Weyl quantization for gauge theories in terms of projective limits of graphs

[joint work w/ Thiemann]

Outline:

1) Motivation (time-dependent Born-Oppenheimer approx., (de)quantization, QFT on curved spacetimes from QG)

space-adiabatic perturbation theory [Panati, Spohn, Teufel; 2003]

2) The basic construction:

(local) strict deformation quantization of $T^*G \simeq G \times \mathfrak{g}$

compact Lie group

[Rieffel, Landsman] "Weyl type"

! $\exists$ (global) version on $G \times \hat{G}$!

equivalence classes of unitary irrep's

[Ruzhansky, Turunen]

3) A projective phase space for Yang-Mills theory and its quantum counterpart

equivariance of the Weyl quantisation

1) Motivation:

originally: How can we extract QFTs on curved spacetime from (L)QG?

$\longrightarrow$ mathematical framework?
(beyond $\hbar \to 0+$, Hamiltonian approach)

$\longrightarrow$ time dependent Born-Oppenheimer approx.?
(systematics beyond $\mathcal{O}(t)$,
no fibered Hamiltonian)

$\downarrow$

$$H_\varepsilon = \int^{\oplus} d\xi \; H_0(\xi) + f(-i\varepsilon\nabla_x) \otimes \mathbb{1}$$

$$D(H_\varepsilon) \subseteq L^2(\mathbb{R}^d, \mathcal{H}_f)$$

Idea: Use/adapt space-adiabatic perturbation theory
[Panati, Spohn, Teufel; 2003]

Basic ingredient: Pseudo-differential calculus (Weyl-Moyal formalism)

Space-adiabatic perturbation theory: (wish list!)

- coupled quantum dynamical system $\mathcal{H}$, $(\hat{H}, D(\hat{H}))$
- splitting of the dynamics (controlled by parameter ε)

$$\mathcal{H} = \mathcal{H}_{slow} \otimes \mathcal{H}_{fast}$$

- deformation (de)quantization (ε-dependent)

$$\hat{\cdot}^{\varepsilon} : S^{\infty}(\varepsilon; \Gamma, \mathcal{B}(\mathcal{H}_{fast})) \subset C^{\infty}(\Gamma, \mathcal{B}(\mathcal{H}_{fast})) \longrightarrow \mathcal{L}(\mathcal{H})$$

symbols

"phase space"

symbol orders

- $H_{\varepsilon} \asymp \sum_{k=0}^{\infty} \varepsilon^k H_k$; $H_k \in S^{\rho-k}(\Gamma, \mathcal{B}(\mathcal{H}_{fast}))$

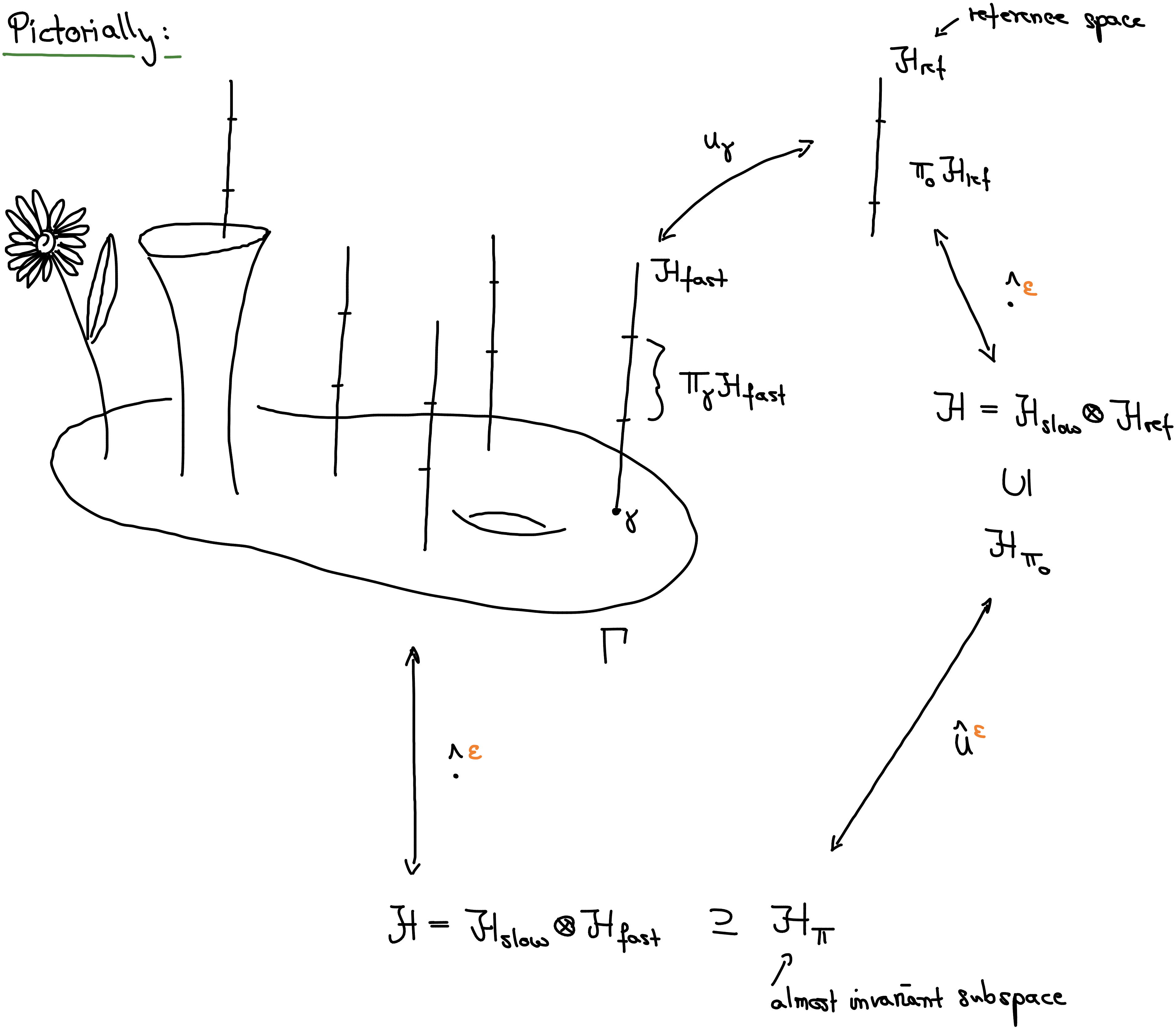
asymptotic expansion (up to $S^{-\infty}(\Gamma, \mathcal{B}(\mathcal{H}_{fast}))$)

smoothing operators

$$\hat{H} = \hat{H}_{\varepsilon}^{\varepsilon}$$

- conditions on (point-wise) spectrum $\sigma(H_0) = \{\sigma_0(H_0(\gamma))\}_{\gamma \in \Gamma}$

Pictorially:



Upshot: Construct effective dynamics in $\mathcal{H}_{\pi_0}$

ε-independent space

2) The basic construction

- Γ will be modeled on T^*G ← compact Lie group G (structure group of a principal bundle)

- pseudo-differential calculus on T^*G ?

↓

start from a (local) strict deformation quantization
[Rieffel, Landsman]

We have: $T^*G \underset{\text{left/right translation}}{\cong} G \times \mathfrak{g}$; $\mathfrak{g} = \mathrm{Lie}(G)$; $n = \dim(G)$

$\exp : \mathfrak{g} \longrightarrow G$ gives a diffeomorphism for some

$$\mathfrak{g} \supseteq \underset{0}{U} \longrightarrow \underset{e}{V} \subseteq G$$

relate the Haar measure on G and Lebesgue measure on $\mathfrak{g}$

$$f \in C_c(V): \quad \int_V f(g)\, dg = \int_U f(\exp(X))\, j(X)^2\, dX$$

$$H \in \mathfrak{t} \subseteq \mathfrak{g} : \quad j(H) = \prod_{\alpha \in R^+} \frac{\sin(\alpha(H))}{\alpha(H)}$$

Lie algebra of a max. torus ↗ ↖ positive roots

Construct operators in $L^2(G)$ from convolution kernels:

$$\varepsilon\in(0,1]: \quad \mathcal{F}^{\varepsilon}_{\sigma}(h,g) = \int_{\mathfrak{g}^*} \sigma(\theta,g)\, e^{\frac{i}{\varepsilon}\theta(X_h)} \frac{d\theta}{(2\pi\varepsilon)^n}$$

for $X_h = \exp^{-1}(h)$; $\sigma\in C^{\infty}_{PW,U_\varepsilon}(\mathfrak{g}^*)\,\hat{\otimes}\, C^\infty(G)$; $U_\varepsilon = \varepsilon^{-1}U$

$$\Longrightarrow \mathcal{F}^{\varepsilon}_{\sigma} \in C^\infty(G)\,\hat{\otimes}\, C^\infty(G)$$

gives the kernel of a Kohn-Nirenberg ΨDO

Deform construction to a $\boxed{\text{Weyl-type}}$ ΨDO:

- define a (smooth) square root $\sqrt{\cdot}: V \subseteq G \longrightarrow V \subseteq G$

$$g \longmapsto \sqrt{g} := \exp\left(\tfrac{1}{2}X_g\right)$$

- $$\mathcal{F}^{W,\varepsilon}_{\sigma}(h,g) = \mathcal{F}^{\varepsilon}_{\sigma}(h, \sqrt{h^{-1}}\,g)$$

Weyl quantisation:

$$\forall \psi\in C^\infty(G): \quad (Q^{W}_{\varepsilon}(\sigma)\psi)(g) = \int_G \mathcal{F}^{W,\varepsilon}_{\sigma}(h,g)\,\psi(h^{-1}g)\,dh$$

Theorem: [Landsman]

$$Q^W_\varepsilon : \overset{\displaystyle C(T^*G)}{\overset{\cup}{C^\infty_{PW,u}(\mathfrak{g}^*) \otimes C^\infty(G)}} \longrightarrow \overset{\displaystyle C(G) \rtimes_L G}{\overset{\cong}{\mathcal{K}(L^2(G))}}$$

is a nondegenerate strict deformation quantization on $(0,1]$ w.r.t. the canonical Poisson structure on T^*G.

$$\sigma, \tau \in C^\infty(T^*G) : \{\sigma,\tau\}_{T^*G} = \langle \partial_\theta \sigma, R\tau \rangle - \langle R\sigma, \partial_\theta \tau \rangle + \{\sigma,\tau\}_-$$

(minus) Lie-Poisson structure

Remark: For special choices of σ, we find "CCR" relations:

$$\sigma_f(\theta,g) = f(g) \quad ; \quad f \in C^\infty(G)$$

$$\sigma_X(\theta,g) = J_X^{L^*(\cdot)^{-1}}(\theta,g) \quad ; \quad X \in \mathfrak{g}$$

bi-equivariant momentum map of the strongly Hamiltonian left pullback action of G on T^*G

$$Q^W_\varepsilon(\{\sigma_f, \sigma_g\}_{T^*G}) = \frac{i}{\varepsilon}[Q^W_\varepsilon(\sigma_f), Q^W_\varepsilon(\sigma_g)] = 0$$

$$Q^W_\varepsilon(\{\sigma_X, \sigma_f\}_{T^*G}) = \frac{i}{\varepsilon}[Q^W_\varepsilon(\sigma_X), Q^W_\varepsilon(\sigma_f)] = R_X f$$

$$Q^W_\varepsilon(\{\sigma_X, \sigma_Y\}_{T^*G}) = \frac{i}{\varepsilon}[Q^W_\varepsilon(\sigma_X), Q^W_\varepsilon(\sigma_Y)] = i\varepsilon R_{[X,Y]}$$

Define ΨDO on $C^\infty(G) \subseteq L^2(G)$ via Q_ε^ω

$\downarrow$

Problem: $\exp : \mathfrak{g} \longrightarrow G$ is not a diffeomorphism

We need to deal with support properties of $\sigma \in C^\infty(T^*G)$

Solution: Exploit Paley-Wiener-Schwartz characterisation of distributions w/ compact support

Definition: $K \sqsubset \mathfrak{g}$ convex, compact

$m \in \mathbb{R}$; $0 \le \delta \le \rho \le 1$

A function $\sigma \in C^\infty(\mathfrak{g}^*_{\mathbb{C}}, C^\infty(G))$ belongs to the space

complexification of $\mathfrak{g}$

of Paley-Wiener-Schwartz symbols $S^{K,m}_{PW,\delta,\rho}$ if:

i) $\sigma : \mathfrak{g}^*_{\mathbb{C}} \longrightarrow C^\infty(G)$ is (weakly) holomorphic

ii) $\forall \alpha, \beta \in \mathbb{N}_0^n : \exists C_{\alpha\beta} > 0 : \forall \theta \in \mathfrak{g}^*_{\mathbb{C}}$:

supporting function of K

$$\sup_{g \in G} |(R^\alpha \partial_\theta^\beta \sigma)(\theta, g)| \le C_{\alpha\beta} \langle \theta \rangle^{m - |\beta|\rho + |\alpha|\delta} e^{H_K(\mathrm{Im}(\theta))}$$

standard regularized distance

We have the expected inclusions:

$$S^{\kappa,m}_{PW,\rho,\delta} \subseteq S^{\kappa,m}_{PW,\rho',\delta} \quad ; \quad \rho \geq \rho'$$

$$S^{\kappa,m}_{PW,\rho,\delta} \subseteq S^{\kappa',m}_{PW,\rho,\delta} \quad ; \quad \kappa \subseteq \kappa'$$

$$S^{\kappa,m}_{PW,\rho,\delta} \subseteq S^{\kappa,m}_{PW,\rho,\delta'} \quad ; \quad \delta \leq \delta'$$

$$S^{\kappa,m}_{PW,\rho,\delta} \subseteq S^{\kappa,m'}_{PW,\rho,\delta} \quad ; \quad m \leq m'$$

Set:
$$S^{\kappa,\infty}_{PW,\rho,\delta} := \bigcup_{m\in\mathbb{R}} S^{\kappa,m}_{PW,\rho,\delta} \quad ; \quad S^{\kappa,-\infty}_{PW} := \bigcap_{m\in\mathbb{R}} S^{\kappa,m}_{PW,\rho,\delta}$$

<u>Corollary:</u> $\sigma \in S^{\kappa,m}_{PW,\rho,\delta}$ defines a continuous operator

$$Q^{W}_{\varepsilon}(\sigma) : C^\infty(G) \longrightarrow C^\infty(G) \quad \text{for some } 0<\varepsilon_0\leq 1$$

extends to $\mathcal{D}'(G)$
for $m=-\infty$ (smoothing)

and all $\varepsilon \in (0,\varepsilon_0]$.

<u>Proposition:</u> The optimal constants $C_{\alpha\beta}>0$ turn the symbol spaces $S^{\kappa,m}_{PW,\rho,\delta}$ into Fréchet spaces.

The strict inductive limit topology turns $S^{\kappa,\infty}_{PW,\rho,\delta}$ into an LF-space.

Lemma: Differentiation, multiplication, and the Poisson bracket become continuous maps between suitable symbol spaces.

Theorem: (Asymptotic completeness of Paley-Wiener-Schwartz symbols)

The symbol spaces are asymptotically complete in the following sense:

Given $\{m_k\}_{k=1}^{\infty} \subset \mathbb{R}$ s.t. $\lim_{k\to\infty} m_k = -\infty$ and $m := \max_k m_k$,

and $a_k \in C^{\infty}(G, S^{\kappa, m_k}_{PW,\rho,\delta})$, there exists $a \in C^{\infty}(G, S^{\kappa,m}_{PW,\rho,\delta})$

such that:

$$a \sim \sum_{k=1}^{\infty} a_k \quad , \quad \text{i.e.}$$

resummation

$$\forall \mu \in \mathbb{R} : \exists k_0 \in \mathbb{N} : \forall k' \geq k_0 : \quad a - \sum_{k=1}^{k'} a_k \in S^{\kappa,\mu}_{PW,\rho,\delta}$$

(uniqueness up to smoothing symbols)

Remark: Non-trivial constructions necessary because of holomorphicity.

↳ no excision functions!

⟶ kernel cut-off operators (Volterra-Mellin ΨDOs, [Krainer])

Observations: • ε-Expansion of the (formal) identity

$$Q_\varepsilon^\omega(\sigma)\, Q_\varepsilon^\omega(\tau) = Q_\varepsilon^\omega(\sigma *_\varepsilon \tau)$$

becomes an asymptotic series for suitable

$$\sigma \in S^{k,m}_{PW,\rho,\delta} \quad , \quad \tau \in S^{k',m'}_{PW,\rho',\delta'}$$

But: Is there a resummation of $\sigma *_\varepsilon \tau$ s.t. the identity is implemented?

• restrictions for compositions of symbols (ρ, δ)

$$\sigma *_\varepsilon \tau \in S^{k+k', m+m'}_{PW,\rho,\delta}$$

for: • $\rho > \delta$ (similar to $\mathbb{R}^{2n}$ – case)

• $\rho > \frac{1}{2}$ (from Lie-Poisson structure)

3) A projective phase space for Yang-Mills theory

$$G \circlearrowright P \xrightarrow{\pi} \Sigma$$

right, semi-analytic, principal G-bundle

Cauchy surface

Field configurations in terms of initial data

$$(A,E) \in T^*\mathcal{A}_P \quad \text{or} \quad |\Lambda^1|T^*\mathcal{A}_P =: \Gamma$$

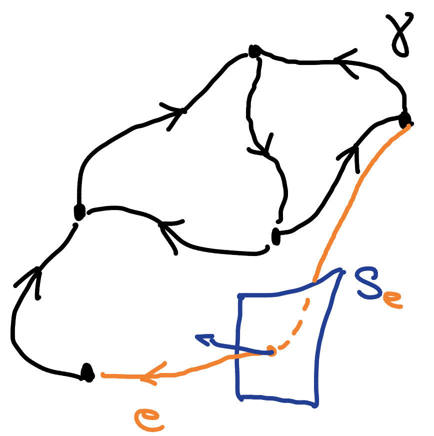
1-densities

Proceed by construction of finite dimensional projections

$$\Gamma \xrightarrow{p_\gamma} \Gamma_\gamma$$

by localizing (A,E) on finite, compactly supported, semi-analytic oriented, embedded graph $\gamma \in \Gamma_0^{sa,\uparrow}$

$$(A,E) \overset{p_\gamma}{\longmapsto}$$



$g_e(A)$ – parallel transport along e

$P^e_X(A,E)$ – flux through S_e weighted by $X \in \mathfrak{g}$

Observation: • Using a suitable regularisation scheme, the symplectic structure on Γ descends to the natural symplectic structure on Γ_γ.

$$(\Gamma, \omega) \xrightarrow{p_\gamma} \left(\Gamma_\gamma = T^*G^{\times |E(\gamma)|} \, ; \, \omega_{T^*G}^{\times |E(\gamma)|}\right)$$

is symplectic.

• the functionals $(g_e \, , \, P_X^e)_{e \in E(\gamma)}$ behave naturally w.r.t. operations on graphs:

i) $e = e_2 \circ e_1$: $g_e(A) = g_{e_2}(A) \, g_{e_1}(A)$

(composition)

ii) $e \longmapsto e^{-1}$: $g_{e^{-1}}(A) = g_e(A)^{-1}$

$$P_X^{e^{-1}}(A,E) = -P^e_{\mathrm{Ad}_{g_e(A)}(X)}(A,E)$$

(inversion)

Composition is subtler for P_X^e :

$$P_X^e(A,E) = c \; P_X^{e_2}(A,E) + (1-c) \, P^{e_1}_{\mathrm{Ad}_{g_{e_2}^{-1}(A)}(X)}(A,E)$$

compatible w.r.t. right action, but only $c = 0,1$ valid.

iii) $e \longrightarrow \emptyset$: drop the corresponding factor in $\Gamma_\gamma = T^*G^{\times|E(\gamma)|}$

(removal)

$\Rightarrow$ • $\{\Gamma_\gamma\}_{\gamma \in \Gamma_0^{sa,r}}$ becomes a directed set w.r.t.

inclusion of oriented, $\mathfrak{g}^*$-labeled subgraphs.

• the relative projections $p_{\gamma\gamma'} : \Gamma_{\gamma'} \longrightarrow \Gamma_\gamma$; $\gamma \subseteq \gamma'$

are symplectic surjections. $p^*_{\gamma\gamma'} : C^\infty(\Gamma_\gamma) \longrightarrow C^\infty(\Gamma_{\gamma'})$ is continuous.

this imposes $c = 0, 1$. We need to choose coherently.

but breaks reversion symmetry.

coming from $C(G) \rtimes_L G$

• the relative projections $\{p_{\gamma\gamma'}\}_{\gamma \subseteq \gamma'}$ induce injective $*$-morphisms

by Weyl quantisation

$$Q^\omega_\varepsilon(p^*_{\gamma\gamma'}) =: \alpha_{\gamma'\gamma} : L(C^\infty(\Gamma_\gamma)) \longrightarrow L(C^\infty(\Gamma_{\gamma'}))$$

continuous linear operators on $C^\infty(\Gamma_{\gamma'})$

Viewing elements of $L(C^\infty(\Gamma_\gamma))$ in terms of Schwartz kernels,

the action on $C^\infty(\Gamma_\gamma)$ comes from the

integrated left regular action ρ_L

$(Q^\omega_\varepsilon = \rho_L \circ \mathcal{F}^{\omega,\varepsilon})$

Some inductive constructions:

We would like to construct an inductive-limit $*$-algebra to represent the noncommutative analog of Γ.

- First try: $\mathfrak{A}_\gamma = C(G_\gamma) \rtimes_L G_\gamma \cong \mathcal{K}(L^2(G_\gamma))$

$$G_\gamma = \pi_1(\Gamma_\gamma) \cong G^{|E(\gamma)|}$$

to small, cannot support the $*$-morphisms $\{\alpha_{\gamma'\gamma}\}_{\gamma \leq \gamma'}$

- Second try: $\mathcal{B}_\gamma := M(\mathfrak{A}_\gamma)$

multiplier algebra

works and has some nice extension properties regarding $\mathfrak{A}_\gamma$:

i) unique extension of morphisms

ii) embedding of $C(G_\gamma)$ and G_γ

iii) recovery of $S(\mathfrak{A}_\gamma)$

state space

as strictly continuous states on $\mathcal{B}_\gamma$

$$\mathcal{B} := C^*\text{-}\varinjlim_{\gamma \leq \gamma'} \mathcal{B}_\gamma \quad (\text{also } W^*\text{-}\varinjlim \text{ possible})$$

Problems/Questions:

- different extension of $\mathfrak{A}_\gamma$? K-theory of $C(\mathcal{C}_\gamma) \rtimes_L \mathcal{C}_\gamma$?

- control on the state space of the inductive limit?

  The only know state comes from gives the inductive-limit Hilbert space:

$$L^2(\overline{\mathcal{A}}) := \varinjlim_{\gamma \leq \gamma'} L^2(\mathcal{C}_\gamma)$$

(spin network Hilbert space)

Final remark:

- gauge transformations act on vertices of γ:

$$\alpha^\lambda_\gamma(\{g_v\}_{v\in V(\gamma)})(F)(\{(h_e, g_e)\}_{e\in E(\gamma)})$$

$$= F(\{\alpha_{g^{-1}_{e(1)}}(h_e), g^{-1}_{e(1)} g_e g_{e(0)}\}_{e\in E(\gamma)})$$

in a compatible way.