Higher structures in Dijkgraaf-Witten theories

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based on work with
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Overview

Part 1

Dijkgraaf-Witten theories as a gauge theory

G a finite group, M a closed oriented 3-manifold.
Configuration space: $\mathrm{Bun}_G(M)$, a finite groupoid. "Path integral":

$$\mathrm{tft}_G(M) = \int_{\mathrm{Bun}_G(M)} e^0 = |\mathrm{Bun}_G(M)|$$

A 3d TFT from a gauge theory

Definition

Groupoid cardinality is the unique function assigning to a finite groupoid X a rational number $|X|$ such that:

- (Normalization): $|\mathrm{pt}| = 1$, where $\mathrm{pt} = *//\mathrm{id}_*$
- (Homotopy invariance): $X \cong Y$ implies $|X| = |Y|$.
- (Gluing): $|X \sqcup Y| = |X| + |Y|$.
- (Covering): If $F : X \to Y$ is an n-sheeted covering map, then $|X| = n|Y|$. Covering of groupoids is a functor $F : X \to Y$, surjective on objects and satisfying the unique path lifting property.
  If $p : y_1 \to y_2$ is a morphism in Y and $x_1 \in X$ such that $F(x_1) = y_1$, then $\exists!\ p' : x_1 \to x_2$ in X such that $F(p') = p$.
  A covering map is n-sheeted, if the preimage of every object in Y consists of n objects in X.

Remark

Homotopy invariance and gluing axiom $\Rightarrow$ groupoid cardinality is completely determined by $|BG| = |*//G|$, where G is a finite group.
$EG := G\backslash\backslash G$ the action groupoid for the left regular action. Then $EG \to BG$ sending $g \xrightarrow{hg^{-1}} h$ to the element $hg^{-1} \in G$ is a $|G|$-sheeted covering. On the other hand, $EG \cong pt$, since the left action is transitive and free. Thus, $|EG| = 1$ by the normalization axiom; the covering axiom then implies that $|BG| = \frac{1}{|G|}$.

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Definition

Let Γ be an essentially finite groupoid, i.e. assume that Γ is equivalent to a groupoid with only finitely many morphisms. The groupoid cardinality of the groupoid Γ is the rational number

$$|\Gamma| := \sum_{[g]\in\pi_0(\Gamma)} \frac{1}{|Aut(g)|}.$$

Properties of groupoid cardinality

Remarks

1. $|X \times Y| = |X| \cdot |Y|$.
   Proof: $\frac{|- \times Y|}{|Y|}$ satisfies the axioms of groupoid cardinality.
2. Let S be a finite G-set and $S//G$ the action groupoid. Then $|S//G| = \frac{|S|}{|G|}$.
   Proof: by the gluing axiom, restrict to transitive actions and use the orbit-stabilizer theorem.

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Multiplicative invariant for closed three-manifolds: $\mathrm{tft}_G(M) := |\mathrm{Bun}_G(M)|$

Remark

If M is connected and $m \in M$, replace $\mathrm{Bun}_G(M)$ by the action groupoid $\mathrm{Hom}(\pi_1(M,m),G)//G$ with G acting by conjugation. Then

$$\left|\mathrm{Bun}_G(M)\right| = \left|\mathrm{Hom}(\pi_1(M),G)\right| \, / \, |G| \, .$$

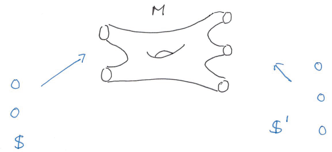
expresses $\mathrm{tft}_G(M)$ in terms of the fundamental group of M.

The statement that $\mathrm{tft}_G(M)$ is an invariant of 3-manifolds is not a deep statement. The relevant statement is locality of the invariant.

Definition of topological field theories

Definition (Cobordism category)

- Objects of $\mathrm{cob}_{n,n-1}$: closed oriented $(n-1)$-manifolds.
- Morphisms: spans $\mathbb{S} \to M \leftarrow \mathbb{S}'$ with M oriented n-manifold with boundary $\partial M \cong \overline{\mathbb{S}} \sqcup \mathbb{S}'$, up to diffeomorphism relative boundary.
- Monoidal product is disjoint union.



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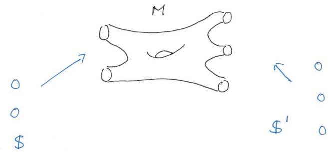

Definition (Topological field theory)

An (oriented) topological field theory is a symmetric monoidal functor $\mathrm{tft} : cob^{or}_{n,n-1} \to \mathrm{vect}$. (It is a *representation* of $cob^{or}_{n,n-1}$.)

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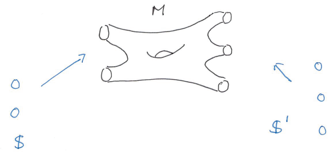

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Remark

Topological fact: $cob^{or}_{2,1}$ is the free symmetric monoidal category on a commutative Frobenius object.
Corollary: $\mathrm{tft}(\mathbb{S}^1)$ is a commutative Frobenius algebra. Any such Frobenius algebra gives a two-dimensional TFT.

Universal construction [BHMV] =GNS construction

General construction of a TFT from an invariant of closed manifolds.
Given $\mathcal{C}$ a category and $1 \in \mathcal{C}$ an object. Two functors:

$$\mathcal{F}_1 : \quad \mathcal{C} \xrightarrow{\mathrm{Hom}(1,-)} \mathit{Set} \xrightarrow{\text{free}} \mathrm{vect}_k \quad \text{and} \quad \mathcal{F}_1^{co} : \quad \mathcal{C} \xrightarrow{\mathrm{Hom}(-,1)} \mathit{Set} \xrightarrow{\text{free}} \mathrm{vect}_k$$

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Invariant=State: $I : \ \mathrm{End}_{\mathcal{C}}(1) \to K$ gives pairing

$$\mathcal{F}_1(c) \otimes \mathcal{F}_1^{co}(c) \to \mathrm{vect} \qquad \text{defined on a basis} \quad \delta_f \otimes \delta_g \mapsto I(g \circ f)$$

GNS functors

$$\mathcal{F}_{1,I} = \mathcal{F}_1/\mathrm{lRad} \quad \text{and } \mathcal{F}_{1,I}^{co} = \mathcal{F}_1/\mathrm{rRad}$$

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Examples: one-object category $\mathcal{C} = *//A$ with algebra A;
$\mathcal{C} = \mathrm{cob}_{n,n-1}$ and $I = \emptyset$ invariant of closed manifolds

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$\mathcal{C} = \mathrm{cob}_{n,n-1}$ and $I = \emptyset$ invariant of closed manifolds

Theorem (L. Müller, CS 2017)

1. *If $\mathcal{C}$ is monoidal and $I : \mathrm{End}_{\mathcal{C}}(1) \to K$ is a morphism of monoids, then the GNS functor $\mathcal{F}_{1,I}$ is lax monoidal.*
2. *If $\mathcal{F}_{1,I}$ is monoidal, then it is symmetric.*
3. *A 3-dimensional (abelian) DW theory can be constructed from a symmetric monoidal category $\mathrm{cob}_{3,2}$ with Wilson lines.*

Vector spaces for surface, alternative geometric construction

Σ closed oriented 2-manifold.
For any 3-manifold M with $\partial M = \Sigma$ and $P \in \mathrm{Bun}_G(\Sigma)$, we can consider

$$\psi_M : \quad P \mapsto |\mathrm{Bun}_G(M, P)|$$

This is a gauge-invariant function on space of field configurations $\mathrm{Bun}_G(\Sigma)$
Thus assign to Σ the space of such functions:

$$\mathrm{tft}_G(\Sigma) = \mathbb{C}[\pi_0(\mathrm{Bun}_G(\Sigma))]$$

Remark

In DW theories, vector spaces are obtained by linearization from categories of bundles

Dijkgraaf-Witten theories as an extended TFT

Implement more locality by pair-of-pants decomposition: cut surfaces along circles.
S a closed oriented 1-manifold.
For any surface Σ with $\partial\Sigma = S$ and consider

$$\psi_\Sigma : \quad \mathrm{Bun}_G(S) \to \mathrm{vect} \quad \text{with} \quad P \mapsto \mathbb{C}[\pi_0(\mathrm{Bun}_G(\Sigma, P)]$$

This is a vector bundle over the space $\mathrm{Bun}_G(S)$ of field configurations on S.
Thus assign to S the category of such bundles:

$$\mathrm{tft}_G(S) = [\mathrm{Bun}_G(S), \mathrm{vect}]$$

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Remarks

1. An extended 3d TFT assigns assigns a $\mathbb{C}$-linear category to a 1-manifold.
2. In DW theories, the category is obtained by a 2-step procedure:

$$S \mapsto \mathrm{Bun}_G(S) \mapsto [\mathrm{Bun}_G(S), \mathrm{vect}]$$

   Assign field configurations, followed by linearization.
3. In general: $\mathrm{tft}_G : \quad \mathrm{cob}_{3,2,1} \xrightarrow{\widetilde{\mathrm{Bun}_G}} \mathrm{SpanGrp} \xrightarrow{[-,\mathrm{vect}]} 2-\mathrm{vect}$

Extended three-dimensional topological field theories

Definition (2-vector spaces)

Denote by 2-vect the symmetric monoidal bicategory

- Objects: finitely semisimple k-linear abelian categories.
- 1-Morphisms: k-linear functors, 2-morphisms: k-linear natural transformations.
- The monoidal product is given by the Deligne product.

Definition (Cobordism bicategory)

- Objects: closed oriented 1-manifolds.
- 1-Morphisms: spans $\mathbb{S} \to M \leftarrow \mathbb{S}'$ with M oriented 2-manifold with boundary $\partial M \cong \overline{\mathbb{S}} \sqcup \mathbb{S}'$.
- 2-Morphisms: 3-manifolds with corners up to diffeomorphisms
- The monoidal product is given by disjoint union.

Definition (Extended topological field theory)

A 3-2-1 extended oriented topological field theory is a symmetric monoidal 2-functor $\mathrm{tft} : cob^{or}_{3,2,1} \to 2\text{-vect}$.

Evaluation of a 3-2-1 TFT

Definition (Extended topological field theory)

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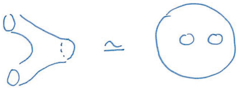
Examples:

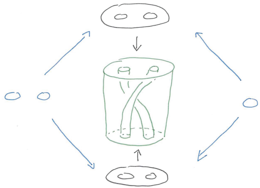
1-morphism



gives tensor product

$\otimes : \mathrm{tft}(\mathbb{S}^1) \times \mathrm{tft}(\mathbb{S}^1) \longrightarrow \mathrm{tft}(\mathbb{S}^1)$

2-morphism



gives braiding

$\otimes \Rightarrow \otimes^{opp}$

$\Rightarrow$ Modular category $\mathrm{tft}(\mathbb{S}^1)$

For Dijkgraaf-Witten theories: $[G//G, \mathrm{vect}] = \mathcal{D}(G) - \mathrm{mod}$

Part 2

Equivariant Dijkgraaf-Witten theories from twisted bundles

Observation

- Bantay constructed in 2005 a premodular category $\mathcal{B}(G \lhd H)$ from a normal subgroup $G \subset H$ (indeed, for a crossed module of groups).
- $\mathcal{B}(G \lhd H)$ is modularizable; Its modularization turns out to be the Drinfeld double $[G//G, \mathrm{vect}] = \mathcal{D}(G) - \mathrm{mod}$.
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- This category inherits an action of $J := H/G$.

Question from representation theory:
Is the modularization the neutral component of a J-equivariant modular category?

Definition

An action of a group J on a category $\mathcal{C}$ is a monoidal functor $\underline{J} \longrightarrow \mathcal{C}$.
A weak action of a group J on a group G is a monoidal functor $\underline{J} \longrightarrow *//G$.

Theorem (Dedecker 1960, Schreier 1926)

There is a bijection between weak J-actions on G and group extensions

$$1 \longrightarrow G \longrightarrow H \xrightarrow{\pi} J \longrightarrow 1$$

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Definition

Let J be a finite group.

1. Denote by $\mathrm{cob}^J_{3,2,1}$ a cobordism bicategory where all manifolds are endowed with a J-cover.
2. An extended 3d J-equivariant TFT is a symmetric monoidal 2-fuctor

$$\mathrm{tft}^J : \quad \mathrm{cob}^J_{3,2,1} \to 2-\mathrm{vect}$$

Goal: construct J a equivariant TFT from a weak J-action on G.

Idea: use a different stack: J-twisted G-bundles for $1 \to G \to H \xrightarrow{\pi} J \to 1$

Definition

Let J act weakly on G. Let $P \xrightarrow{J} M$ be a J-cover over M.

- A P-twisted G-bundle over M is a pair (Q, φ), consisting of an H-bundle Q over M and a smooth map $\varphi : Q \to P$ over M that is required to obey

$$\varphi(q \cdot h) = \varphi(q) \cdot \pi(h)$$

  for all $q \in Q$ and $h \in H$.
- A morphism of P-twisted bundles (Q, φ) and (Q', φ') is a morphism $f : Q \to Q'$ of H-bundles such that $\varphi' \circ f = \varphi$.
- We denote the category of P-twisted G-bundles by $\mathrm{Bun}_G(P \to M)$.

Remarks

- Twisted G-bundles can be pulled back along maps $f : M \to N$

$$f^* : \mathrm{Bun}_G(P \to N) \to \mathrm{Bun}_G(f^* P \to M)$$

- $\mathrm{Bun}_G(M \times J \to M) \cong \mathrm{Bun}_G(M)$.

Theorem (Maier, Nikolaus, S, 2012)

$1 \to G \to H \xrightarrow{\pi} J \to 1$

1. *J-twisted G-bundles give a symmetric monoidal 2-functor*

$$\widetilde{\mathrm{Bun}}_G : \quad \mathrm{cob}^J_{3,2,1} \to \mathrm{SpanGrp}$$

2. *Upon linearization, this functor gives a J-equivariant TFT tft^J_G.*

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Remarks

- We have $\mathrm{tft}^J_G(\mathbb{S}^1_j) = [H_j// \ G, \mathrm{vect}]$ with $H_j := \pi^{-1}(j)$.
- The category $[H//G, \mathrm{vect}]$ carries the structure of a J-equivariant modular category.
- $$\begin{tikzcd} {[G//G,\mathrm{vect}]} \arrow[loop above, "J"] \arrow[r, hook] \arrow[d, shift left, "\text{orbifold}"] & {[H//G,\mathrm{vect}]} \arrow[loop above, "J"] \arrow[d, shift left, "\text{orbifold}"] \quad \text{mod} \\ \mathcal{B}(G \triangleleft H) \quad \text{mod} \arrow[u, shift left, "\text{modularization}"] \arrow[r, hook] & {[H//H,\mathrm{vect}]} \quad \text{mod} \arrow[u, shift left] \end{tikzcd}$$

Theorem (S., Woike 2017)

1. *Reformulate a J-equivariant TFT as*

$$Z: \quad \mathrm{cob}_{3,2} \longrightarrow \mathrm{RepGrpd}_{\mathbb{C}}$$

2. *Let* Γ *and* Ω *be additive, essentially finite homotopy invariant presheaves satisfying a gluing condition and* $\phi : \Gamma \longrightarrow \Omega$ *a morphism of presheaves. Then the associated symmetric monoidal functor*

$$Z_\phi : \mathrm{cob}_{3,2} \longrightarrow \mathrm{RepGrpd}_{\mathbb{C}}$$

is an Ω*-equivariant topological field theory.*

Compute orbifold theory (at the level of invariants of 3-manifolds):

$$\frac{Z_\lambda}{J}(M) = \int_{\mathrm{Bun}_J(M)} Z_\lambda(M,P)\, dP = \int_{\mathrm{Bun}_J(M)} |\pi_*^{-1}[P]|\, dP = |\mathrm{Bun}_H(M)|,$$

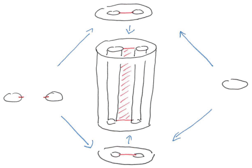
where Cavalieri's principle for groupoid cardinality enters in the last step.

Part 3: Topological field theories with defects

Dijkgraaf-Witten theories with defects from relative bundles

Consider a larger symmetric monoidal bicategory $cob_{3,2,1}^{\partial,or}$:

- Objects: 1-manifolds with marked points
- 1-Morphisms: 2-manifolds with boundary
- 2-morphisms: 3-manifolds with corner



Definition (TFT with defects)

An (oriented) 3-2-1 TFT with defects is a symmetric monoidal functor $\mathrm{tft} : cob_{3,2,1}^{\partial,or} \to \text{2-vect}$.

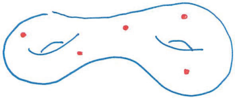
Goal: construct Dijkgraaf-Witten theories with defects

Two applications of TFT with defects: quantum codes and CFT

1. Some problems with quantum codes:
   - Realistic samples have simple topology (disc).
     Dimension of distinguished subspace sensitive to genus (Verlinde formula).
   - Relevant representations of mapping class groups too small to admit no universal quantum gates.

Possible way out: two-layer systems with twist defects
Effectively conformal blocks at higher genus

- Increased dimension
- Representations of braid groups admitting universal gates



• twist defect (e.g. from lattice dislocation)

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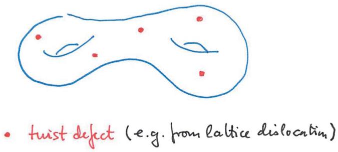
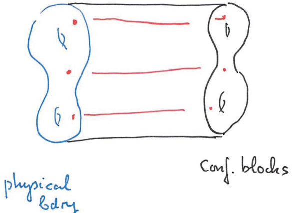


2. Holographic construction of CFT correlators
(including ultimately non-ssi CFTs)



Defects and boundaries in Dijkgraaf-Witten theories

Idea: keep the same 2-step procedure,

$$\mathrm{tft}_G : \quad \mathrm{cob}_{3,2,1} \xrightarrow{\widetilde{\mathrm{Bun}}} \mathrm{SpanGrp} \longrightarrow 2-\mathrm{vect}$$

but allow for more general bundles as field configurations.

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Definition

Given a relative manifold $j : Y \to X$ and a group homomorphism $i : H \to G$,

$$\mathrm{Bun}_{H\to G}(Y \to X) := \left\{ \begin{array}{c} P_G \\ \downarrow \\ X \end{array} , \begin{array}{c} P_H \\ \downarrow \\ Y \end{array} , \alpha : \mathrm{Ind}_H^G P_H \xrightarrow{\sim} j^* P_G \right\}$$

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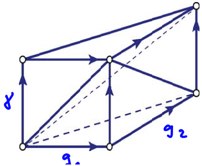
Topological Langrangian
$\omega \in Z^3(G, \mathbb{C}^*)$
Transgress to $\tau(\omega) \in Z^2(G//G, \mathbb{C}^*)$
to get twisted linearization of
$\mathrm{Bun}_G(\mathbb{S}^1) \cong G//G$



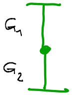
$\tau(\omega)(\gamma; g_1, g_2)$

Categories for 1-manifolds

(Fuchs, S., Valentino, 2014)

2.6 Categories from 1-manifolds

Example: Interval



$\mathcal{I}$

$i_1 : H_1 \to G_1$

$H_{12} \to G_1 \times G_2$

$i_2 : H_2 \to G_2$

$$\mathrm{Bun}(\mathcal{I}) \cong {}_{G_1 \times G_2}\backslash\!\backslash \, G_1 \times G_1 \times G_2 \times G_2 \, /\!/_{H_1 \times H_{12} \times H_2}$$

Transgress to 2-cocycle on $\mathrm{Bun}(\mathcal{I})$

(for twisted linearization)

Data:

$\omega_a \in Z^3(G_a, \mathbb{C}^\times)$ bulk Lagrangian

$\theta_a \in C^2(H_a, \mathbb{C}^\times)$ bdry Lagrangian

$d\theta_a = i^* \omega_a$

$\theta_{12} \in C^2(H_{12}, \mathbb{C}^\times)$

$d\theta_{12} = \omega_1 (\omega_2)^{-1}$

Check:

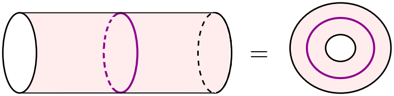
Module category $\mathcal{M}(H, \theta)$ over $(G\text{-vect})^\omega$

$$\#\left(\mathcal{I}_{H_1}^{G}{}^{H_2} \right) = \mathrm{Fun}_{(G\text{-vect})^\omega}\left(\mathcal{M}(H_1, \theta_1), \mathcal{M}(H_2, \theta_2) \right)$$

This "explains" representation theoretic results:
classification of module categories, cf. [O]

Functors for surfaces: the transmission functor

Extended TFT: surface with boundaries, $\rightsquigarrow$ left exact k-linear functor.
Special case: to cylinder with a circular defect $\mathcal{D}$



associate the transmission functor: for $\mathcal{D}$ $\mathcal{A}$-bimodule category get functor

$$F_{\mathcal{D}}: \quad \mathcal{Z}(\mathcal{A}) \to \mathcal{Z}(\mathcal{A})$$

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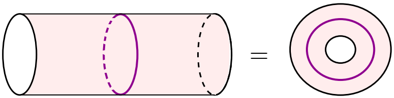
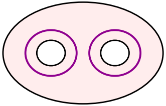


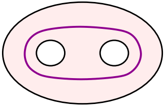
associate the transmission functor: for $\mathcal{D}$ $\mathcal{A}$-bimodule category get functor

$$F_{\mathcal{D}}: \quad \mathcal{Z}(\mathcal{A}) \to \mathcal{Z}(\mathcal{A})$$

General principle of field theory:
Invertible codimension 1 defects describe symmetries.
3d TFT of TV type: Brauer-Picard group describes symmetries
TFT axioms: if $\mathcal{D}$ is invertible, then $F_{\mathcal{D}}$ is braided. Compare two functors



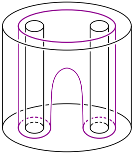
$\otimes \circ (F_{\mathcal{D}} \times F_{\mathcal{D}})$



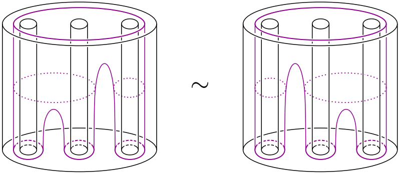
$F_{\mathcal{D}} \circ \otimes$

The transmission functor for invertible defects is braided

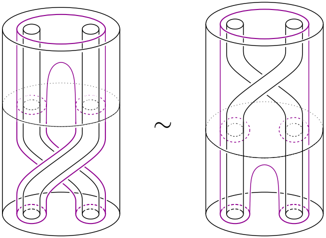
Natural isomorphism
$\otimes \circ (F_{\mathcal{D}} \times F_{\mathcal{D}}) \Rightarrow F_{\mathcal{D}} \circ \otimes$



is monoidal



and braided



Explicit calculation for Dijkgraaf-Witten theories: $\mathcal{D} \mapsto F_{\mathcal{D}}$ is the map in the description of the Brauer-Picard group

(Fuchs, Priel, S., Valentino, 2015)

2.8 Symmetries for abelian DW

Special case: : $G = A$ abelian , $\omega = 1$

$$\mathrm{BrPic}(A\text{-vect}) = O_q(A \oplus A^*)$$

with $q(g, \chi) = \chi(g)$

quadratic form

Obvious symmetries:

1) Symmetries of Bun_A

$$\varphi \in \mathrm{Aut}(\mathrm{Bun}_A) = \mathrm{Aut}(A)$$

Subgroup:

$$H_\varphi = \mathrm{graph}\,\varphi \subset A \oplus A, \quad \Theta = 1$$

Braided equivalence:

$$\varphi \oplus (\varphi^*)^{-1} : A \oplus A^* \to A \oplus A^*$$

2) Automorphisms of CS 2-gerbe

1-gerbe on Bun_A "B-field"

$$H^2(A, \mathbb{C}^*) \xrightarrow{\sim} AB(A, \mathbb{C}^*) \ni \beta$$

(transgression)

Subgroup: $A_{diag} \subset A \oplus A \quad \Theta = \beta$

Braided equivalence:

$$A \oplus A^* \to A \oplus A^*$$
$$(g, \chi) \mapsto (g, \chi + \beta(g, -))$$

3) Partial e-m dualities:

Example: A cyclic, fix $\delta : A \xrightarrow{\simeq} A^*$

Braided equivalence:

$$A \oplus A^* \longrightarrow A \oplus A^*$$
$$(g, \chi) \mapsto (\delta^{-1}\chi, \delta g)$$

Subgroup: $A_{diag} \subset A \oplus A$

$$\beta(a_1, a_2) = \frac{\delta(a_1)(a_2)}{\delta(a_2)(a_1)} \in AB(A, \mathbb{C}^*)$$

Theorem [FPSV]

These symmetries form a set of generators for $\mathrm{BrPic}(A\text{-vect})$.

Outlook

Dijkgraaf-Witten Gauge theory	$\subset$	Turaev-Viro TFT state sum model	$\subset$	Reshetikhin-Turaev TFT surgery

- Fact:
  Topological boundary conditions only exist in Turaev-Viro theories
  $\Longrightarrow$ natural framework for holographic constructions of 2d CFTs
- Goal:
  $\mathrm{tft} : \mathit{cob}^{\partial,or}_{2+\epsilon,2,1} \to 2\text{-vect}$ based on finite tensor categories (not necessarily semisimple)
- Applications:
  - Holographic construction of logarithmic conformal field theories with dualities
  - TFT understanding of "'categorified representation theory"
  - TFT understanding of other facts in representation theory, e.g. $\mathrm{SL}_2(\mathbb{Z})$-equivariant Frobenius-Schur indicators.