Scattering of Atoms and Non-Locality of the Vacuum in QED

(based on CMP **375**, 2017)

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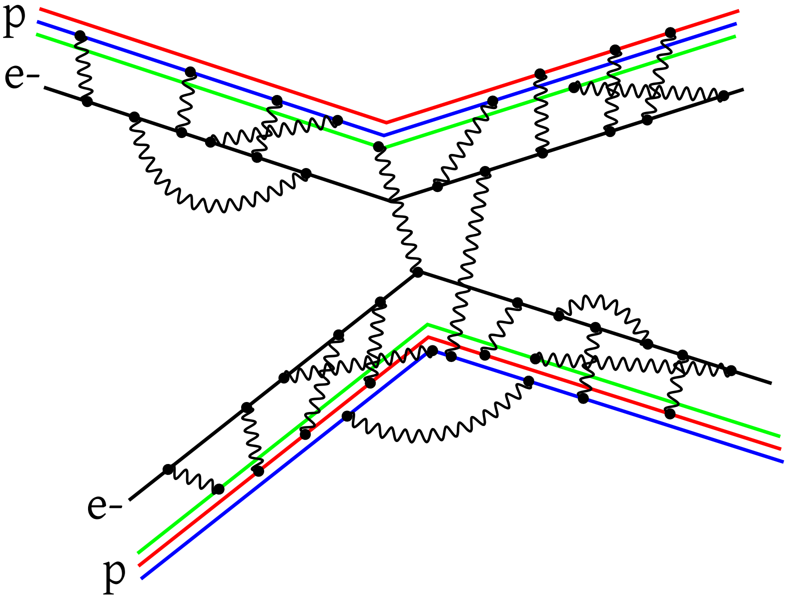
(joint work with Wojciech Dybalski)

Zentrum Mathematik
Technische Universität München

Foundational and Structural Aspects of Gauge Theories
May 31st @ Mainz Institute for Theoretical Physics,
Johannes Gutenberg Universität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Interacting Quantum Field Theory

Interacting Quantum Field Theory, Non-Perturbatively

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Exercise 1 Quantum Mechanics:

(a) Find $\mathscr{H}$, Hamiltonian H_0 and Observables for free particles

(b) Born probability interpretation $|\Psi(x)|^2$

(c) Add interaction $H := H_0 + H_{\text{int}}$

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(h) ω defines new Hilbert space $\mathscr{H}$ on which interact. model lives
(change of rep.), and where $H = \lim\limits_{R\to\infty} H^R$ is well-defined.

The Particle Spectrum

Vacuum $\Omega \in \mathscr{H}$ translation invariant, Space-time translations α_x unitarily implemented

$$\mathscr{H} \ni \Psi \longmapsto U(t, \mathbf{x})\Psi$$

SNAG-Theorem → strongly commut. self-adjoint generators $(H, \boldsymbol{P})$
$\hat{=}$ energy-momentum op.

Spectral Resolution of $(H, \boldsymbol{P})$ by POVM $E(\Delta)$ for Borel $\Delta \subset \mathbb{R}^4$.

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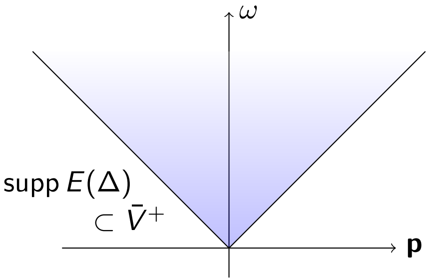

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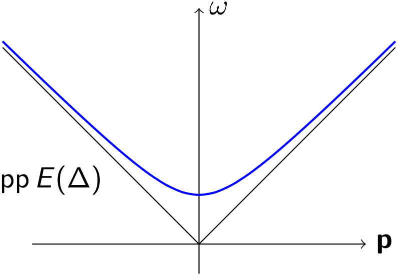

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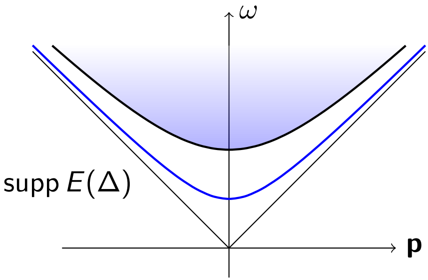

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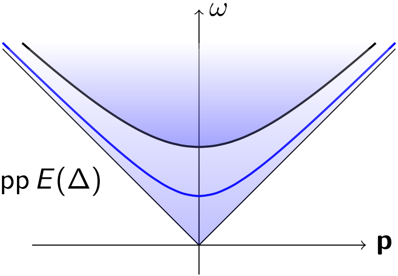

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- Haag '54, Lehmann-Symanzik-Zimmermann '54
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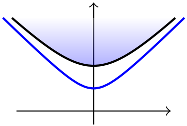

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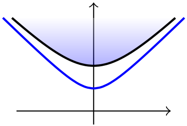

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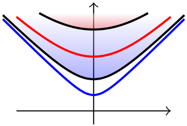
  i.e. need local operator $A \in \mathfrak{A}(\mathcal{O})$ s.t. $A\Omega$ has "nicely behaved" spectrum near mass shell



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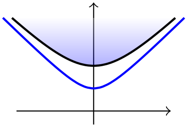

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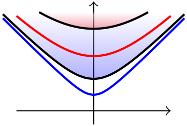

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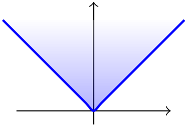

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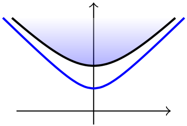

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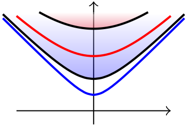

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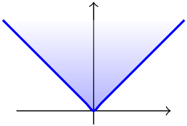
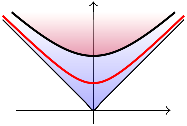


- Dybalski '05 — (SC) + non-isolated vacuum
- Duch, Herdegen '13 — (SC) weakened, $m \geq 0$



Remarks: Other Aspects of the Infrared Problem

Charges, Particles and Infraparticles in AQFT

Our present assumptions restrict us to **neutral**-particle states Ψ_1.

(Electrical) **Charges** are expected to have

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Current Research and Tentative Approaches

- Scattering of Infraparticles? [Buchholz et al.'91–] [Herdegen'13]
- Space-like asymptotics of $F^{\mu\nu}$ experimentally not accessible, suitable Infravacuum-states conjectured to "stabilize" infraparticles [Kraus, Polley, Reents'77] [Buchholz, Roberts'13]

  $\rightarrow$ Feasible to describe Compton-scattering [Alazzawi, Dybalski'15]
- Perturbation Theory with String-local Quantum Fields [Schroer et al.'04–] [Mund, de Oliveira'16]
- Study infrared problem in more tractable non-relativistic models [Fröhlich'73] [Chen, Fröhlich, Pizzo'07]... [Dybalski, Pizzo'12–]

Non-Locality of the Vacuum: Reeh-Schlieder Property

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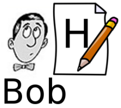

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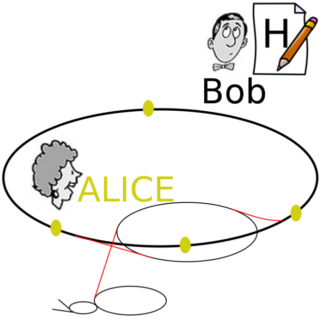

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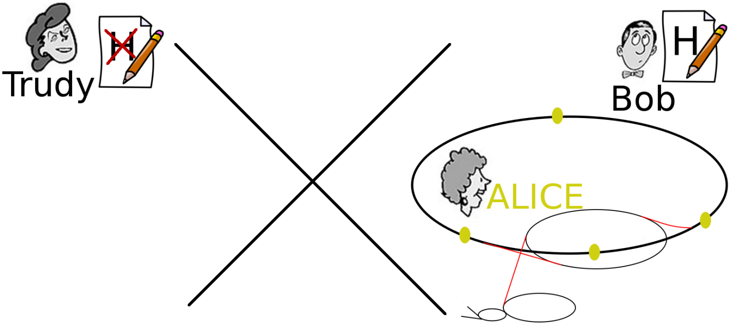

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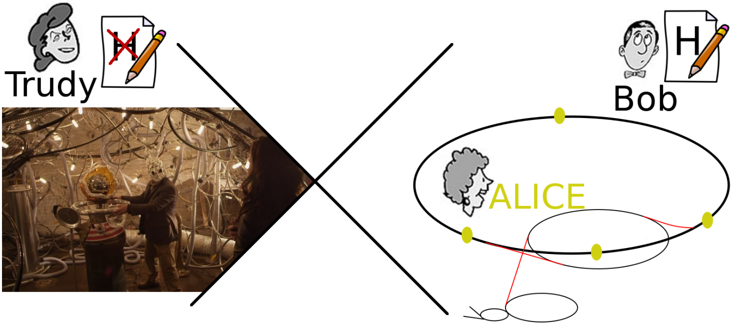

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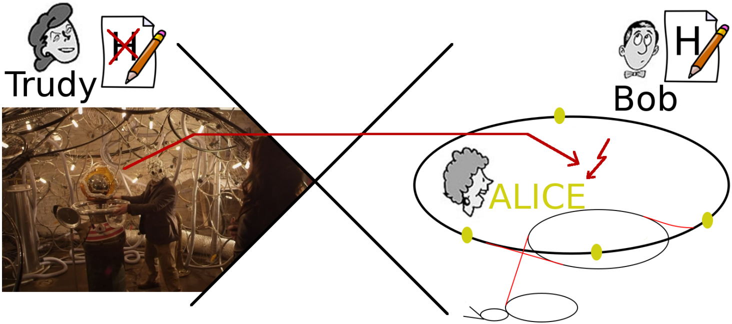

Overview

Algebraic Framework for Local Quantum Theory

Mathematical Objects

Haag-Kastler QFT $(\mathfrak{A}, \alpha, \Omega, \mathscr{H})$ in the vacuum sector.

Described by mathematical entities. . .

- Hilbert space $\mathscr{H}$ of pure states
- distinguished *vacuum* $\Omega \in \mathscr{H}$
- net of von Neumann algebras $\mathbb{R}^{3+1} \supset \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \subset \mathrm{B}(\mathscr{H})$
- space-time translations of states $(t, \mathbf{x}) \mapsto U(t, \mathbf{x}) = e^{\mathrm{i}tH - \mathrm{i}\mathbf{x}\cdot\boldsymbol{P}}$
- translations of observables $\alpha_x A := A(x) := U(x)\, A\, U(x)^*$

Algebraic Framework for Local Quantum Theory

The Haag-Kastler Axioms

...which are subject to

(HK1)	$\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$	**(Isotony)**
(HK2)	$\mathcal{O}_1 \subset \mathcal{O}_2' \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)'$	**(Locality)**
(HK3)	$\alpha_x \mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O} + x),\ \forall x \in \mathbb{R}^4$	**(Covariance)**
(HK4)	$E_{(H,\boldsymbol{P})}(\{0\})\mathscr{H} = \mathbb{C}\Omega$	(**Uniqueness of $\boldsymbol{\Omega}$**)
(HK5)	$\operatorname{supp} E_{(H,\boldsymbol{P})} \subset \bar{V}^+$	(**Spectrum Condition**)
(HK6)	$\overline{\mathfrak{A}(\mathcal{O})\Omega} = \mathscr{H}$	(**Reeh-Schlieder Property**)

Preparing Single-Particle States

Single-particle states $\Psi_1, \Psi_2 \in E_{\{M=m\}}\mathscr{H}$ are non-local objects:

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Instead now fix **one** bounded space-time region $\mathcal{O} \subset \mathbb{R}^4$.

Reeh-Schlieder (HK6) $\Rightarrow \exists (A_{k\beta})_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$: $\|A_{k\beta}\Omega - \Psi_k\| = \beta$.

Preparing Single-Particle States

Single-particle states $\Psi_1, \Psi_2 \in E_{\{M=m\}}\mathscr{H}$ are non-local objects:

$$\Psi_1 = E_m A\Omega = \chi(\tfrac{M^2-m^2}{\epsilon})A\Omega \sim A(\hat{\chi}_\epsilon)\Omega, \quad (\chi \in \mathscr{S}, \epsilon \searrow 0).$$

Instead now fix **one** bounded space-time region $\mathcal{O} \subset \mathbb{R}^4$.

Reeh-Schlieder (HK6) $\Rightarrow \exists (A_{k\beta})_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$: $\|A_{k\beta}\Omega - \Psi_k\| = \beta$.

> Def.: We call a family of local operators $(A_{k\beta})_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$ s.t.
>
> $$\|A_{k\beta}\Omega - \Psi_k\| \leq \beta \text{ and } \|A_{k\beta}\| \leq \beta^{-\gamma}$$
>
> a Reeh-Schlieder family for Ψ_k of degree $\gamma > 0$.

Assumption: **Strengthened Reeh-Schlieder Property** (HK6$^\sharp$)

Reeh-Schlieder families of finite degree generate
a total subset of the single-particle space $\mathscr{H}_1 \subset \mathscr{H}$.

Strengthened Reeh-Schlieder yields Scattering States

Strengthened Reeh-Schlieder Property ($\gamma > 0$)
$(A_{k\beta})_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$, s.t. $\|A_{k\beta}\Omega - \Psi_k\| \leq \beta$ and $\|A_{k\beta}\| \leq \beta^{-\gamma}$

Theorem (MD'15) Let Ψ_k be single-particle states admitting Reeh-Schlieder families $A_{k\beta}$ of finite degree. Then for any regular positive-energy Klein-Gordon sol. f_k with disjoint velocity supports

$$\Psi_\tau := \mathcal{B}_{1\tau} \dots \mathcal{B}_{n\tau}\Omega \stackrel{\tau\to\pm\infty}{\longrightarrow} \Psi^{\pm}.$$

The scalar products of any two such Ψ^+, Ψ'^+ can be computed using the Fock prescription (similarly for incoming states).

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Previous results (Herbst '71, Dybalski '05, Herdegen '13) require spectral condition of Herbst-type, e.g. for some $\epsilon > 0$,

$$\Psi_k = E_{\{M=m\}}A_k\Omega, \quad A_k \in \mathfrak{A}(\mathcal{O}), \quad \left\|E_{\{0<|M-m|<\delta\}}A_k\Omega\right\| \leq \delta^{\epsilon}.$$

Construction of Scattering States

Reeh-Schlieder and Haag-Ruelle Creation Operators

Reference Dynamics: Klein-Gordon solutions f_k with disjointly and compactly supported wave packets $\tilde{f}_k \in \mathscr{C}_c^\infty(\mathbb{R}^3)$ (“*regular*”)

Creation-Operator Approximants: with $\hat{\chi} \in \mathscr{C}_c^\infty(\mathbb{R}^4 \setminus \bar{V}^-)$, set

$$B_{k\beta} := A_{k\beta}(\chi) := \int \mathrm{d}^4x \; \chi(x) \; A_{k\beta}(x),$$

$$\mathcal{B}_{k\tau} := \int \mathrm{d}^3x \; f_k(\tau, \mathbf{x}) \; B_{k\beta}(\tau, \mathbf{x}), \quad (\tau \in \mathbb{R}).$$

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Haag-Ruelle/LSZ: $\mathcal{B}_{k\tau}\Omega \rightharpoonup \Psi'_k(f_k) := \tilde{f}_k(\mathbf{P})\Psi'_k \in \mathscr{H}_1$ for fixed small enough β.

Reeh-Schlieder: $\boxed{\beta = \beta(\tau) := |\tau|^{-\mu}, \; \mu > 0}$ then $\mathcal{B}_{k\tau}\Omega \to \Psi_k(f_k)$.

Candidate Scattering States: Limits $\tau \to \pm\infty$ of $\Psi_\tau := \mathcal{B}_{1\tau}\mathcal{B}_{2\tau}\Omega$.

Mathematical Tools (1) — Discretized Cook's method

$$\|\Psi_{\tau_2} - \Psi_{\tau_1}\| = \left\| \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \, \partial_\tau \Psi_\tau \right\| \leq \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \ \|\partial_\tau \Psi_\tau\| \stackrel{!}{<} \infty \quad (\tau_2 \to \pm\infty)$$

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Recall: $\mathcal{B}_{j\tau}\Omega \to \Psi_j \in \mathscr{H}_1$ (by construction)

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For best possible **summability** as $N \to \infty$ we should

- choose $(\tau_k)_{k \in \mathbb{N}}$ as sparse as possible, $\tau_k := (1+\rho)^k \tau_0$, $\rho > 0$
- control equal- and non-equal-time commutators in $(\star\star)$
- control estimation of unbounded leftmost $\mathcal{B}_{j\tau_k}$ in $(\star)$

Tools (2) — Non-Equal-Time Commutator Estimates

$$f_k(t,\mathbf{x}) = \int \mathrm{d}^3k \; \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega_m(\mathbf{k})t} \; \tilde{f}_k(\mathbf{k}), \quad \tilde{f}_k \in \mathscr{C}_c^\infty(\mathbb{R}^s), \quad \omega_m(\mathbf{k}) := \sqrt{\mathbf{k}^2 + m^2}$$

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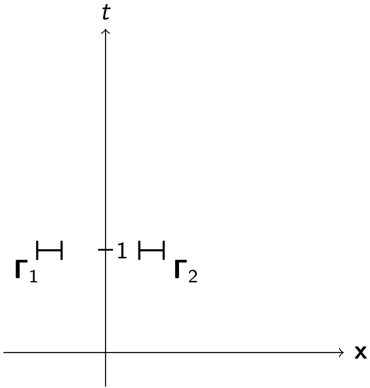
  $$\mathbf{\Gamma}_f := \mathbf{v}(\operatorname{supp} \tilde{f})$$



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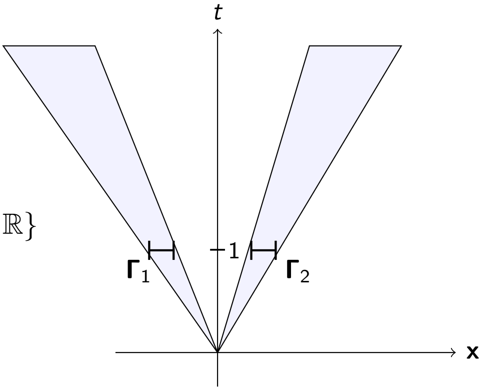

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  $\mathcal{A}_{k\tau} = \int \mathrm{d}^3x\, f_k(\tau,\mathbf{x})\, A_{k\beta}(\tau,\mathbf{x}),$

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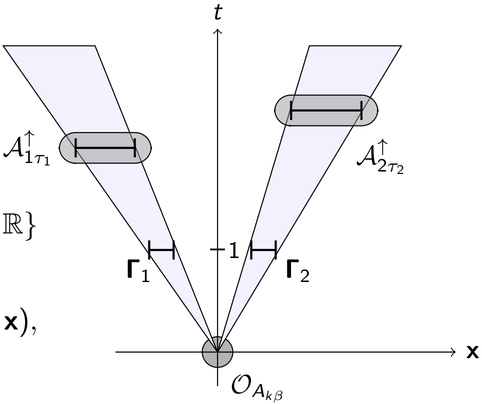

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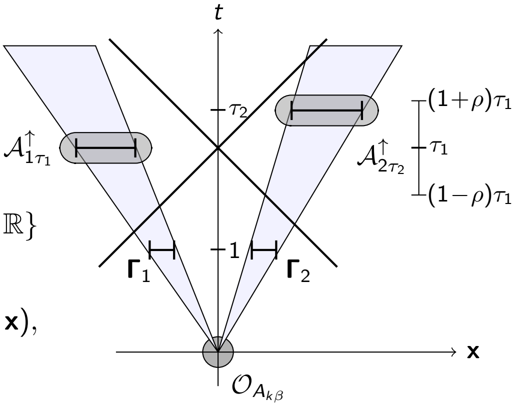

Lemma: Let f_k be regular s.t. $\mathbf{\Gamma}_1 \cap \mathbf{\Gamma}_2 = \emptyset$ and $A_{k\beta}$ have finite degree.

$$\exists\, \rho > 0\ \forall\ |\tau_1 - \tau_2| \le \rho\,|\tau_1| : \quad \|[\mathcal{B}_{1\tau_1}, \mathcal{B}_{2\tau_2}]\| \le \frac{C_N \left\|A_{1\beta(\tau_1)}\right\| \left\|A_{2\beta(\tau_2)}\right\|}{1+|\tau_1|^N + |\tau_2|^N}$$

Assembling the Mathematical Arsenal

The reason why Discrete Cook works may be summarized:

Lemma (local difference estimate) Let $A_{k\beta}$ be RS families of finite degree, and f_k regular positive-energy Klein-Gordon solutions with disjoint velocity supports. Then for sufficiently small scaling $\mu > 0$, $\exists\, \rho > 0\ \forall\ |\tau_1 - \tau_2| \le \rho\,|\tau_1|$,

$$\|\Psi_{\tau_2} - \Psi_{\tau_1}\|^2 \le C_1 \sum_{k=1}^{n} \|\mathcal{B}_{k\tau_2}\Omega - \mathcal{B}_{k\tau_1}\Omega\|^2 + C_2\,|\tau_1|^{-\delta}$$

Proof based on **non-equal-time** commutator estimates, **energy-bounds** [Buchholz'90], and **Clustering** arguments from [Dybalski'05], [Buchholz'77], and [Araki, Hepp, Ruelle'62].

Is it useful?

Wave Operators and S-Matrix

Let $\mathscr{F}$ denote Fock space over finite RS-degree 1-particle vectors and $\mathscr{F}_{\mathrm{disj}} \subset \mathscr{F}$ the set of product states with disjoint $\mathbf{\Gamma}_k$.

Def. (Møller op.) For $\Psi_{\mathrm{prod}} = \Psi_1(f_1)\Omega \otimes \ldots \otimes \Psi_n(f_n)\Omega \in \mathscr{F}_{\mathrm{disj}}$, $\Psi_k = \lim_{\beta\to 0} \tilde{f}_k(\mathbf{P}) A_{k\beta}\Omega$ define

$$W_\pm : \begin{cases} \mathscr{F}_{\mathrm{disj}} \longrightarrow \mathscr{H}, \\ \Psi_{\mathrm{prod}} \longmapsto \lim\limits_{\tau\to\pm\infty} \mathcal{B}_{1\tau} \ldots \mathcal{B}_{n\tau}\Omega. \end{cases}$$

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Questions:

- is S unitary?
- possibly $S = 0$? $W_\pm = 0$?
- is S an operator?

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- Proposition. Assume there is a regular local $A \in \mathfrak{A}(\mathcal{O})$ with Herbst-exponent $\epsilon > 0$. Then one can construct $A_\beta \in \mathfrak{A}(\mathcal{O} + B_\epsilon)$ s.t.

$$\|E(\Delta)(A_\beta\Omega - \Psi_1)\| < C_\Delta \beta, \quad \ln \|A_\beta\| < \beta^{-\gamma}$$

  for any compact $\Delta \subset \mathbb{R}^{s+1}$, with suitable C_Δ, and $\gamma \sim 1/\epsilon$.

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Thanks for your attention!

Backup

Fock Structure

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Theorem (Fock structure)
For $\Psi, \Psi' \in \mathscr{F}_{\text{disj}}$, $\Psi = \Psi_1 \otimes \ldots \otimes \Psi_n$, similarly for Ψ', we have

$$\langle W_+ \Psi', W_+ \Psi \rangle = \delta_{nn'} \sum_{\pi \in \mathfrak{S}_n} \prod_{k=1}^{n} \langle \Psi'_k, \Psi_{\pi(k)} \rangle$$

without relative velocity support restrictions on the single-particle states between the two families, and similarly for W_-.
Proof. (standard) Clustering and Double Commutator Estimates.

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Unitarity? $S^* S = (W_+^* W_-)^* W_+^* W_- = W_-^* \underbrace{W_+ W_+^*}_{\text{projection onto outgoing space } \mathscr{H}^+ \subset \mathscr{H}} W_-$

$$W_+ W_+^* = \mathbb{1}_{\mathscr{H}} \Leftrightarrow \mathscr{H}^+ = \mathscr{H} \text{ (“asymptotic completeness”)}$$

see e.g. recent discussion [Dybalski, Gerard‘14]

Tools (3) — Energy-Momentum Transfer

Naive Standard Estimation yields ($m > 0$)

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Why $\mathcal{B}_{k\tau} = \mathcal{A}_{k\tau}(\chi)$, $\hat{\chi} \in \mathscr{C}_c^\infty(\mathbb{R}^{3+1})$?

Lemma (Energy-Momentum Transfer or "Arveson Spectrum")

$$\mathcal{B}_{k\tau} E(\Delta) = E(\Delta + \operatorname{supp} \hat{\chi}) \mathcal{B}_{k\tau} E(\Delta) \text{ for compact } \Delta \subset \mathbb{R}^{3+1}.$$

Tools (4) — Energy Bounds

Naive Standard Estimation yields ($m > 0$)

$$\|\mathcal{B}_{k\tau}\| \leq \|B_{k\beta(\tau)}\| \int \mathrm{d}^3x \, |f_k(\tau, \mathbf{x})| \sim C_\chi \, \|A_{k\beta(\tau)}\| \cdot (1 + |\tau|^{3/2}) \quad ↯$$

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Lemma (Energy Bounds)

$$\|\mathcal{B}_{k\tau} E(\Delta)\| \leq C_\Delta \left\|A_{k\beta(\tau)}\right\|$$

$$\left\| \prod_{k=1}^{N} \mathcal{B}_{k\tau_k} E(\Delta) \right\| \leq C_{\Delta,N} \prod_{k=1}^{N} \left\|A_{k\beta(\tau_k)}\right\|$$

(C_Δ depends on f_k and χ, but not on τ).

Proof. Methods of Harmonic Analysis developed in [Buchholz'90].

Tools (5) — Clustering Estimates

Creation-annihilation operators satisfy $a(f)a^{\dagger}(g)\Omega = \langle f, g\rangle\Omega$.
Similar property of $\mathcal{B}_{k\tau}$ would enable us to compute

$$\begin{aligned}
\|\mathcal{B}_{1\tau_2}(\mathcal{B}_{2\tau_2} - \mathcal{B}_{2\tau_1})\Omega\|^2 &= \left\langle \Omega, (\Delta\mathcal{B}_2^*)\mathcal{B}_{1\tau_2}^*\mathcal{B}_{1\tau_2}(\Delta\mathcal{B}_2)\Omega \right\rangle \\
&= \left\langle \mathcal{B}_{1\tau_2}^*\mathcal{B}_{1\tau_2}\Omega, (\Delta\mathcal{B}_2^*)(\Delta\mathcal{B}_2)\Omega \right\rangle + (\text{comm.}) \\
&= \left\langle \Omega, \mathcal{B}_{1\tau_2}^*\mathcal{B}_{1\tau_2}\Omega \right\rangle \left\langle \Omega, (\Delta\mathcal{B}_2^*)(\Delta\mathcal{B}_2)\Omega \right\rangle + (\text{error?})
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Lemma (Clustering of $\mathcal{B}_{k\tau}$) Assume that $\hat{\chi} \in \mathscr{C}_c^\infty(\mathbb{R}^{3+1} \setminus \bar{V}^-)$ and $A_{k\beta}$ differentiable, then

$$\left\| E_\Omega^\perp \mathcal{B}_{1\tau}^*\mathcal{B}_{2\tau}\Omega \right\| \leq \frac{C}{|\tau|^{\kappa/2}} \left\|A_{1\beta(\tau)}\right\|_{\text{AHR}} \left\|A_{2\beta(\tau)}\right\|_{\text{AHR}},$$

where $\|A\|_{\text{AHR}} := \|A\| + \|\partial_t A\|$, C depends on χ, $\mathcal{O}$, f_k and $\kappa := 3/2$ for $m > 0$.

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The proof follows a strategy of [Dybalski'05] [Buchholz'77] relying on space-like decay of $\langle \Omega, A_1 U(t, \mathbf{x}) A_2 \Omega\rangle$ [Araki, Hepp, Ruelle'62].