}\right) = \sum_{n=2}^{\infty}(-1)^n \int_{\beta S_n} dU\, \omega^{\beta,c}\left(\alpha_{iu_1}\left(K_h\right)\otimes\cdots\otimes\alpha_{iu_n}\left(K_h\right)\right)$$

The Microcausal Wavefront Set

$$\mathrm{WF}(\omega) = \left\{ (x, k; x', k') \in \dot{T}^* M^2 \,|\, (x, k) \sim (x', k'), k \in \left(\overline{V_+}\right)_x \right\}$$

where $(x, k) \sim (x', k')$ means that there exists a null geodesic connecting x and x', to which k is cotangent and k' is the parallel transport of k. ▸ back

Non-Equilibrium Steady State

In the framework of perturbation theory it is possible to give a definition of a non-equilibrium steady state (NESS) [Ruelle '00]

Definition (Non-Equilibrium Steady States)

Given a stationary state ω on $(\mathcal{A}, \alpha_t)$ and a self-adjoint perturbation $P \in \mathcal{A}$, we call non-equilibrium steady states the weak* limit points of the set

$$\left\{ \frac{1}{T} \int_0^T \omega \circ \alpha_t^P dt \,\middle|\, T > 0 \right\} .$$

The set of NESS is a non-empty, compact subset of the state space whose elements are α^P-invariant.

back