A noncommutative approach to the quantisation of lattice gauge theories

based on [arXiv:1705.03815]

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Foundational and Structural Aspects of Gauge Theories

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Gauge theories: connections on principal bundles up to equivalence.

Problems with quantisation of gauge theories:

- Connections: infinitely many degrees of freedom.
- Space of connections up to gauge equivalence. Singular quotient.

Wilson (1974): approximate the manifold M with a lattice.

Kogut and Susskind (1975): Hamiltonian formulation.

Let $\Lambda = (\Lambda^0, \Lambda^1, s, t)$ be a directed graph:

- Λ^0 set of vertices;
- Λ^1 set of edges;
- source and target maps:

$$s : \Lambda^1 \to \Lambda^0, \qquad t : \Lambda^1 \to \Lambda^0.$$

Look at Λ^0 as a topological space w.r.t. the discrete topology.

Remark

Assumptions on the graph:

- At most one edge between two points;
- No single loops;
- Connected.

Let $P \to \Lambda^0$ a principal bundle with **compact** structure group G.

The base Λ^0 is discrete. $\Rightarrow$ $P \simeq G^{\Lambda^0}$

For any $e \in \Lambda^1$ we define parallel transporters:

$$\mathcal{A}_e := \{F : P_{s(e)} \to P_{t(e)} \text{ smooth} \mid F(x \cdot g) = F(x) \cdot g\}.$$

Space of connections/gauge fields:

$$\prod_{e \in \Lambda^1} \mathcal{A}_e \simeq G^{\Lambda^1} := \mathcal{K}.$$

Canonical choice of probability measure: the Haar measure μ on G^{Λ^1}.

Group of gauge transformations:

$$\mathcal{G} := \prod_{v \in \Lambda^0} P_v \times_{\mathrm{Ad}} G \simeq G^{\Lambda^0}.$$

Action of $\mathcal{G}$ on $\mathcal{K}$:

$$(g = (g_v)_{v \in \Lambda^0}, a = (a_e)_{e \in \Lambda^1}) \mapsto g \cdot a := ((g \cdot a)_e)_{e \in \Lambda^1},$$
$$(g \cdot a)_e := g_{t(e)} a_e g_{s(e)}^{-1}.$$

Extends to an action of $\mathcal{G}$ on the phase space $T^*\mathcal{K} = (T^*G)^{\Lambda^1}$.

Quantisation of the system:

$$\mathcal{H} = L^2(\mathcal{K}, \mu) = L^2(G^{\Lambda^1}, \mu_G^{\Lambda^1}), \qquad A := B_0(L^2(\mathcal{K}, \mu)).$$

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Motivation: Weyl quantisation.

Look at the operators P, Q on $L^2(\mathbb{R})$ satisfying the commutation relation

$$PQ - QP = -i\hbar.$$

Exponentiating to 1-parameter groups we get the Weyl form of the CCR's:

$$U(t)V(s) = e^{-ist}V(s)U(t).$$

All such pairs of 1-parameter groups on $L^2(\mathbb{R})$ are unitarily equivalent.

Covariant pair of representations of $\mathbb{R}$ and $C_0(\mathbb{R})$.

Theorem (Stone-Von Neumann, Mackey)

Let G be a locally compact group. Then any covariant pair of representations of G and $C_0(G)$ on a Hilbert space $\mathcal{H}$ is a multiple of the standard representation on $L^2(G)$.

For the left regular representation we have

$$C_0(G) \rtimes_\lambda G \simeq B_0(L^2(G))$$

Kijowski and Rudolph in [3] consider the algebra of observables

$$A := C(G^{\Lambda^1}) \rtimes \mathrm{G}^{\Lambda^1} \simeq B_0(\mathcal{H}).$$

This algebra can be realised as a groupoid C*-algebra.

Geometric realisation as a groupoid C*-algebra: **pair groupoid**

$$X \times X \underset{t}{\overset{s}{\rightrightarrows}} X \ .$$

$C_{\mathrm{r}}^*(X \times X)$: completion of the convolution algebra $C_c(X \times X)$ in the norm coming from the left regular representation.

Theorem (Renault)

Let X locally compact and Hausdorff with a Radon measure of full support μ.

$$C_{\mathrm{r}}^*(X \times X) \simeq B_0(L^2(X)). \tag{1}$$

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Define the quantization without the need of a group structure!

For Q compact Riemannian, the quantization of T^*Q is $B_0(L^2(Q))$.

Unitary representation U of the gauge group $\mathcal{G} = G^{\Lambda^0}$ on $\mathcal{H} = L^2(\mathcal{K})$.

$$U((g_x)_{x\in\Lambda^0})\psi((a_e)_{e\in\Lambda^1}) = \psi\left((g_{s(e)}a_e g_{t(e)}^{-1})_{e\in\Lambda^1}\right), \qquad \psi \in \mathcal{H}. \tag{2}$$

Hilbert space $\mathcal{H}^{\mathcal{G}}$ of $\mathcal{G}$-invariant vectors in $\mathcal{H}$.

Push the measure forward to $\mathcal{G}\backslash\mathcal{K}$: we obtain $\mathcal{H}^{\mathcal{G}} \simeq L^2(\mathcal{G}\backslash\mathcal{K}, \mu_G)$.

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$\pi : A \to \mathcal{B}(\mathcal{H})$ induces a representation $\tilde{\pi} : A^{\mathcal{G}} \to \mathcal{B}(\mathcal{H}^{\mathcal{G}})$.

Observable algebra of the reduced system:

$$A^{\mathcal{G}} / \ker\tilde{\pi} \simeq B_0(\mathcal{H}^{\mathcal{G}}).$$

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Open problem: Guillemin-Sternberg conjecture, i.e. quantisation commutes with reduction. (cf. Boeijnk, Landsman, Van Suijlekom).

The free Hamiltonian (Kogut-Susskind [4]) is defined as

$$H_0 = \sum_{e \in \Lambda^{\mathbf{1}}} -\frac{1}{2} l_e \Delta_e \tag{3}$$

where Δ_e is the Laplacian (Casimir) on G.

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H_0 is well-behaved with respect to the action of the gauge group:

The restriction $H_{0,\mathrm{red}}$ to $\mathrm{Dom}(H_0) \cap \mathcal{H}^{\mathcal{G}}$ is a self-adjoint operator on $\mathcal{H}^{\mathcal{G}}$.

Commutative diagram

$$\begin{tikzcd}
\mathrm{Dom}(H_0) \arrow[r, "H_0"] \arrow[d, "p_{\mathcal{H}^{\mathcal{G}}}"'] & \mathcal{H} \arrow[d, "p_{\mathcal{H}^{\mathcal{G}}}"] \\
\mathrm{Dom}(H_0) \cap \mathcal{H}^{\mathcal{G}} \arrow[r, "H_{0,\mathrm{red}}"] & \mathcal{H}^{\mathcal{G}}
\end{tikzcd}$$

Let Λ_i, Λ_j be two graphs. An inclusion $\iota : \Lambda_i \to \Lambda_j$ consists of

- An injective map $\iota^0 : \Lambda_i^0 \to \Lambda_j^0$,
- A map ι^1 that assigns to every $e \in \Lambda_i^1$ a simple path in Λ_j starting at $s(e)$ and ending at $t(e)$.

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Definition

Λ_j is a refinement of Λ_i if there exists an inclusion $\iota : \Lambda_i \to \Lambda_j$.

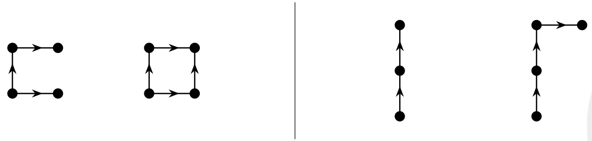
Partial order on the set of (equivalence classes of) finite oriented graphs.

Remark

More rigorously: look at the free category C_Λ generated by a graph and define refinements as functors between such categories.

Two classes :

- Λ_j is obtained from Λ_i by adding an extra edge (possibly an extra vertex):



Λ_i Λ_j Λ_i Λ_{i+1}

Without addition of a vertex. With addition of a vertex.

- Λ_j is obtained from Λ_i by subdividing an edge into two edges:

Λ_i Λ_j

Definition

We let Refine denote the category with the following properties:

- Its set of objects is the class of oriented graphs;
- Given two oriented graphs Λ_i and Λ_j, then the set of morphisms from Λ_i to Λ_j is given by the set of refinements $(\Lambda_i, \Lambda_j, \iota)$.
- Composition is given by composition of refinement functors.
- For each oriented graph Λ, there is a canonical refinement $(\Lambda, \Lambda, \mathrm{Id})$;

Aim: define functors from Refine to the categories appearing in the quantisation: $\mathsf{Top}, \mathsf{Hilb}, \mathsf{C^*Alg}, \mathsf{Grpd}$.

Any refinement $\Lambda_i \preceq \Lambda_j$ induces a map at the level of configuration spaces

$$R_{ij} : \mathcal{K}_j \to \mathcal{K}_i \quad (R_{ij}(a))_e = a_{e_1} \dots a_{e_n}$$

for $e \in \Lambda_i$ with $\iota(e) = (e_1, \dots, e_n)$.

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Equivariance condition

$$(\iota^{(0)})^*(g) \cdot R_{i,j}(a) = R_{i,j}(g \cdot a), \qquad \forall g \in \mathcal{G}_j, a \in \mathcal{K}_j$$

hence it descends to a map $R_{i,j}^{\mathrm{red}} : \mathcal{G}_j \backslash \mathcal{K}_j \to \mathcal{G}_i \backslash \mathcal{K}_i$.

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We obtain a commutative diagram:

$$\begin{tikzcd} \mathcal{K}_j \arrow[r, "R_{i,j}"] \arrow[d, "\pi_j"'] & \mathcal{K}_i \arrow[d, "\pi_i"] \\ \mathcal{G}_j \backslash \mathcal{K}_j \arrow[r, "R^{\mathrm{red}}_{i,j}"] & \mathcal{G}_i \backslash \mathcal{K}_i \end{tikzcd} \tag{4}$$

On the inverse system of Hausdorff spaces $(\mathcal{K}_i, R_{i,j})$ we have an *exact* inverse system of Radon measures for $(\mathcal{K}_i, R_{i,j})$, i.e. for $i \leq j$ we have

$$(R_{i,j})_*(\mu_j) = \mu_i.$$

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Obtain direct systems of Hilbert spaces: we have unitary maps

$$u_{ij} : R_{ij}^* : L^2(\mathcal{K}_i) \to L^2(\mathcal{K}_j) \quad u(\psi) = \psi \cdot R_{ij},$$

$$u_{ij} : (R_{ij}^{\mathrm{red}})^* : L^2(\mathcal{G}_i \backslash \mathcal{K}_i) \to L^2(\mathcal{G}_j \backslash \mathcal{K}_j) \quad u(\psi) = \psi \cdot R_{ij}^{\mathrm{red}}.$$

The isometries between the Hilbert spaces induce injective *-homomorphisms between the algebras of observables.

$$v\colon B_0(L^2(\mathcal{K}_i)) \to B_0(L^2(\mathcal{K}_j)), \qquad b \mapsto ubu^*;$$
$$v^{\mathrm{red}}\colon B_0(L^2(\mathcal{G}_i\backslash\mathcal{K}_i)) \to B_0(L^2(\mathcal{G}_j\backslash\mathcal{K}_j)), \qquad b \mapsto u^{\mathrm{red}}b(u^{\mathrm{red}})^*,$$

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The collections $(B_0(L^2(\mathcal{K}_i)), v_{i,j})$ and $(B_0(L^2(\mathcal{G}_i\backslash\mathcal{K}_i)), v^{\mathrm{red}}_{i,j})$ form direct systems of C*-algebras.

The map $R_{i,j}\colon \mathcal{K}_j \to \mathcal{K}_i$ canonically gives rise to a groupoid morphism

$$\mathsf{R}_{i,j} = \left(\mathsf{R}^{(0)}_{i,j}, \mathsf{R}^{(1)}_{i,j}\right) : \mathsf{G}_j \to \mathsf{G}_i \qquad \mathsf{R}^{(0)}_{i,j} = R_{i,j},\ \mathsf{R}^{(1)}_{i,j} = R_{i,j} \times R_{i,j}$$

(Similarly for $\mathsf{R}^{\mathrm{red}}_{i,j}\colon \mathsf{G}^{\mathrm{red}}_j \to \mathsf{G}^{\mathrm{red}}_i$.)

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The maps $\mathsf{R}_{i,j}$ induce a map $\mathsf{R}^*_{i,j}$ between the groupoid C^*-algebras (pullback).

$$\begin{tikzcd} C^*(\mathsf{G}_i) \arrow[r, "\mathsf{R}^*_{i,j}"'] \arrow[d, "\simeq"'] & C^*(\mathsf{G}_j) \arrow[d, "\simeq"] \\ B_0(L^2(\mathcal{K}_i)) \arrow[r, "v_{i,j}"'] & B_0(L^2(\mathcal{K}_j)) \end{tikzcd} \tag{5}$$

Given two graphs Λ_i, Λ_j, we consider the Hamiltonians

$$H_{0,i} = \sum_{e \in \Lambda_i^1} -\frac{1}{2} l_{i,e} \Delta_e, \quad \text{and} \quad H_{0,j} = \sum_{e \in \Lambda_j^1} -\frac{1}{2} l_{j,e} \Delta_e,$$

let $\mathcal{H}_i := L^2(\mathcal{K}_i)$, let $\mathcal{H}_j := L^2(\mathcal{K}_j)$ with maps $u \colon \mathcal{H}_i \to \mathcal{H}_j$ and $u_{\mathrm{red}} \colon \mathcal{H}_i^{\mathcal{G}_i} \to \mathcal{H}_j^{\mathcal{G}_j}$ between the corresponding Hilbert spaces.

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If we assume that for each $e \in \Lambda_i^1$, we have

$$l_{i,e} = \sum_{k=1}^{n} l_{j,e_k}, \tag{6}$$

we have compatibility between Hamiltonians at different levels.

We can take the projective limit on the space of connections

$$\mathcal{K}_\infty := \varprojlim_{i \in I} \mathcal{K}_i, \qquad R_{i,\infty} : \mathcal{K}_\infty \to \mathcal{K}_i.$$

Note that there is no group structure on $\mathcal{K}_\infty$!

There exists a Radon measure μ_∞ on $\mathcal{K}_\infty$ such that $R_{i,\infty}(\mu_\infty) = \mu_i$ (Prokhorov's theorem).

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Direct limit Hilbert space:

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Direct limit Hilbert space:

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For the direct limit of the observable algebras

$$\varinjlim_{i\in I} A_i \simeq B_0(\mathcal{H}_\infty).$$

The limit groupoid G_∞ is also a pair groupoid and is given by

$$G_\infty = \mathcal{K}_\infty \times \mathcal{K}_\infty .$$

It is by definition a free and transitive groupoid.

The groupoid C*-algebra $C^*(\mathcal{K}_\infty \times \mathcal{K}_\infty)$ is isomorphic to the limit observable algebra:

$$C^*(G_\infty) = C^*(\varprojlim_{i\in I} G_i) \simeq B_0(L^2(\mathcal{K}_\infty, \mu)) \simeq \varinjlim_{i\in I} B_0(L^2(\mathcal{K}_i, \mu_i)) \simeq \varinjlim_{i\in I} C^*(G_i),$$

By equivariance can consder $\mathcal{K}_\infty^{\mathrm{red}}$. Then there exists a Radon measure $\mu_\infty^{\mathrm{red}}$ on $\mathcal{K}_\infty^{\mathrm{red}}$ such that $R_{i,\infty}^{\mathrm{red}}(\mu_\infty^{\mathrm{red}}) = \mu_i^{\mathrm{red}}$.

Next, we can consider the space of square integrable functions on $\mathcal{K}_\infty^{\mathrm{red}}$ with respect to the Radon measure $\mu_\infty^{\mathrm{red}}$ and get

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The algebra of observables satisfies

$$\varinjlim_{i \in I} B_0(\mathcal{H}_i) \simeq B_0(\mathcal{H}_\infty^{\mathrm{red}}).$$

and this is again a groupoid C*-algebra

$$C^*(\mathrm{G}_\infty^{\mathrm{red}}) \simeq B_0(L^2(\mathcal{K}_\infty^{\mathrm{red}}, \mu_\infty^{\mathrm{red}})).$$

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- Goal: incorporate the dynamics:
  - study of the Hamiltonian in the limit;
  - infinitely many degrees of freedom: need to enlarge the algebra of observables.

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