Hopf algebra gauge theories on ribbon graphs

Workshop Foundational and structural aspects of gauge theories
Mainz Institute for Theoretical Physics
June 1, 2017

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Department Mathematik, Universität Erlangen-Nürnberg

Ref:

- C. Meusburger, D. Wise, Hopf algebra gauge theory on a ribbon graph, arXiv:1512.03966
- C. Meusburger, Kitaev models as a Hopf algebra gauge theory, Commun. Math. Phys. 353(1), 413-468

Motivation

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- **Kitaev lattice models**

  [Kitaev, ‘03]
  [Buerschaper et al ‘10]

  [Buerschaper, Aguado ‘09]
  [Kadar ‘09]
  [Kirillov ‘11]

- **Lewin-Wen models**

  [Lewin,Wen ‘05]

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⇨ topological quantum computing

⇨ condensed matter physics

⇨ TQFT

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ingredients

- finite-dim. semisimple Hopf algebra H

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⇨ **output: topological invariant of Σ : $Z_{TV}(\Sigma)$**

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Turaev-Viro TQFT $\mathrm{Rep}(H)$

↕ [Balsam,Kirillov ‘12]

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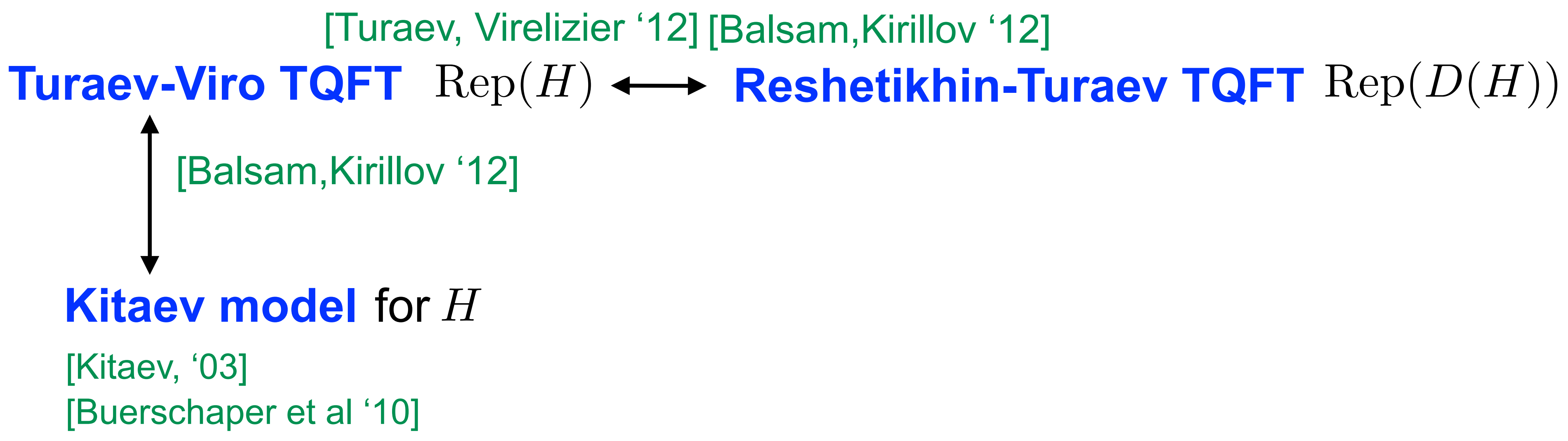

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combinatorial quantisation for $D(H)$

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mathematical structures for lattice gauge theory with values in a Hopf algebra

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➪ generalise concept of lattice gauge theory from group to Hopf algebra

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lattice gauge theory for a group G

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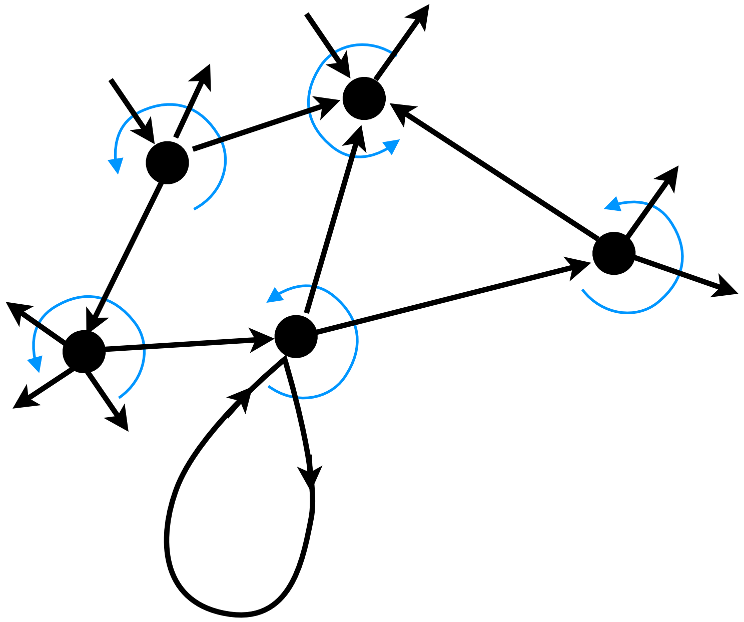
lattice gauge theory for a group G

ribbon graph Γ



1. Hopf algebra gauge theory

lattice gauge theory for a group G

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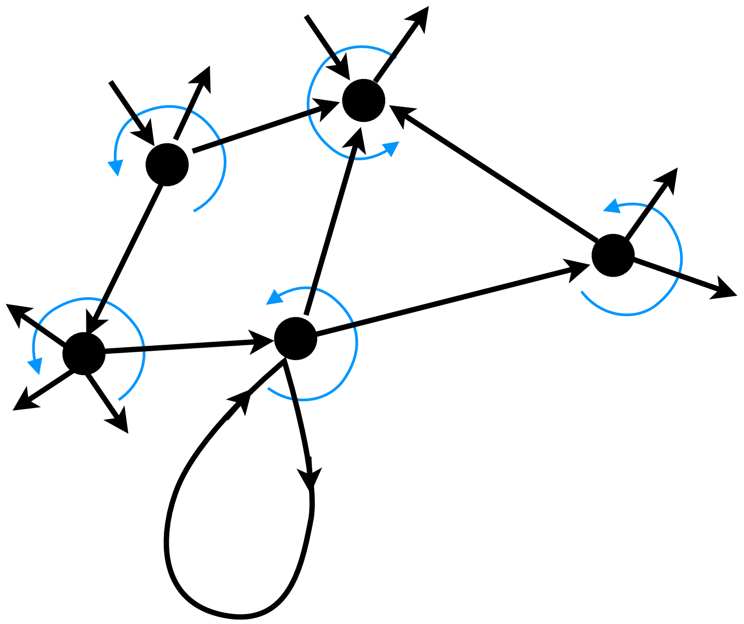

gauge fields **set** $G^{\times E}$

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lattice gauge theory
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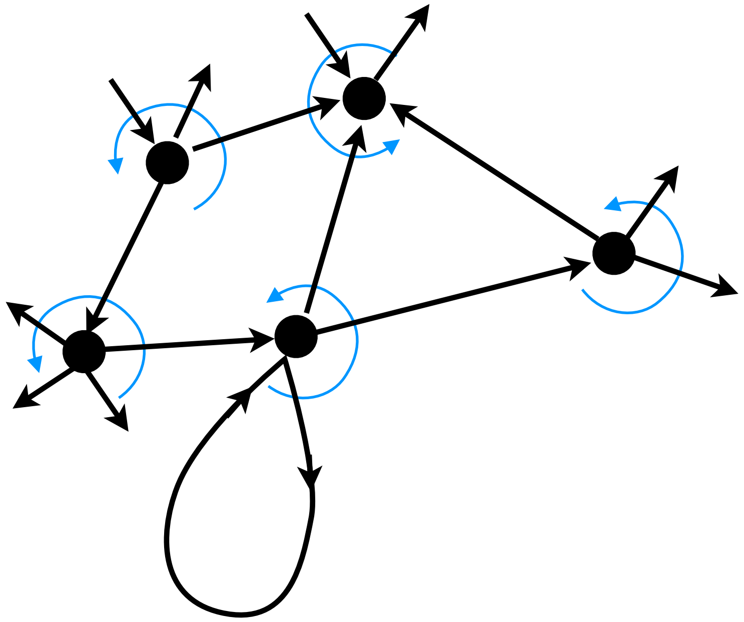

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functions **algebra** $\mathrm{Fun}(G^{\times E})$

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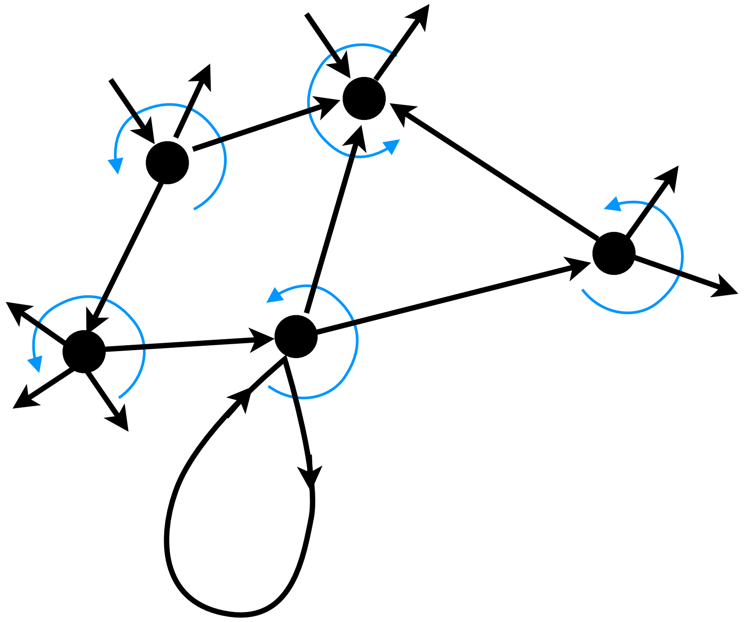

gauge fields **set** $G^{\times E}$

↕ evaluation $(f,g) \mapsto f(g)$

functions **algebra** $\mathrm{Fun}(G^{\times E})$

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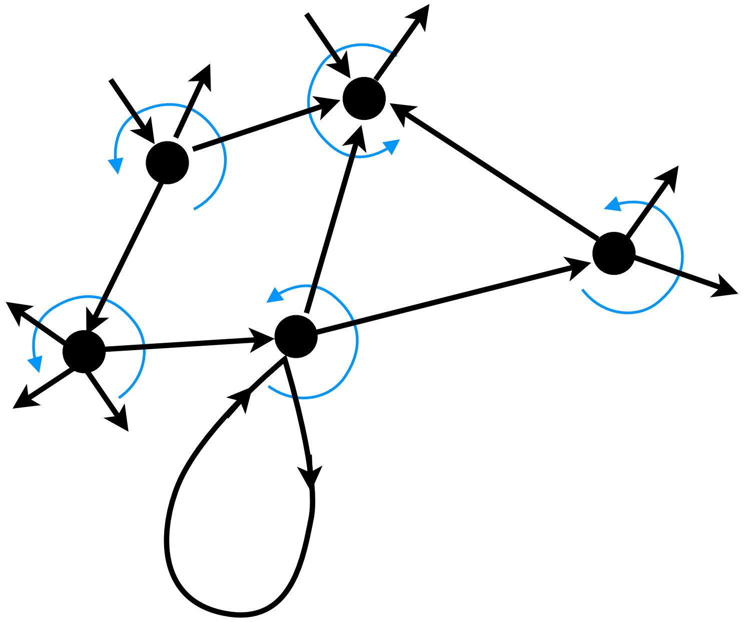

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gauge transformations **group** $G^{\times V}$

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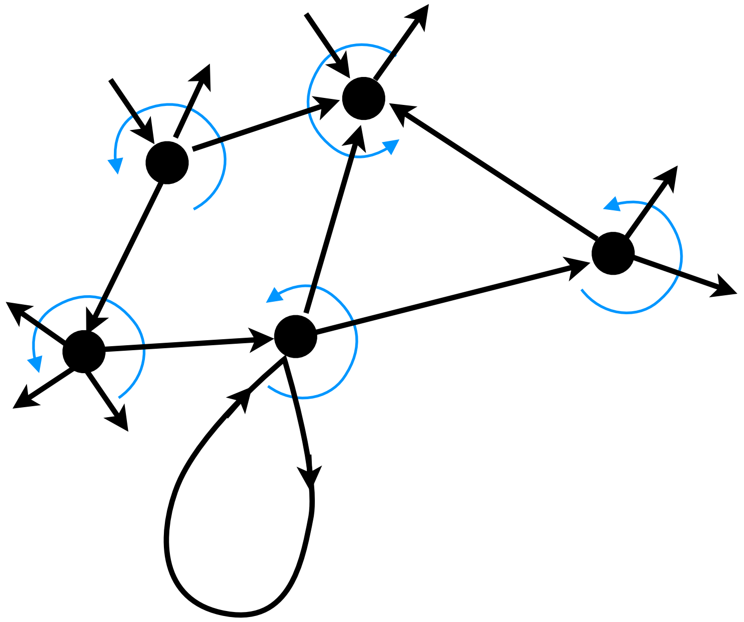

gauge fields **set** $G^{\times E}$

evaluation $(f,g) \mapsto f(g)$

functions **algebra** $\mathrm{Fun}(G^{\times E})$

gauge transformations **group** $G^{\times V}$

action on connections and functions

$$\lhd : \mathrm{Fun}(G^{\times E}) \times G^{\times V} \to \mathrm{Fun}(G^{\times E})$$

$$(f \lhd h)(g) = f(h \rhd g)$$

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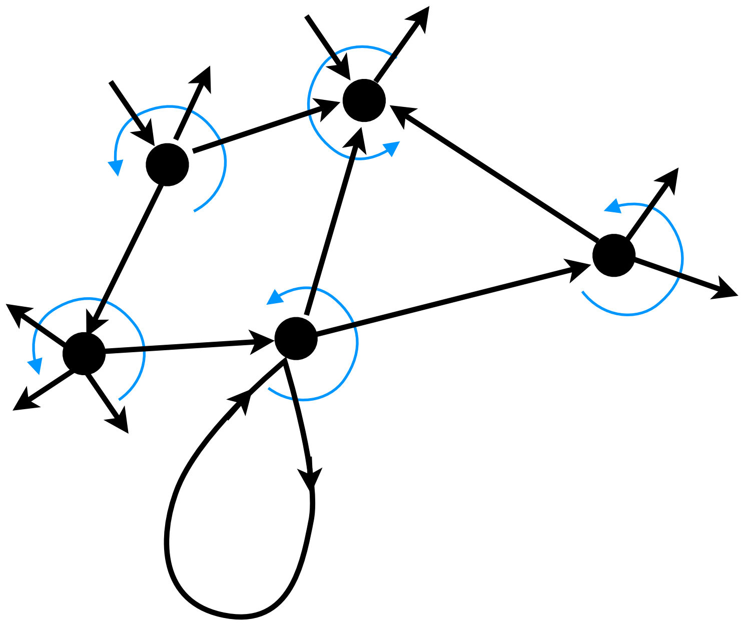

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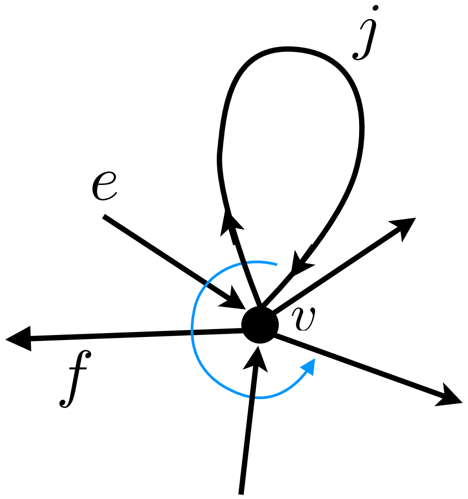
$\updownarrow$ evaluation $(f,g) \mapsto f(g)$

functions **algebra** $\mathrm{Fun}(G^{\times E})$



gauge transformations **group** $G^{\times V}$

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$$g_e \mapsto g_v \cdot g_e$$
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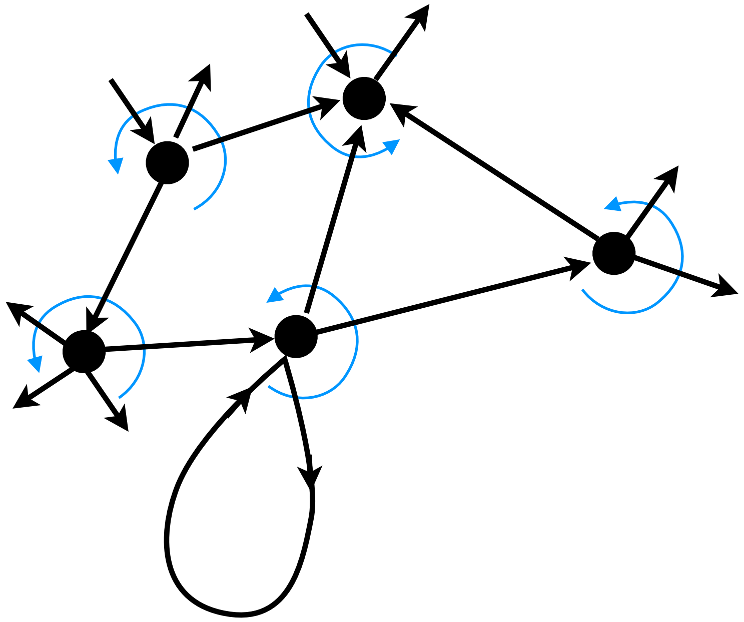

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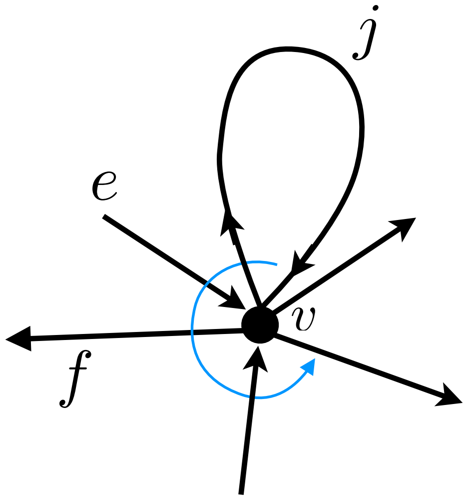

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observables $f \in \mathrm{Fun}(G^{\times E})$

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lattice gauge theory
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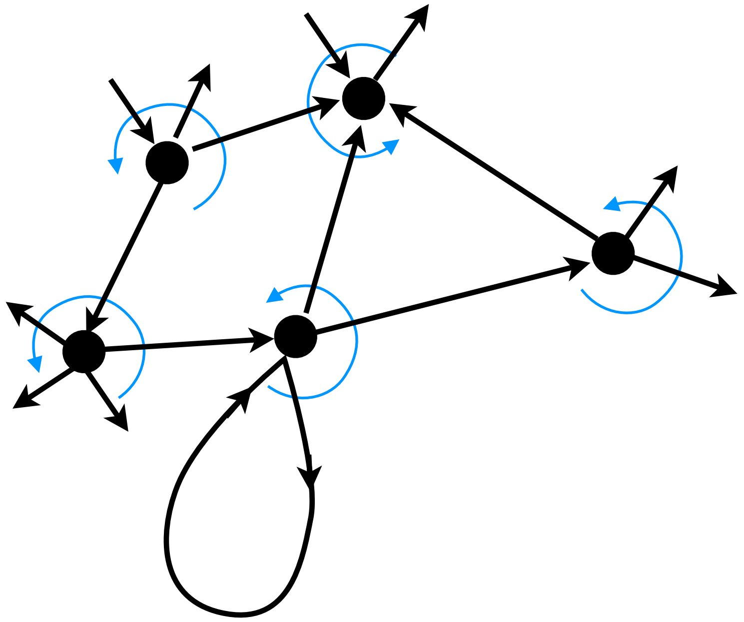

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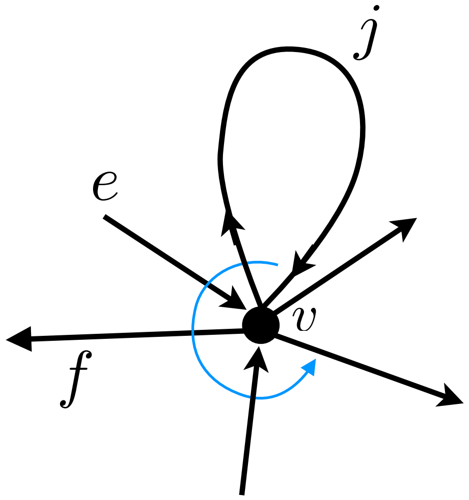

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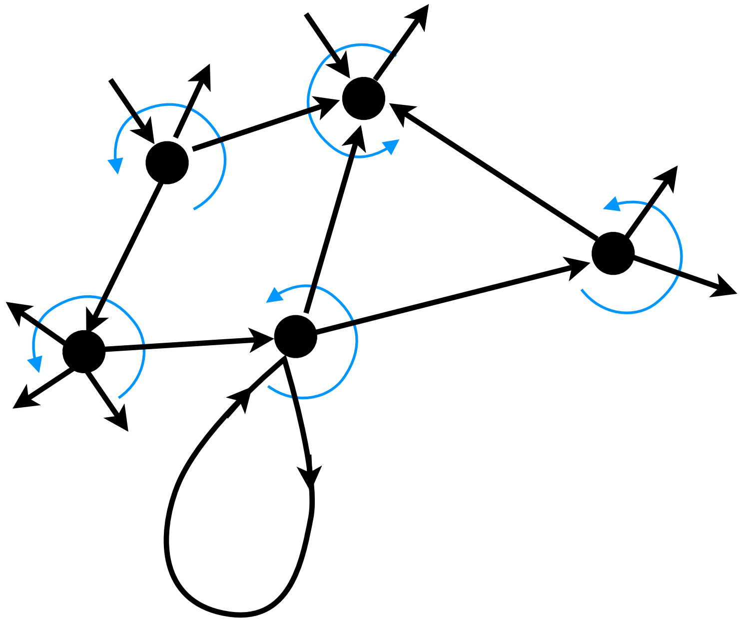

lattice gauge theory
for a finite-dim Hopf algebra K

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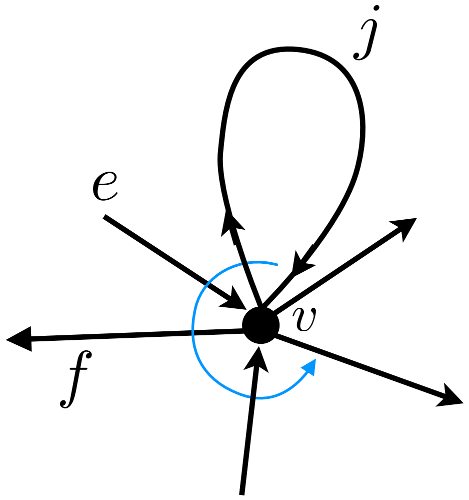

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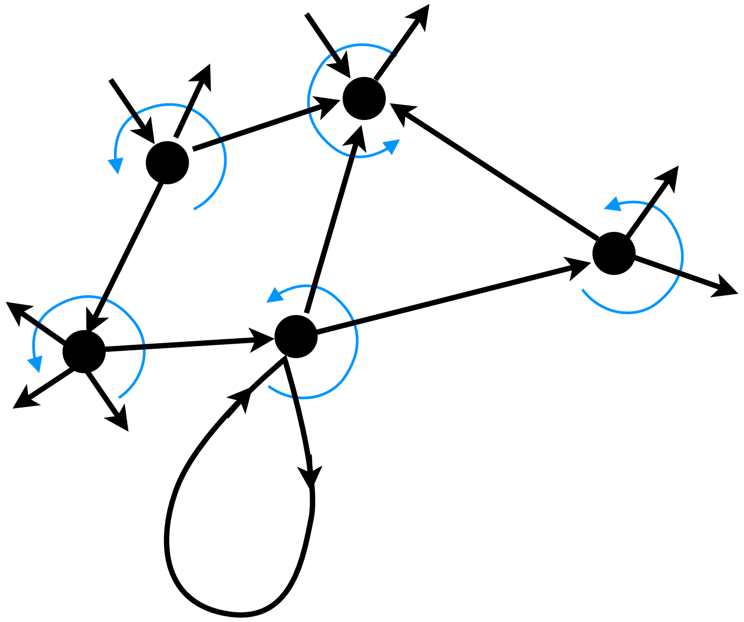

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vector space $K^{\otimes E}$

gauge fields **set** $G^{\times E}$

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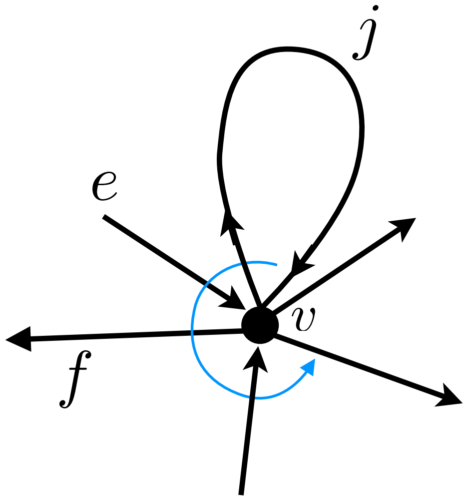

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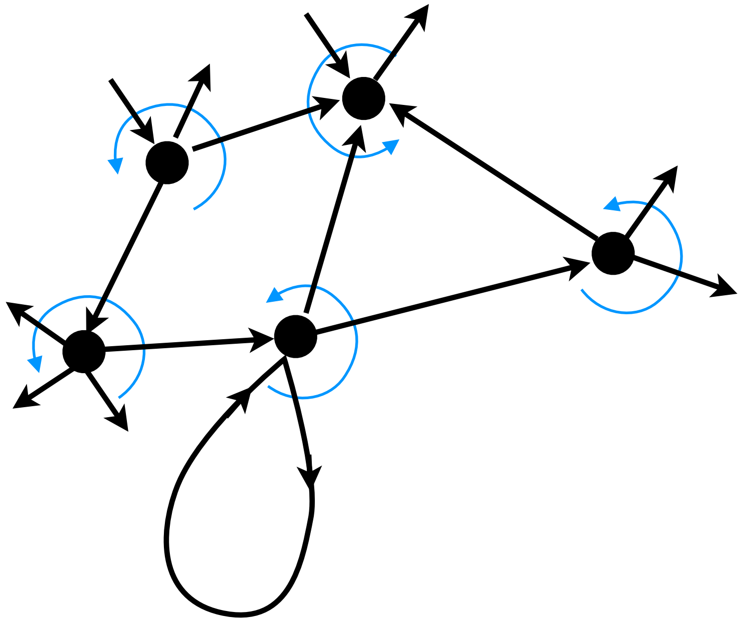

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algebra structure on $K^{*\otimes E}$

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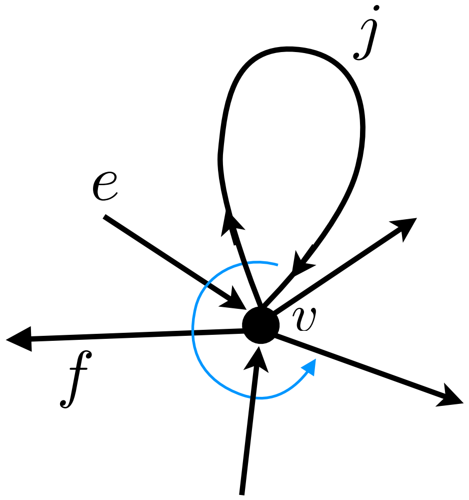

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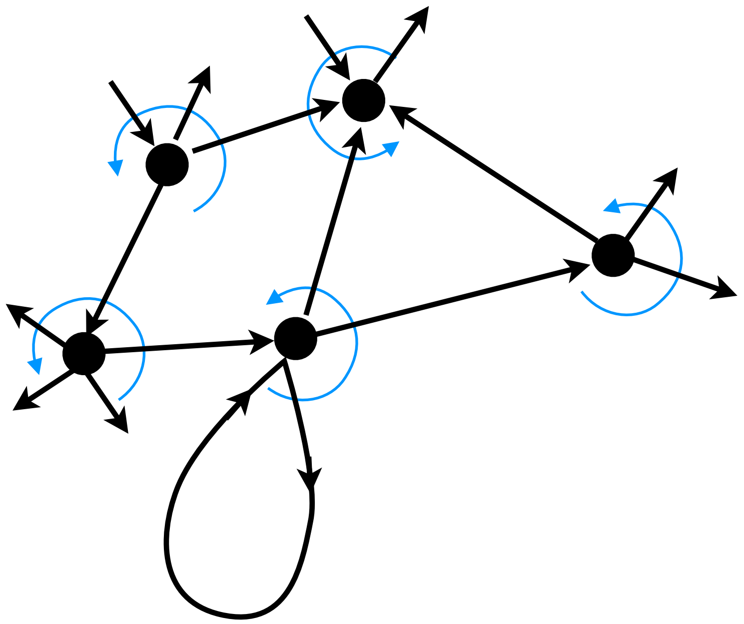

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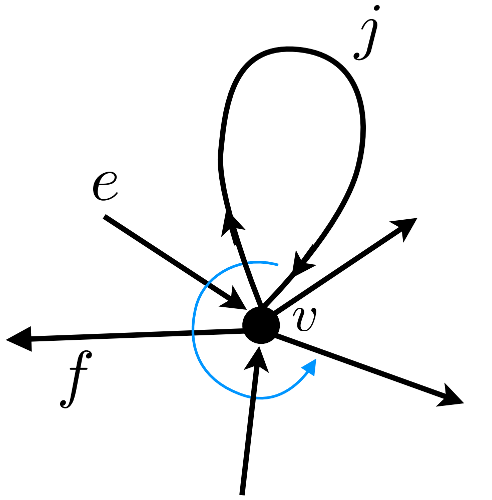

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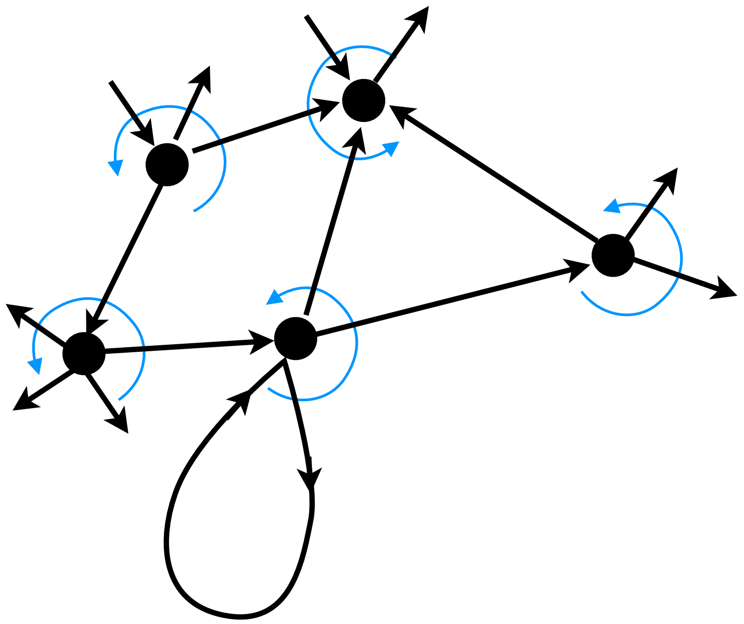
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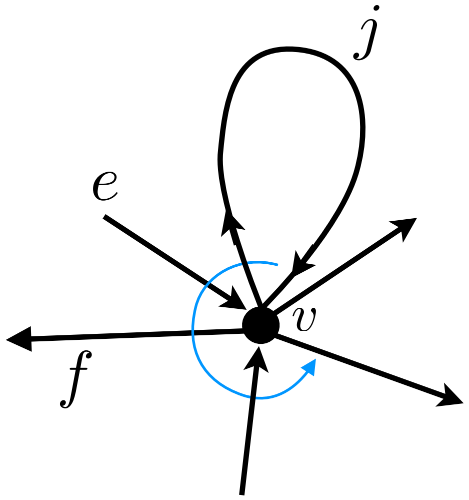

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lattice gauge theory
for a group G

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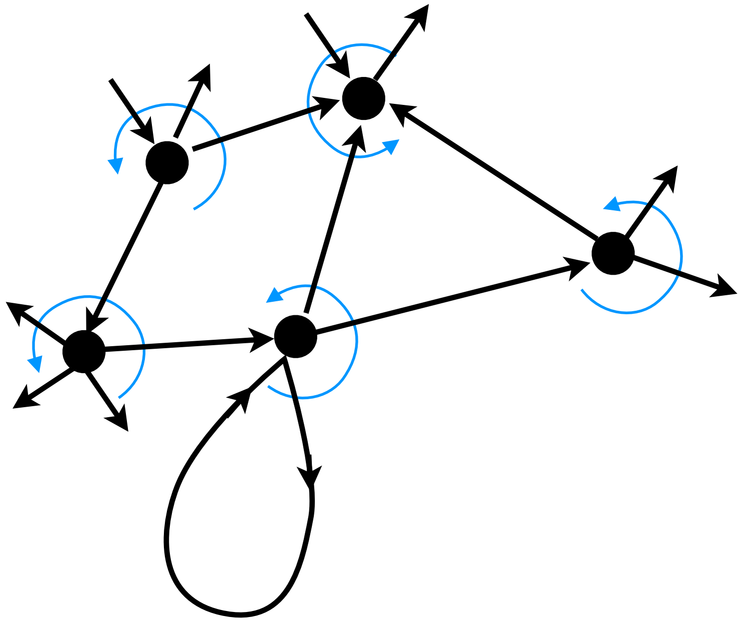
ribbon graph Γ



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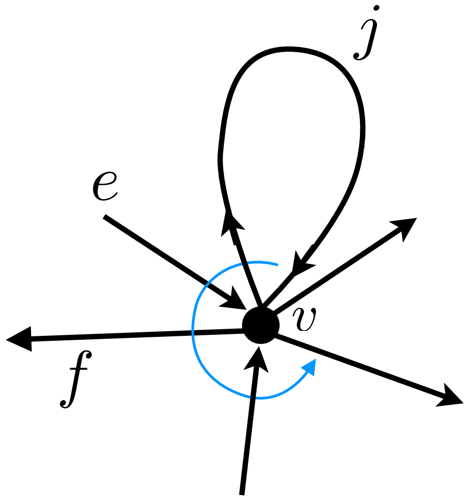

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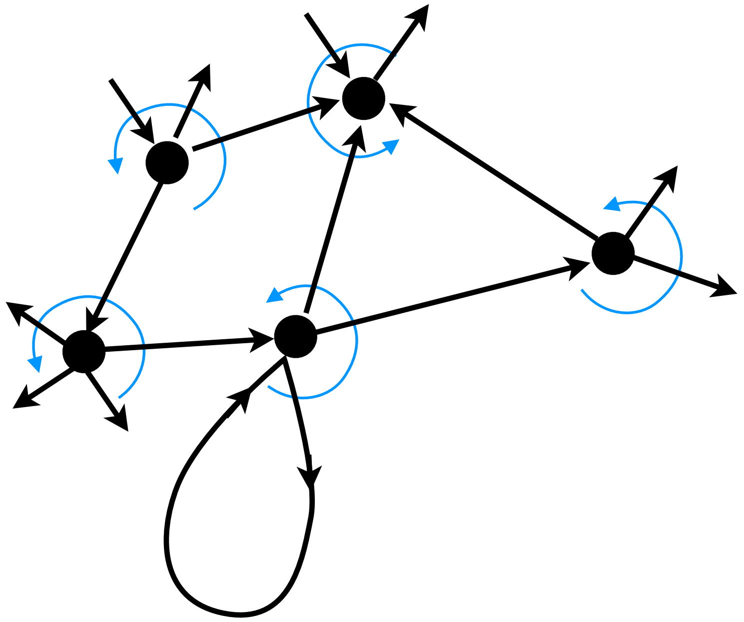

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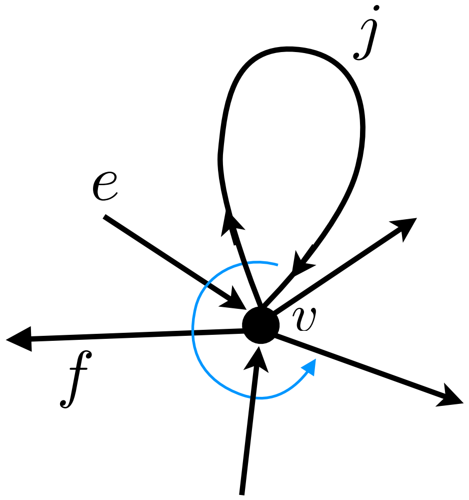

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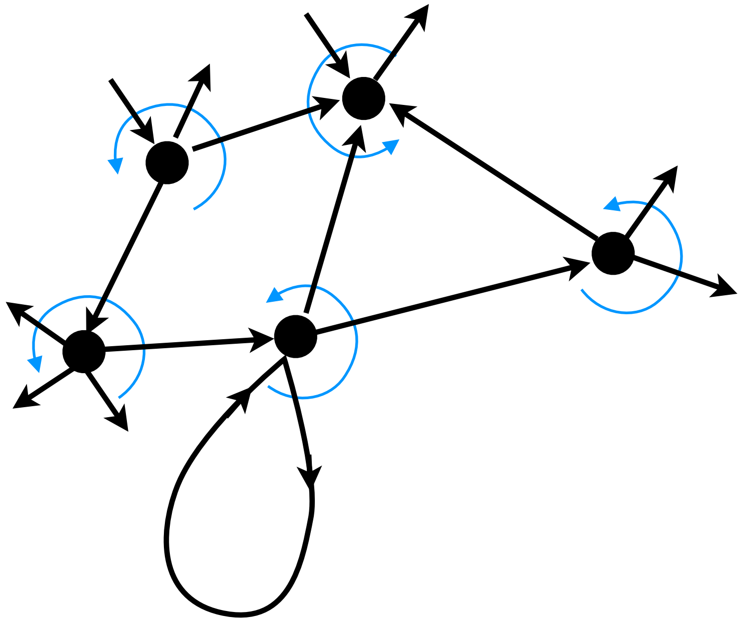

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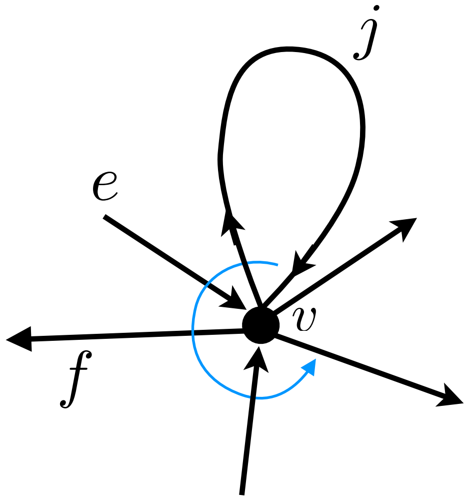

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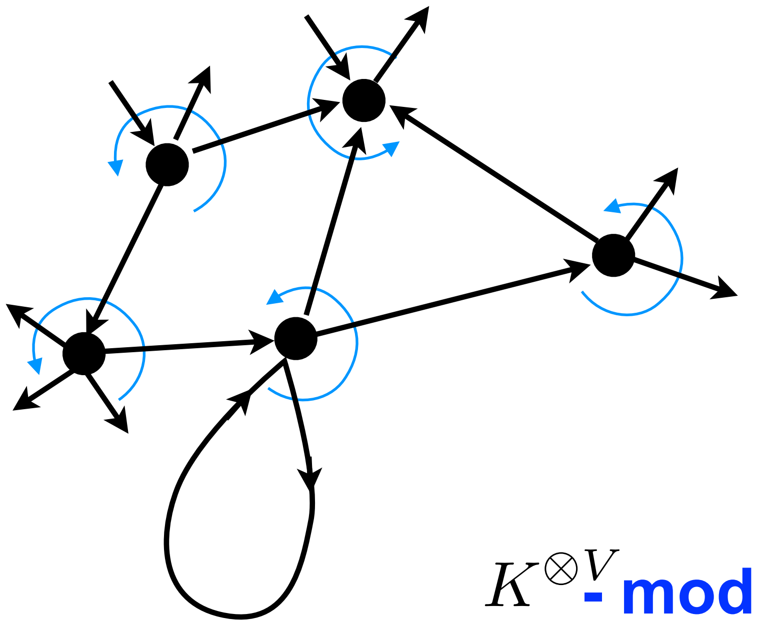

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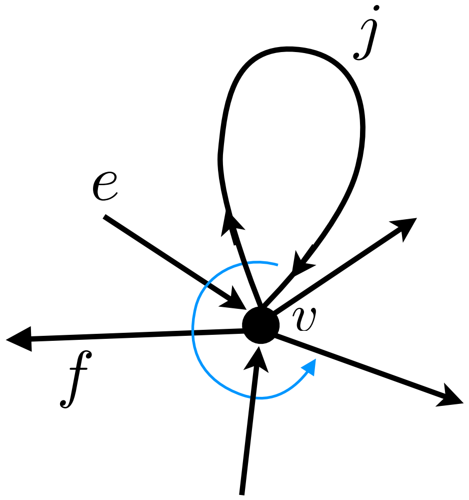

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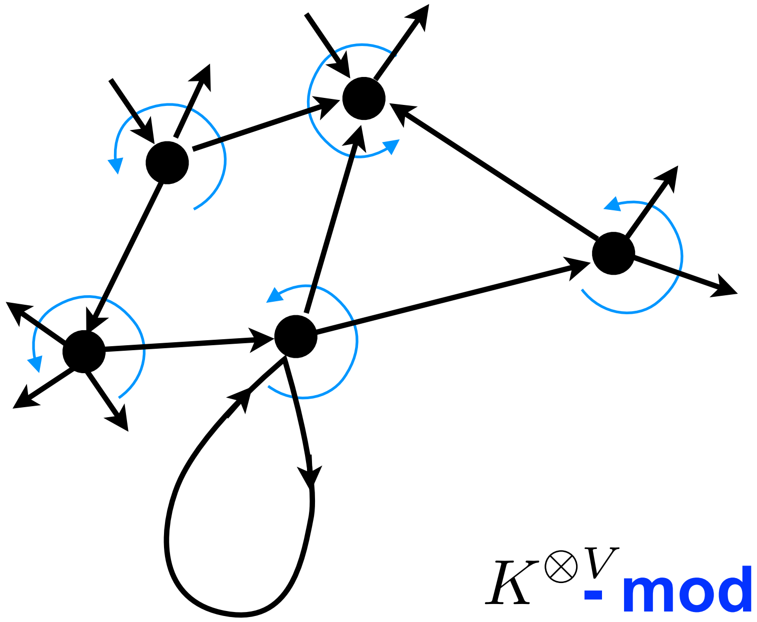

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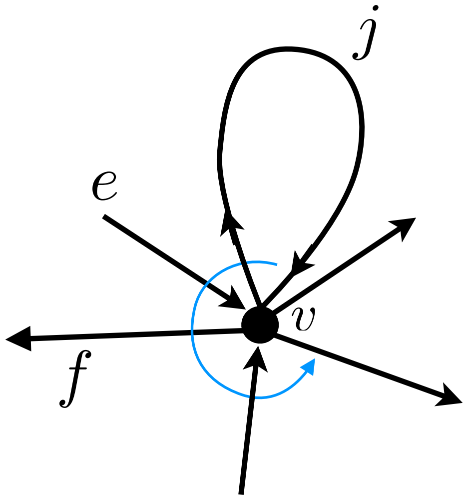

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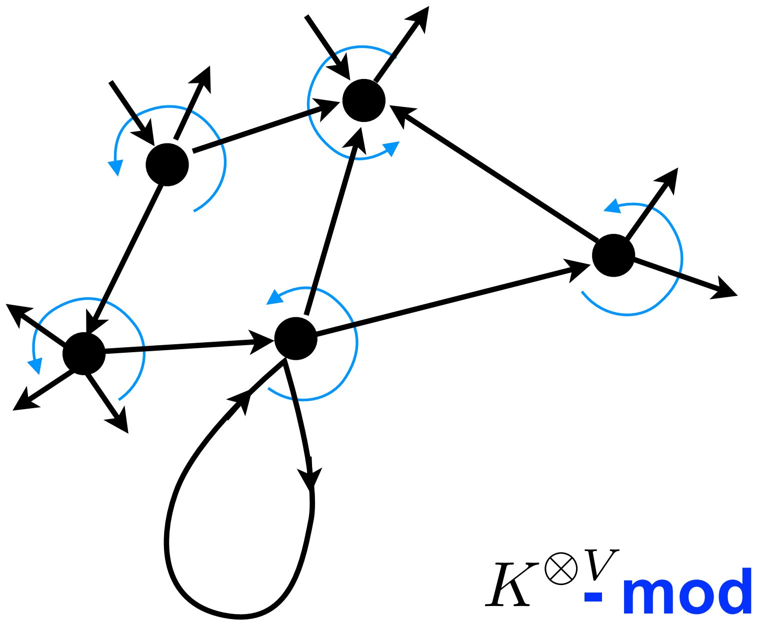

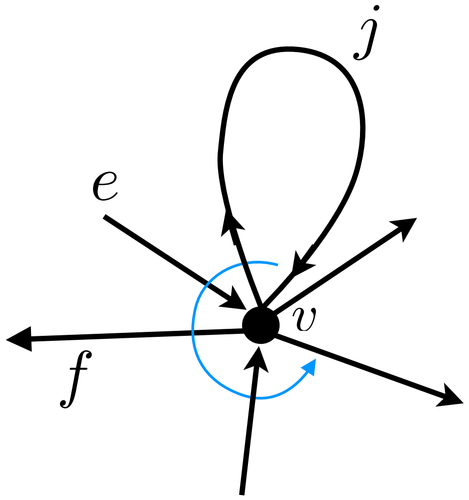

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Locality condition

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- cut ribbon graph in vertex discs

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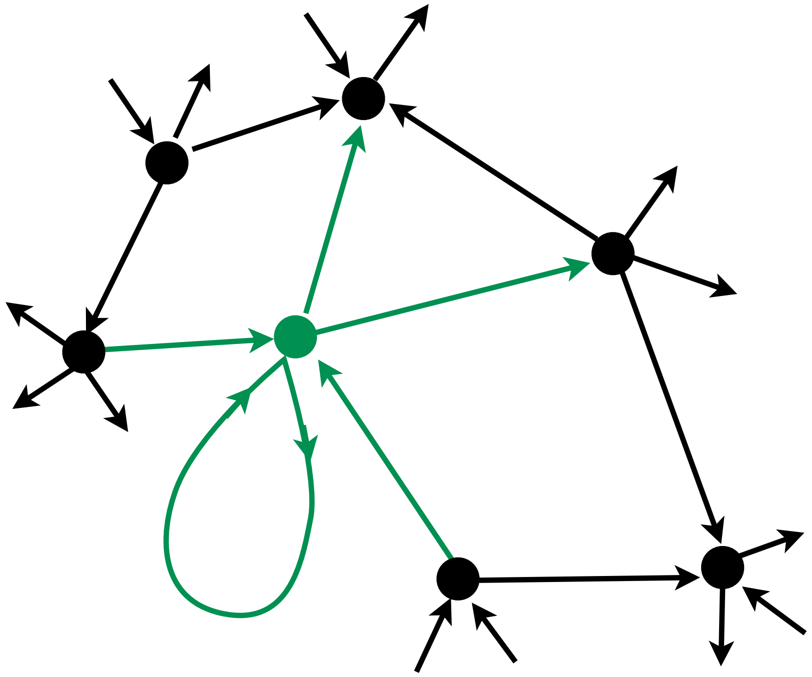

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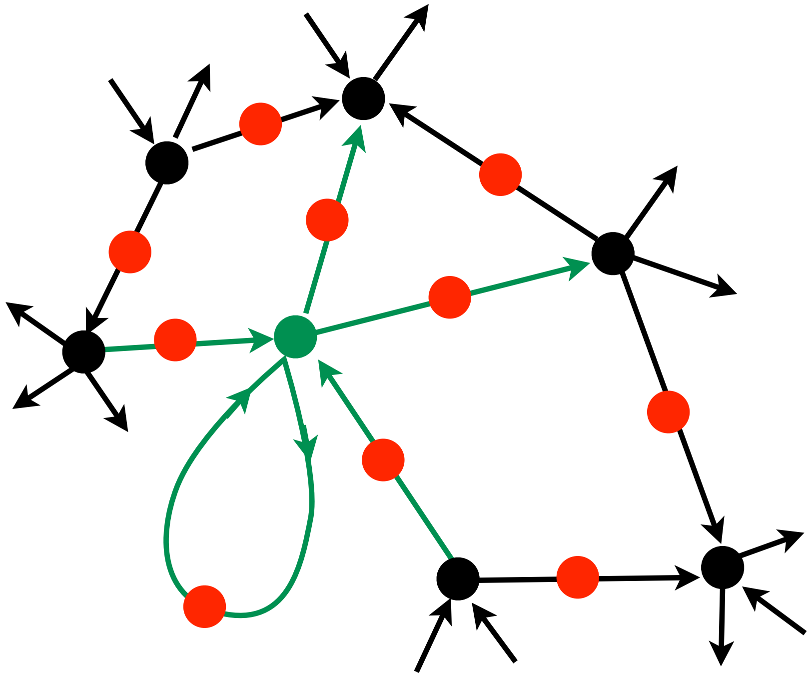

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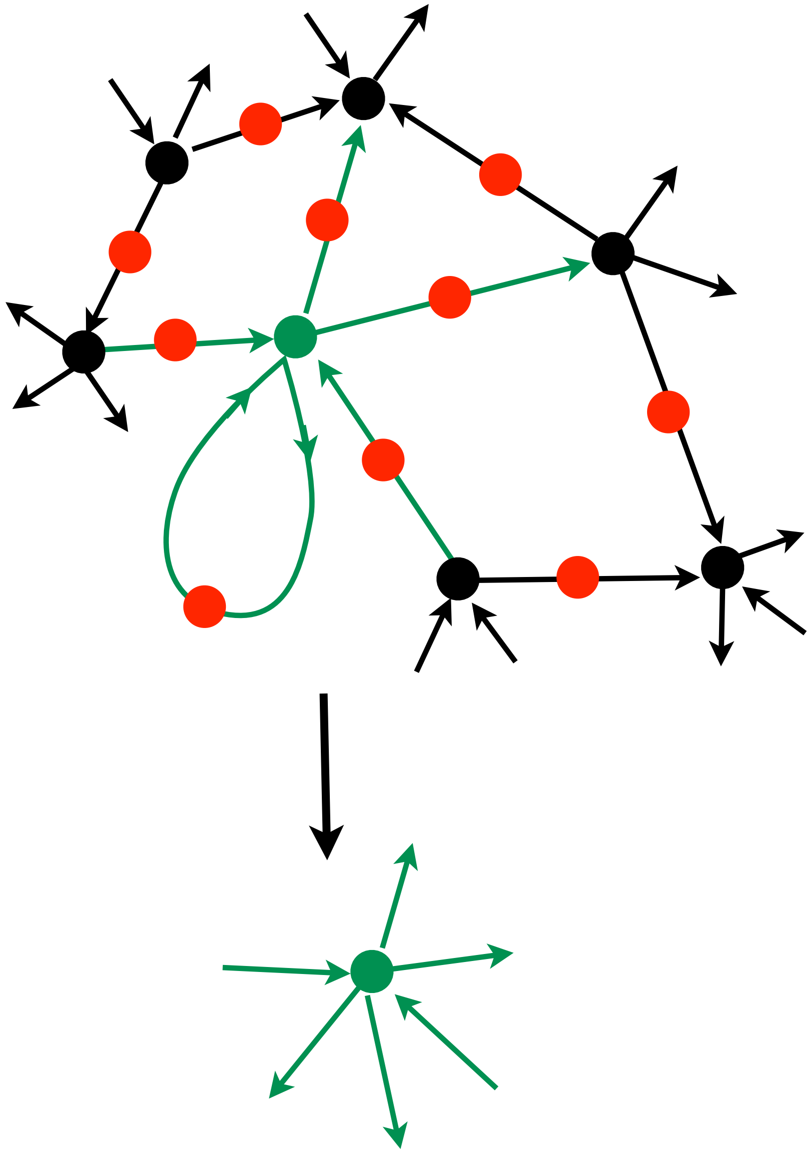

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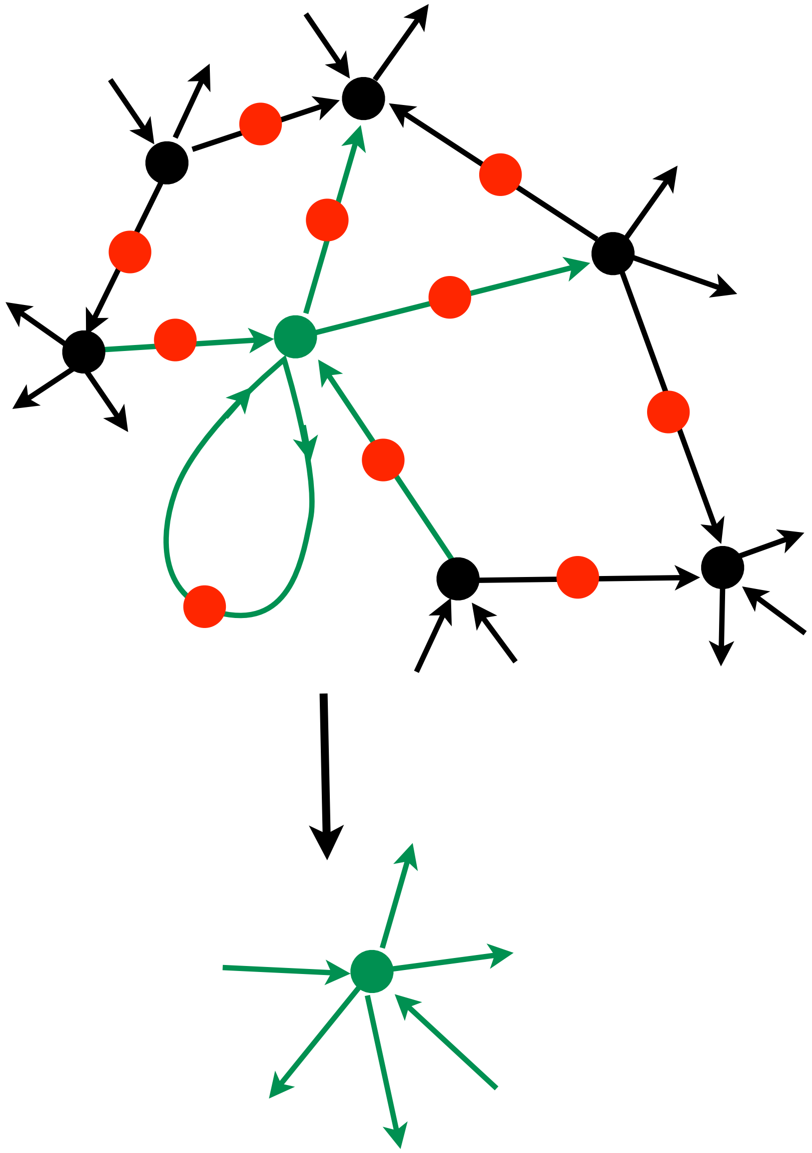
$K^{\otimes V}$-module algebra structure on $K^{*\otimes E}$
induced by K -module algebra structures on vertex discs



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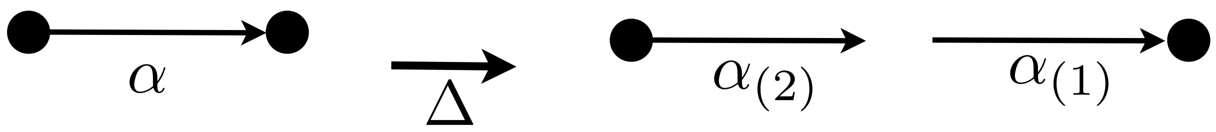

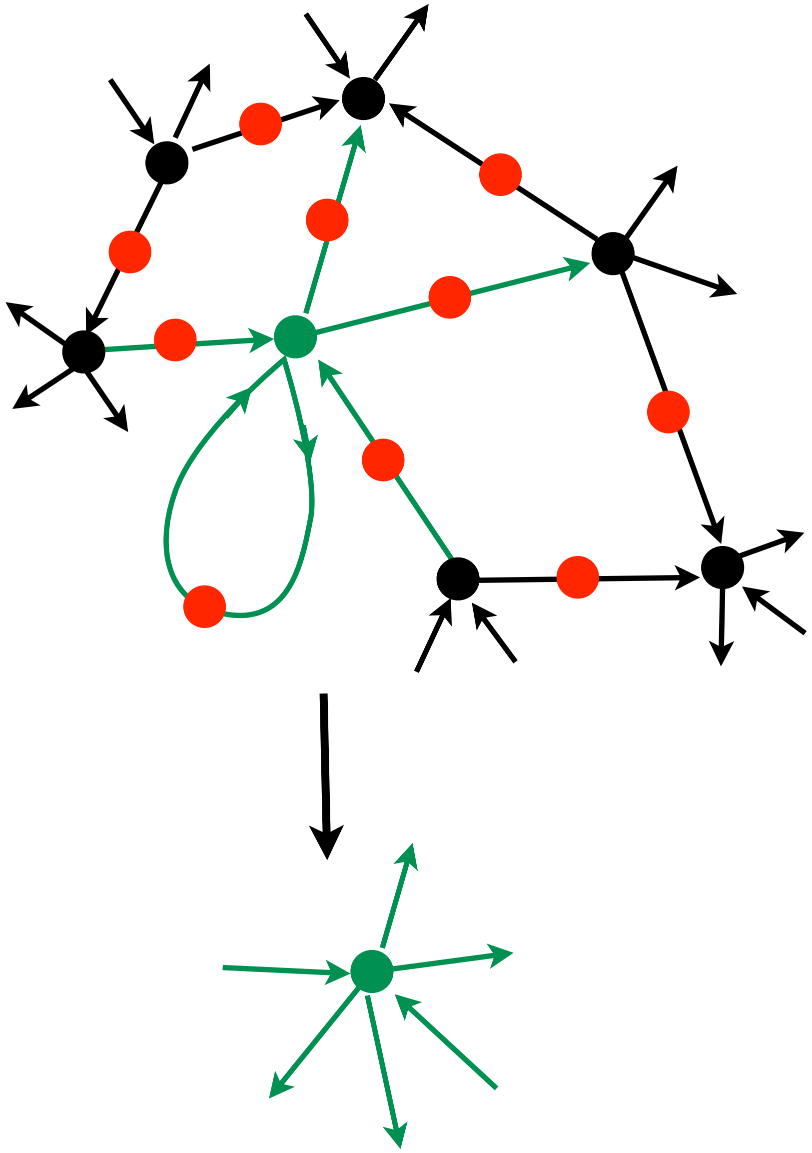

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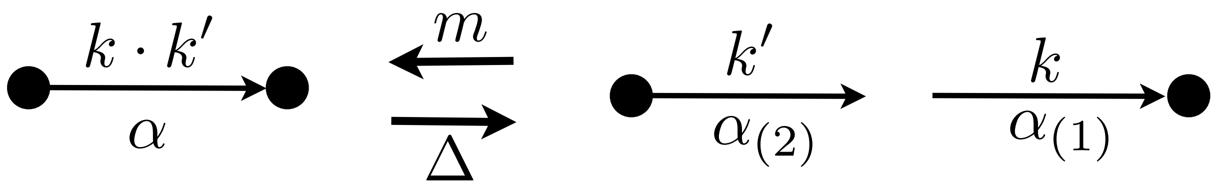

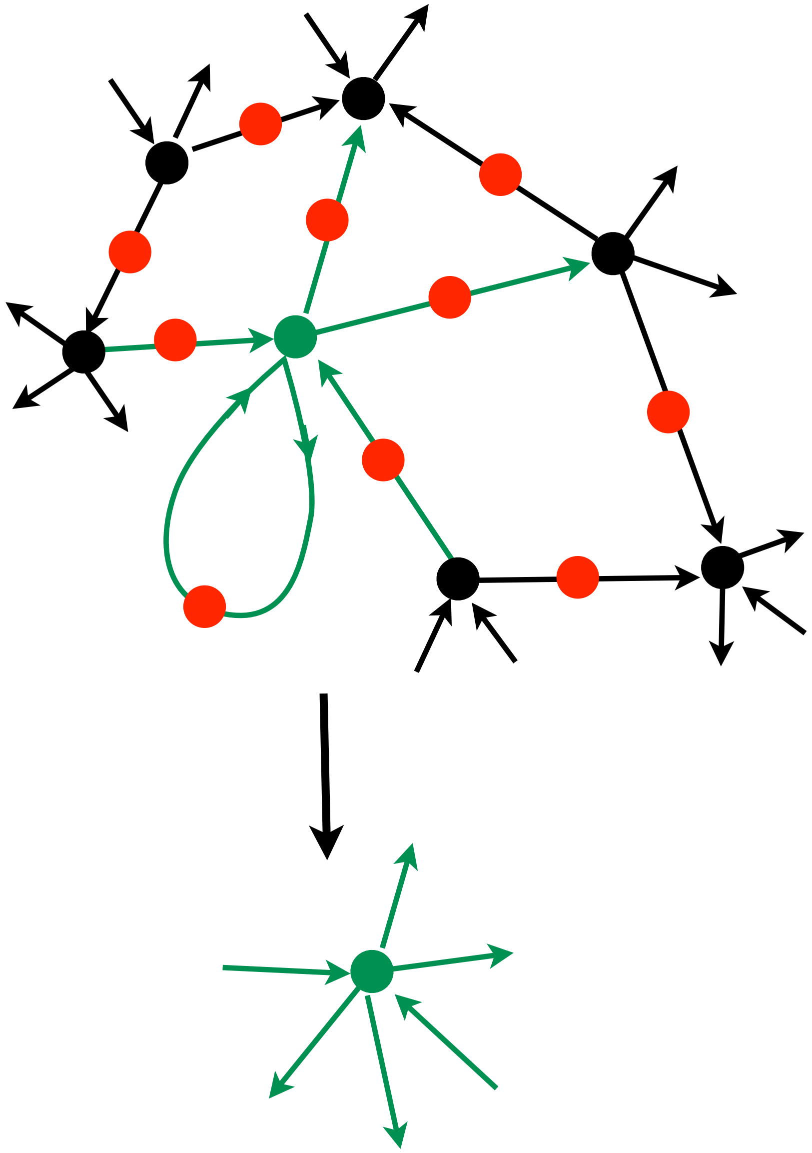

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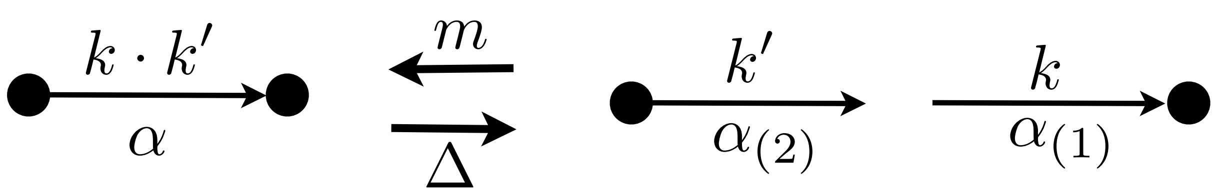
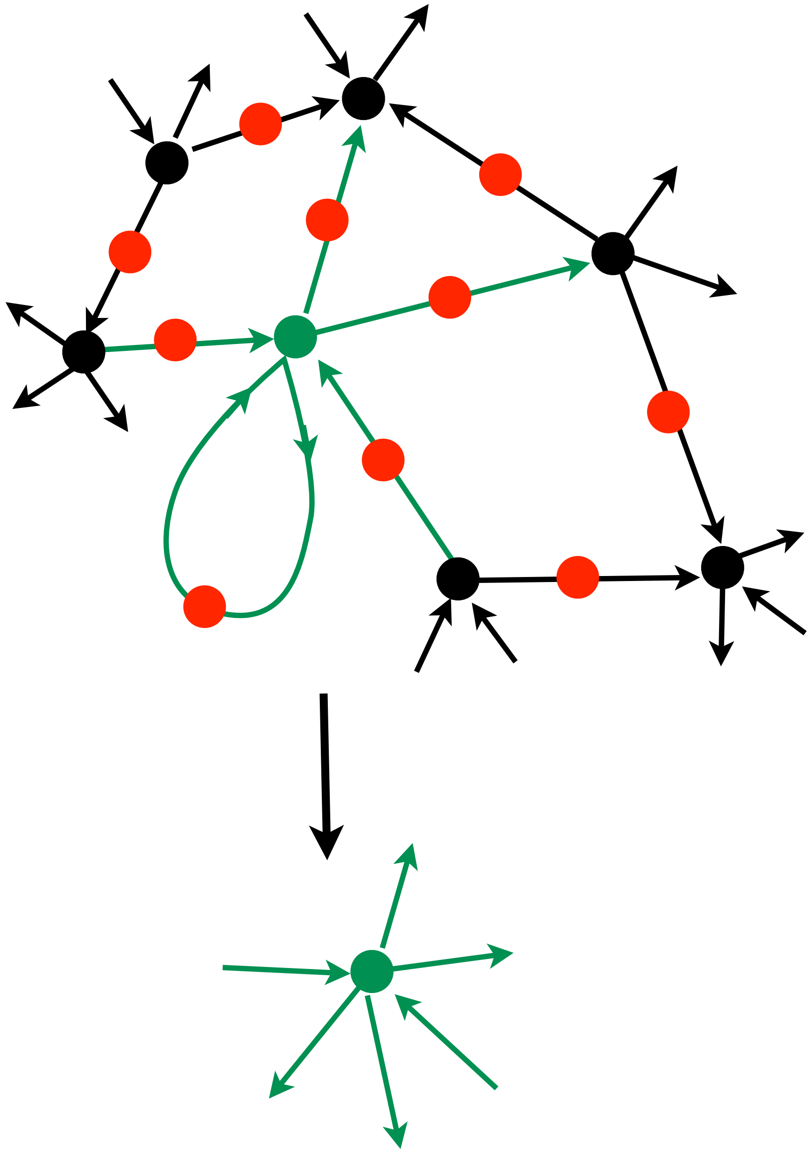
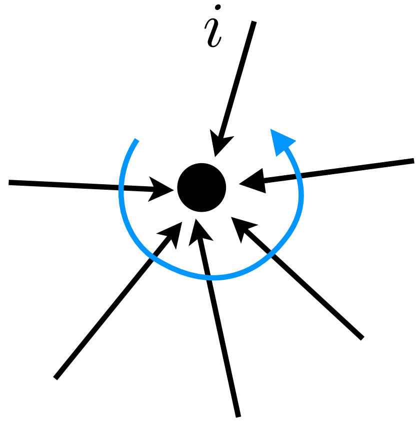




Hopf algebra gauge theory on a vertex disc



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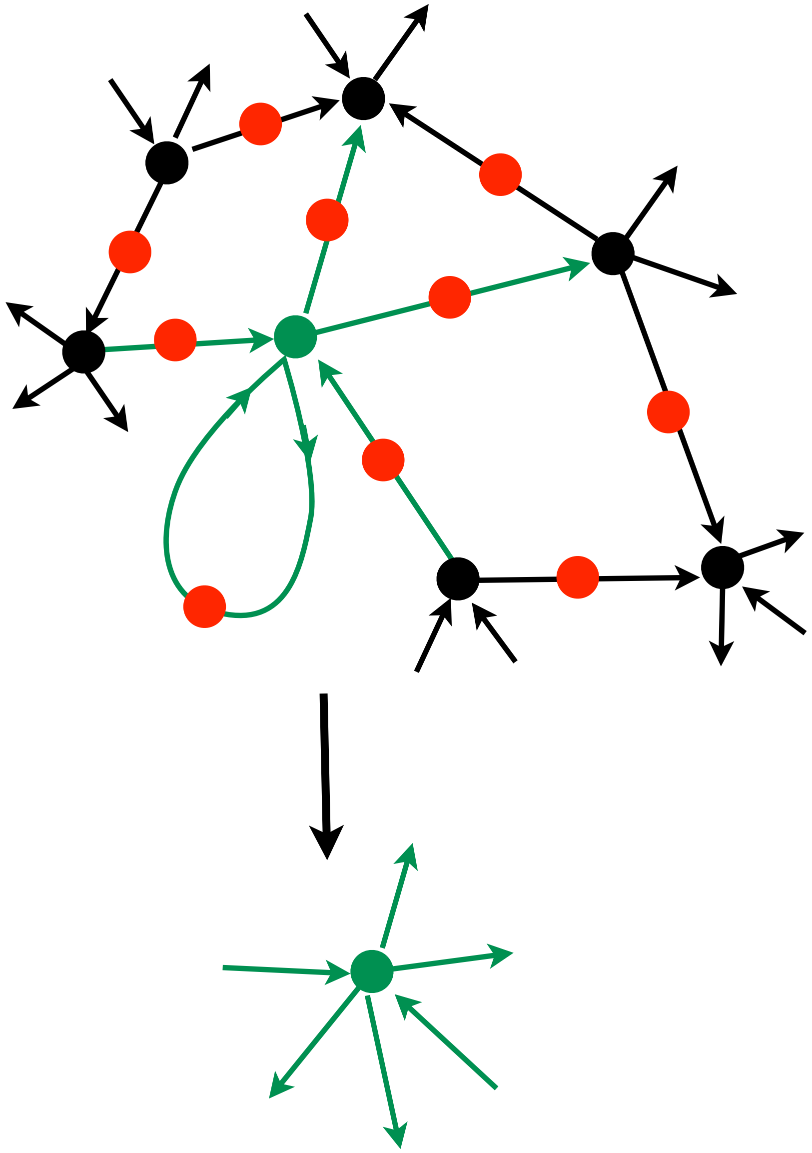

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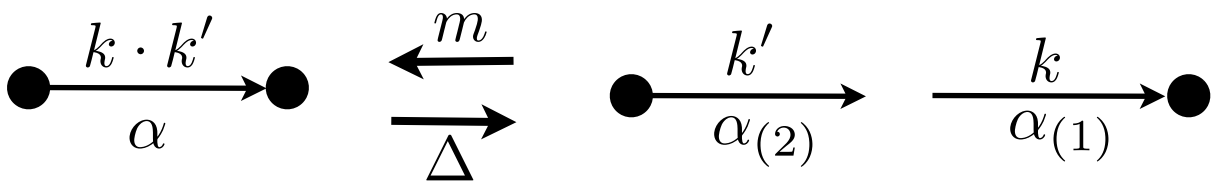

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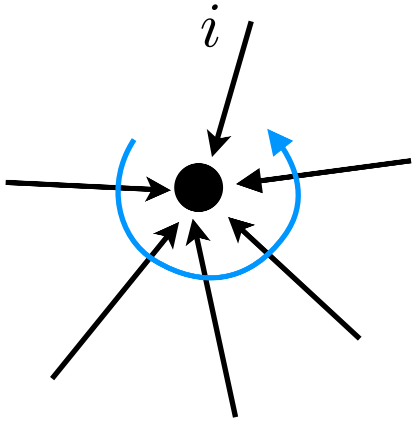

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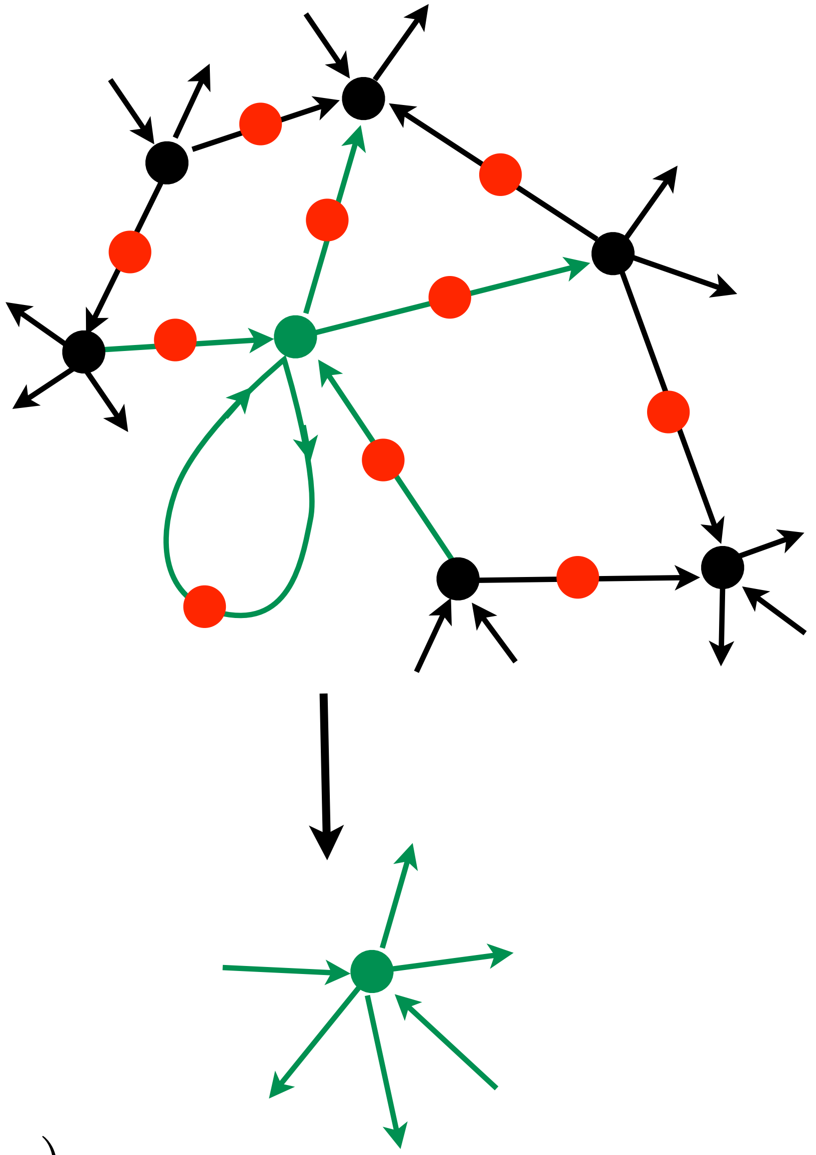

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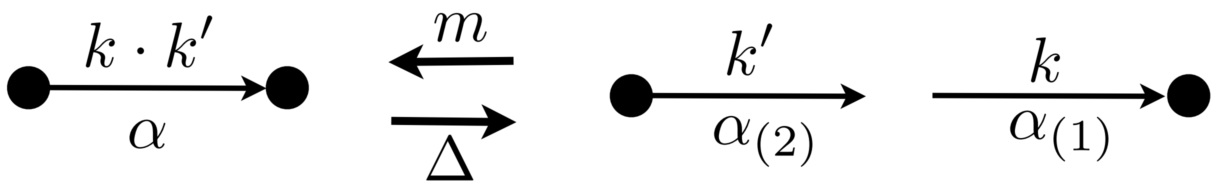

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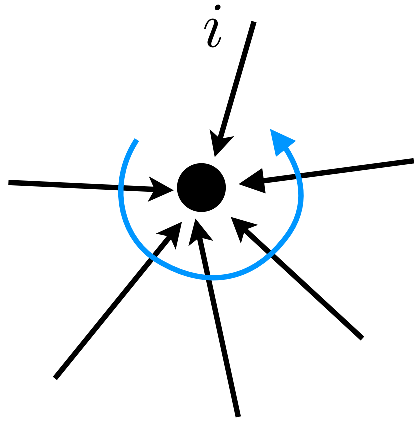

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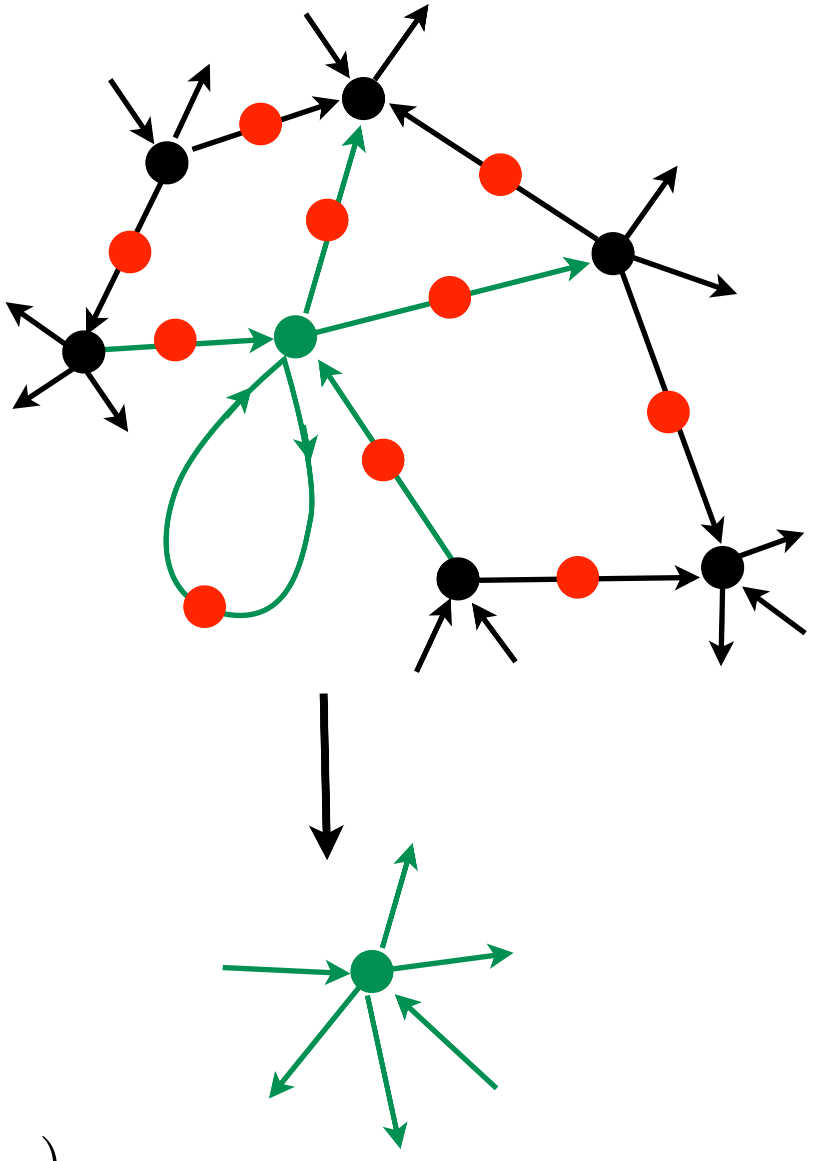

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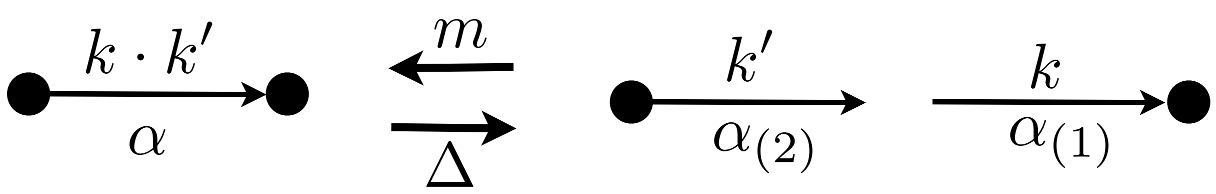

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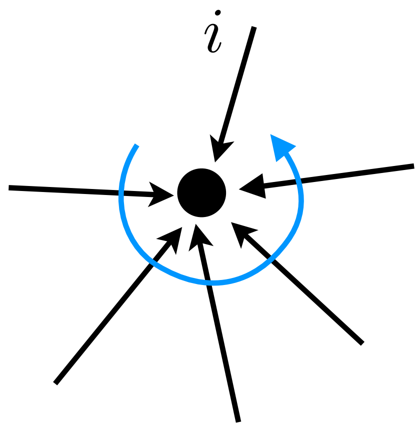

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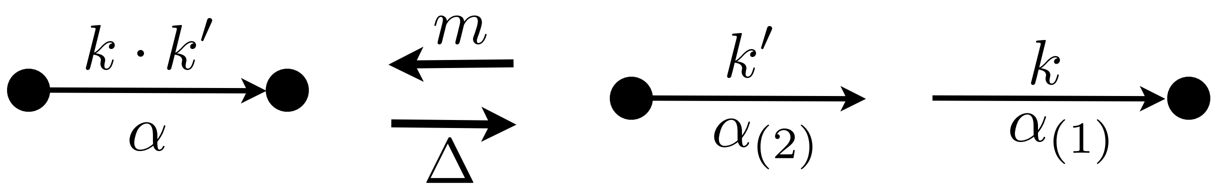

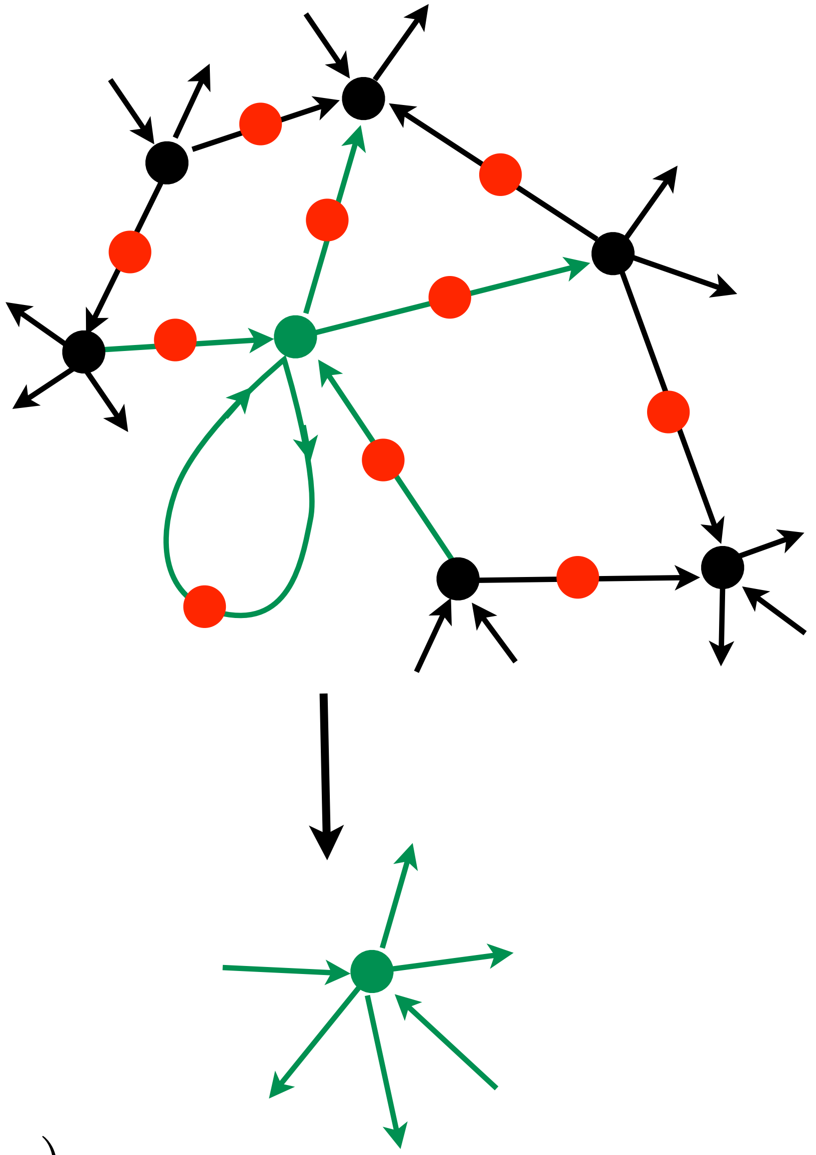

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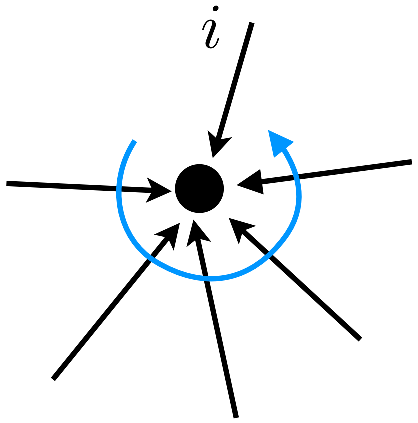

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❌ **works only for K cocommutative**

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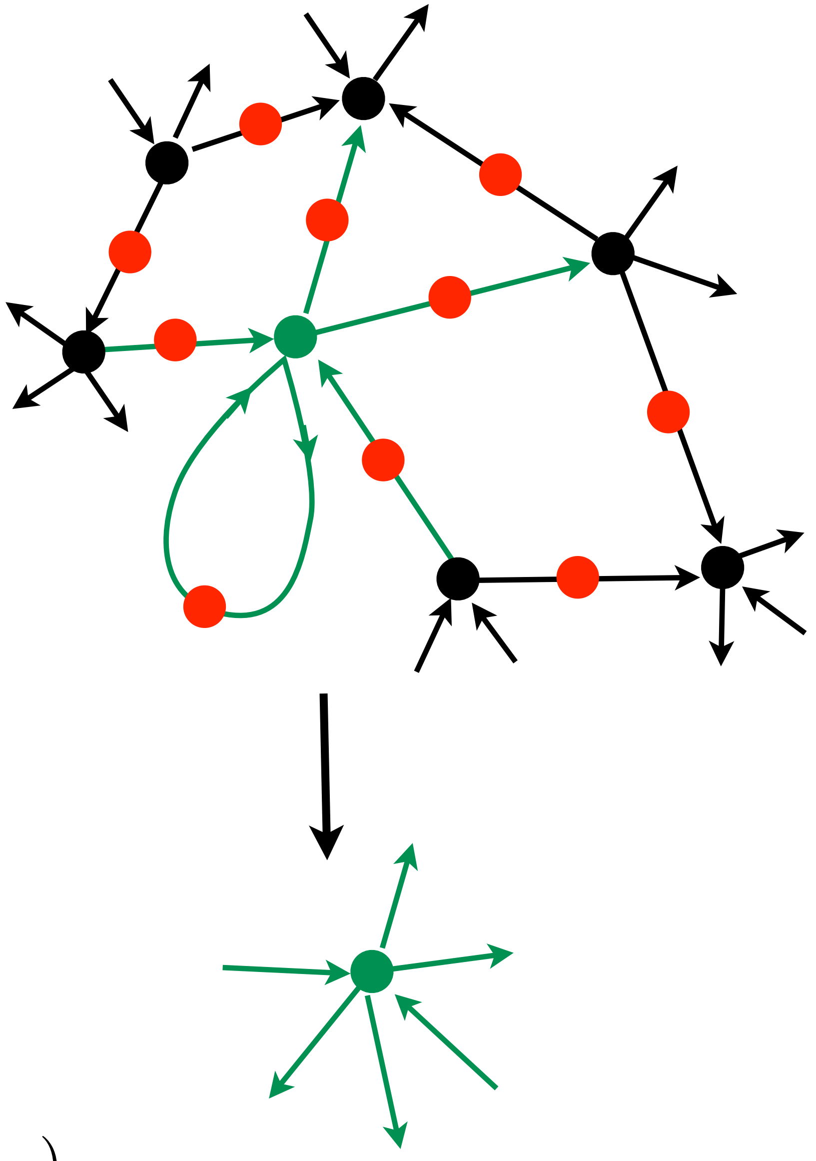

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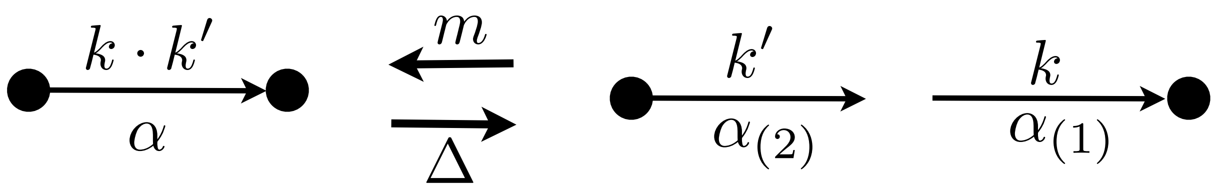

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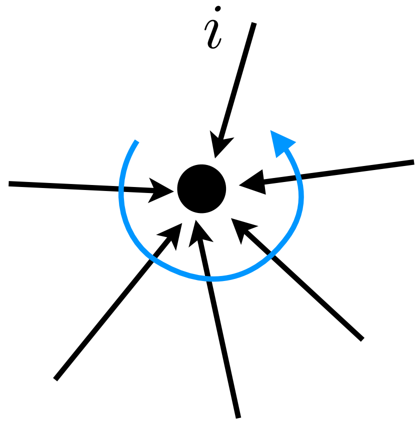

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$$(\alpha\otimes\beta)\lhd h = ((\alpha\otimes 1)\cdot(1\otimes\beta))\lhd h = (\alpha\lhd h_{(1)})\otimes(\beta\lhd h_{(2)})$$

$$= ((1\otimes\beta)\cdot(\alpha\otimes 1))\lhd h = (\alpha\lhd h_{(2)})\otimes(\beta\lhd h_{(1)})$$

Locality condition

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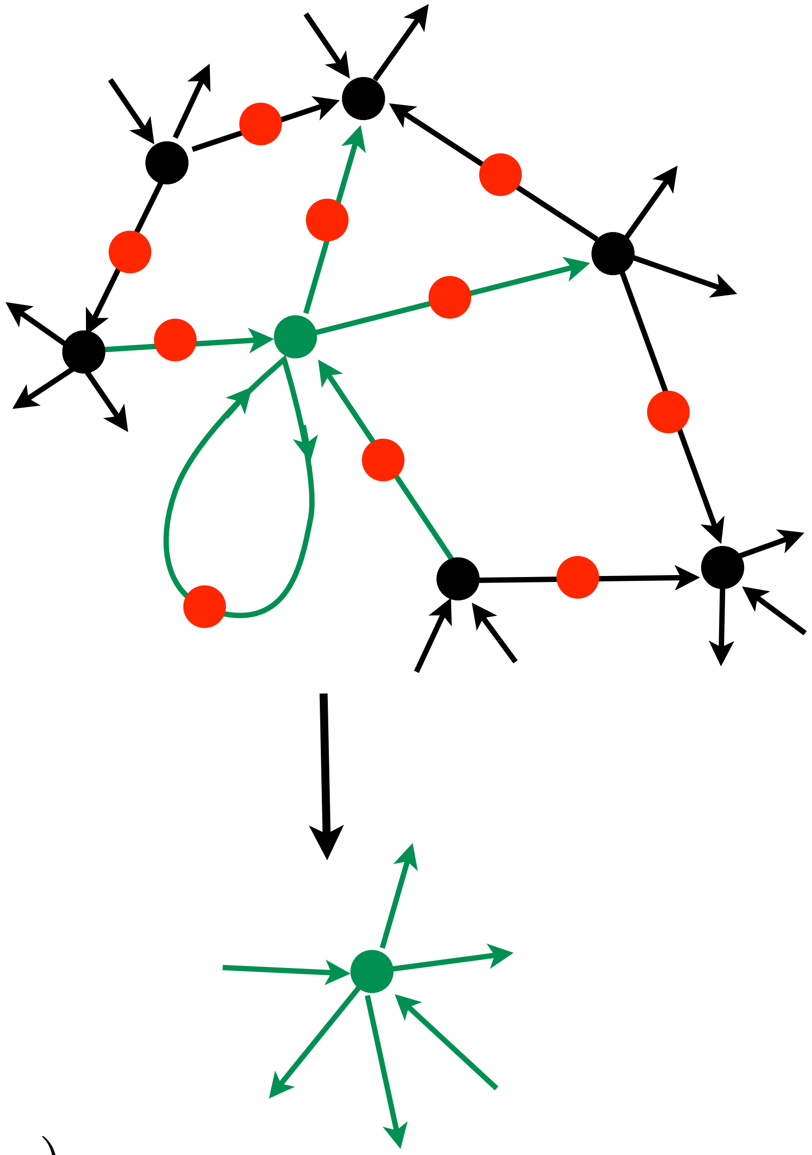
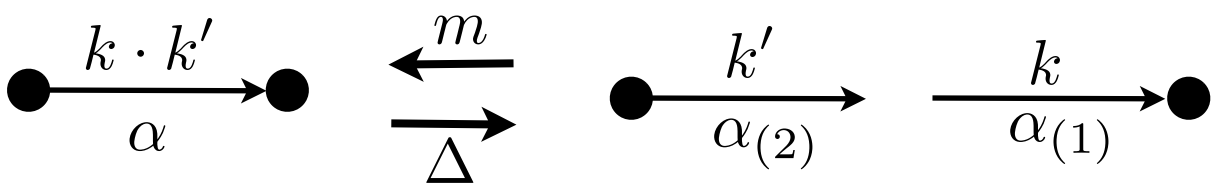
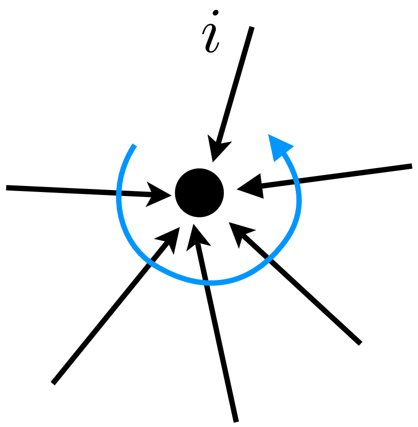


$K^{\otimes V}$-module algebra structure on $K^{*\otimes E}$
induced by K -module algebra structures on vertex discs

$$G^* : K^{*\otimes E} \to K^{*\otimes 2E} \cong \otimes_{v\in V} K^{*\otimes |v|}$$



Hopf algebra gauge theory on a vertex disc



- K -module algebra structure on $K^{*\otimes|v|}$
- action on single edge $(\alpha)_i \lhd h = \langle \alpha_{(1)}, h\rangle\, (\alpha_{(2)})_i$
- algebra structure? **naively:** tensor product $K^{*\otimes|v|}$

❌ **works only for K cocommutative**

$$(\alpha\otimes\beta)\lhd h = ((\alpha\otimes 1)\cdot(1\otimes\beta))\lhd h = (\alpha\lhd h_{(1)})\otimes(\beta\lhd h_{(2)})$$
$$= ((1\otimes\beta)\cdot(\alpha\otimes 1))\lhd h = (\alpha\lhd h_{(2)})\otimes(\beta\lhd h_{(1)})$$

➪ need structure to relate $\Delta(h)=h_{(1)}\otimes h_{(2)}$ and $\Delta^{op}(h)=h_{(2)}\otimes h_{(1)}$

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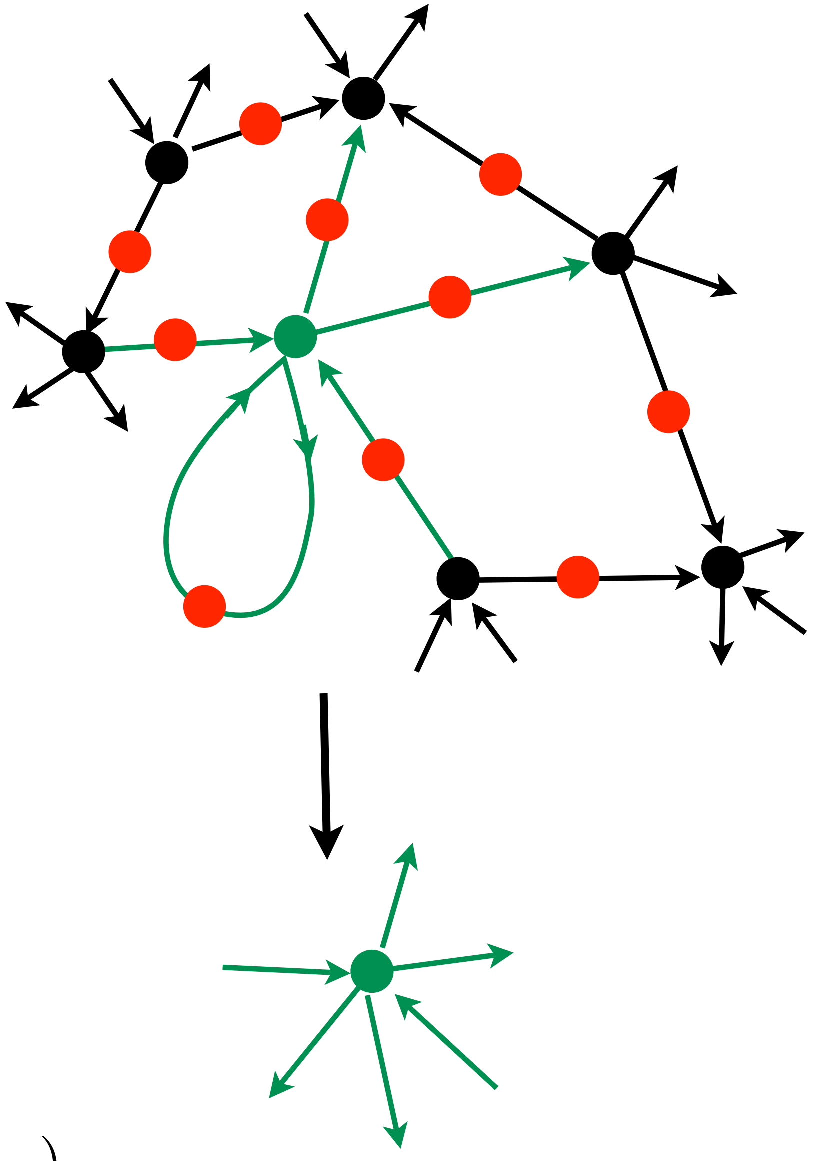

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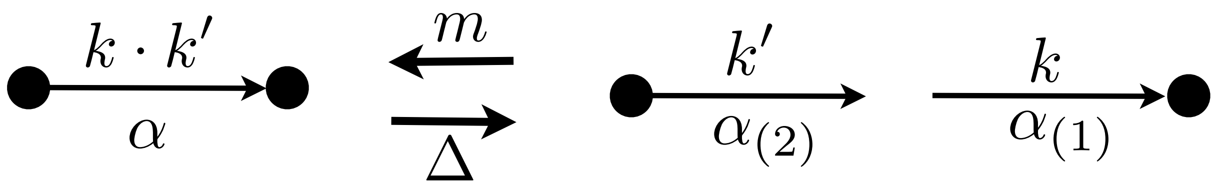

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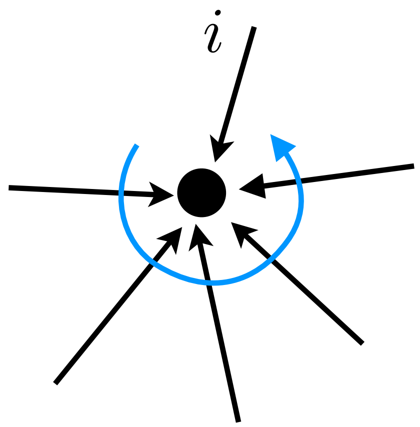

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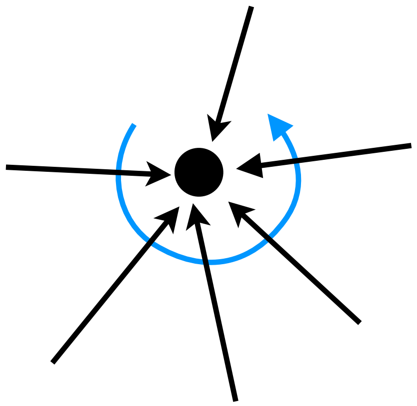
➩ require: K quasitriangular: $\Delta^{op} = R\cdot\Delta\cdot R^{-1}$

Hopf algebra gauge theory on a vertex disc



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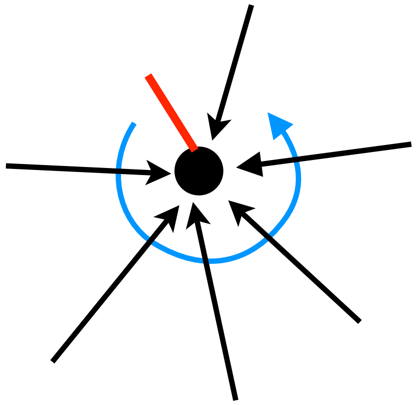

ordering of edge ends at v

Hopf algebra gauge theory on a vertex disc

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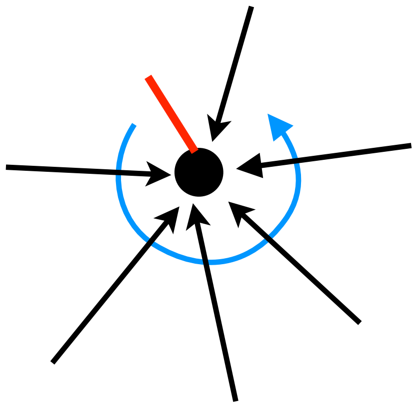

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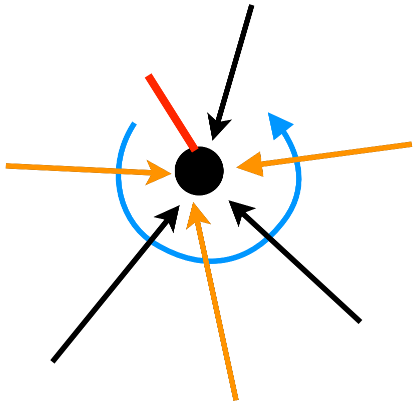

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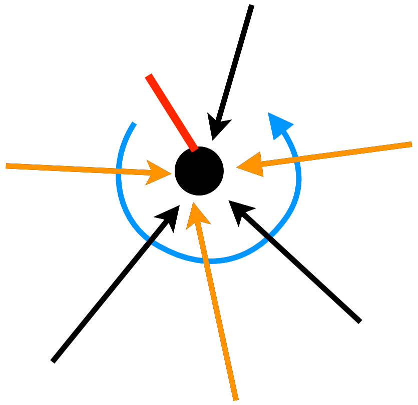

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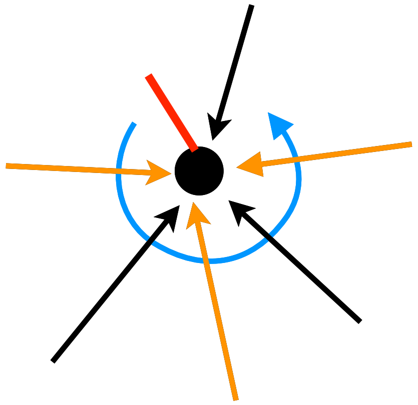

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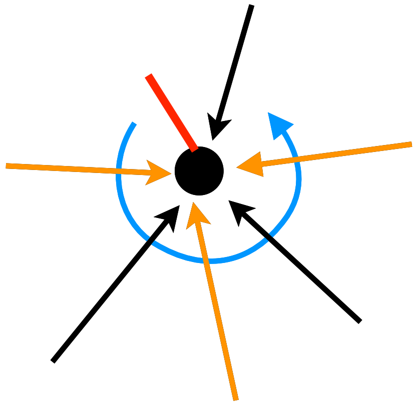

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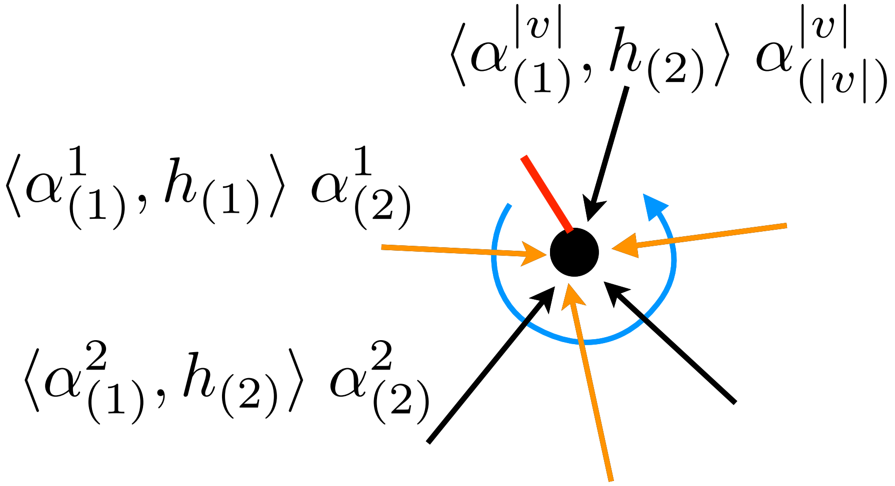

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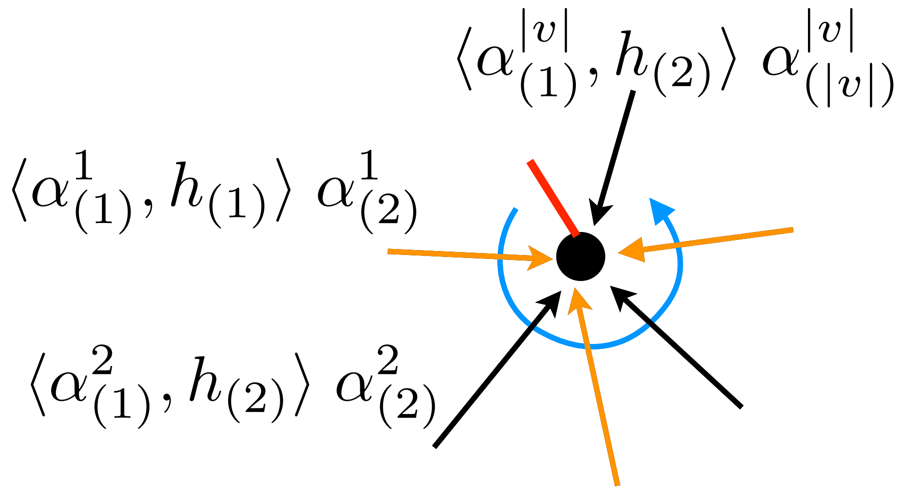

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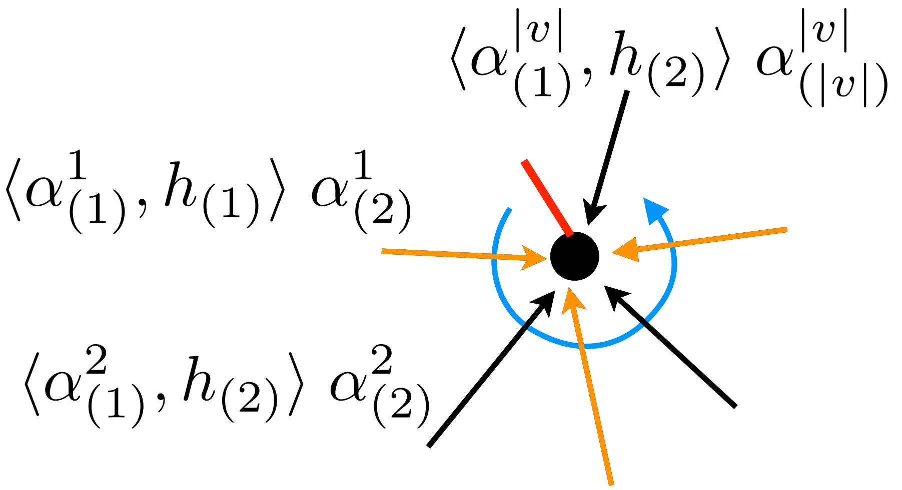

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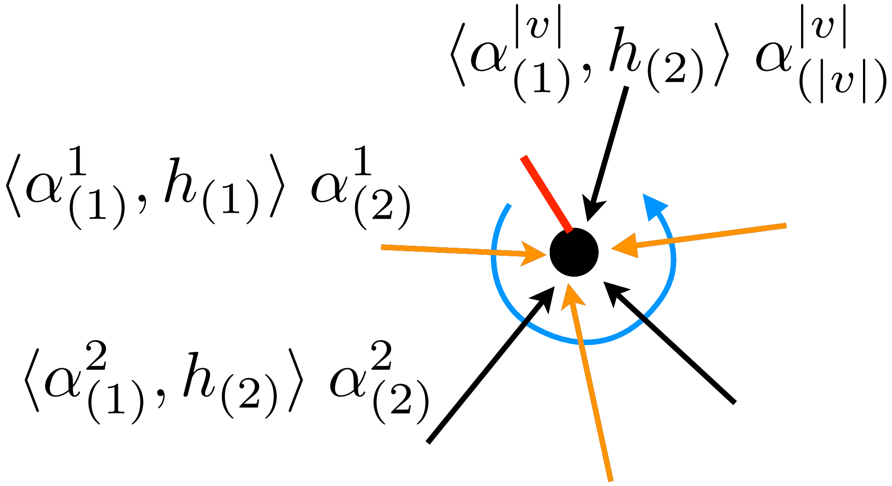

$\langle \alpha^{|v|}_{(1)}, h_{(2)}\rangle \; \alpha^{|v|}_{(|v|)}$

$\langle \alpha^1_{(1)}, h_{(1)}\rangle \; \alpha^1_{(2)}$

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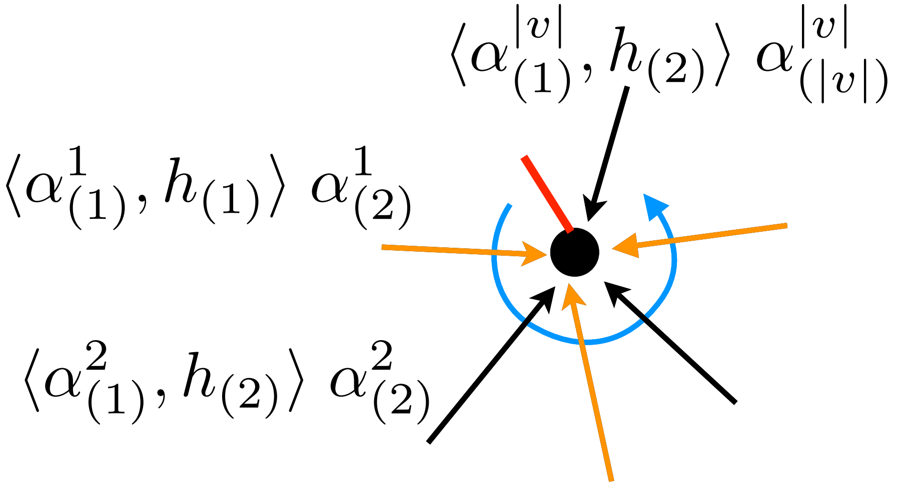

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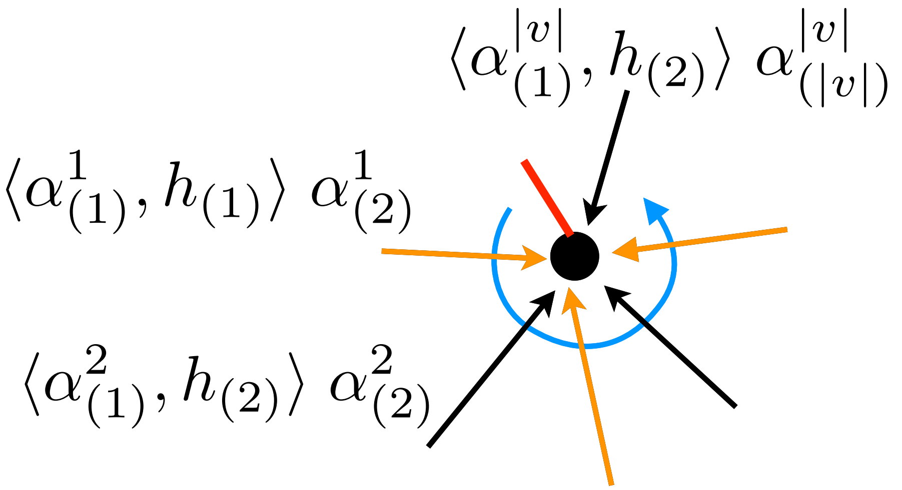

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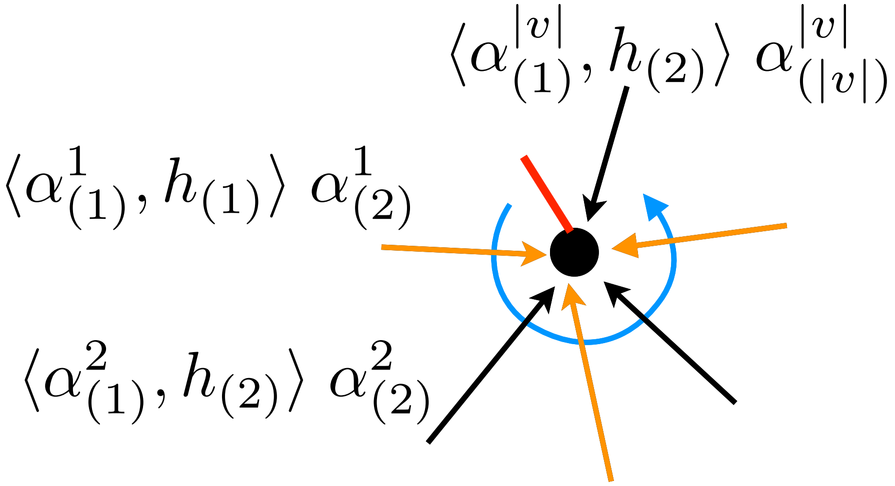

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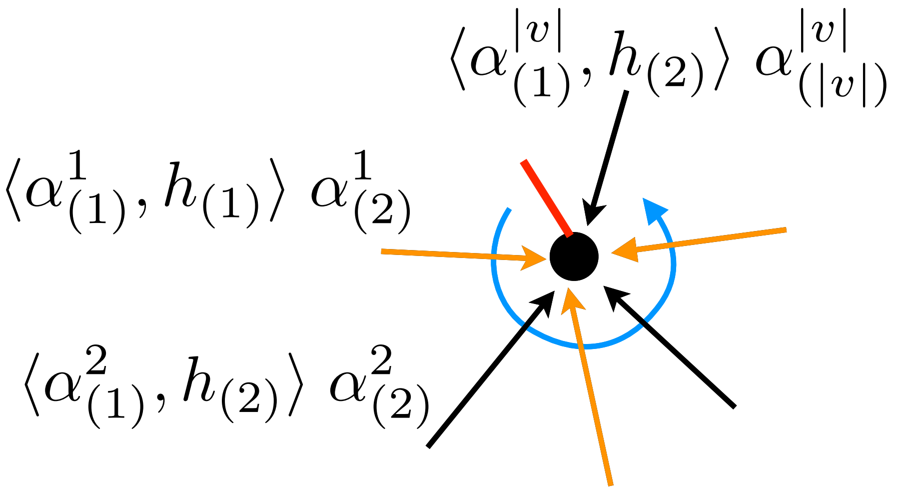

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➪ vertex disc with arbitrary edge orientation:
require that $T^* : K^* \to K^*$ is algebra and module isomorphism

Hopf algebra gauge theory on a ribbon graph

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- select one of the ends for each edge $e \in E(\Gamma)$, sets $I_v \subset \{1, ..., |v|\}$

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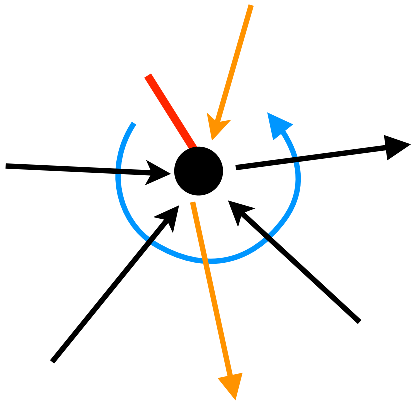

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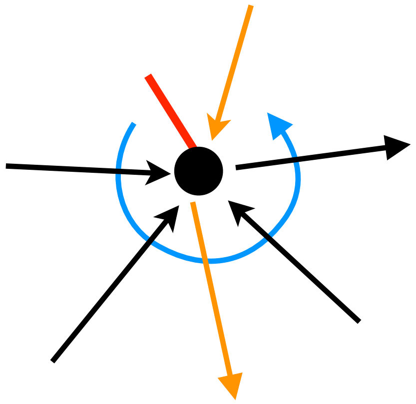

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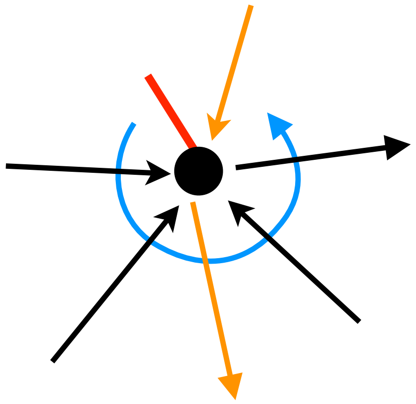

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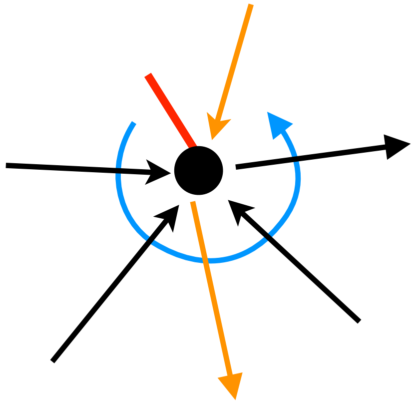
➪ **Hopf algebra gauge theory on vertex disc:** K- module algebra $\mathcal{A}_v^*$

Hopf algebra gauge theory on a ribbon graph



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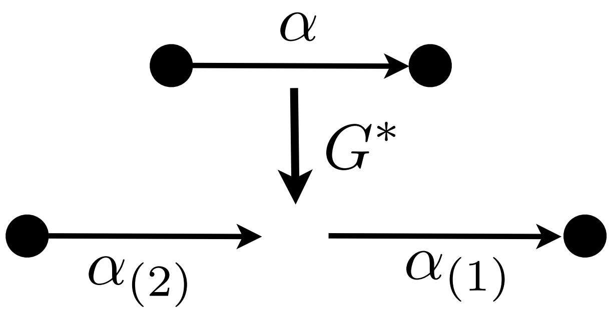
gluing vertex discs

linear map $G^* : K^{*\otimes E} \to \otimes_{v \in V} \mathcal{A}_v^*$



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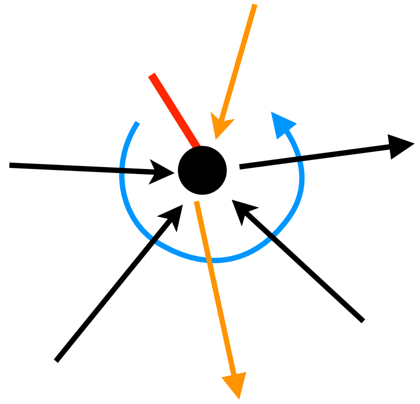

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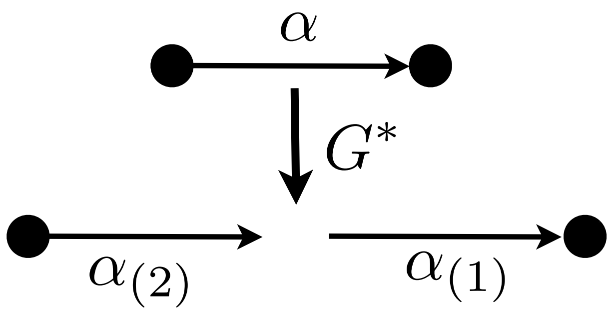
gluing vertex discs ↓ linear map $G^* : K^{*\otimes E} \to \otimes_{v \in V} \mathcal{A}_v^*$



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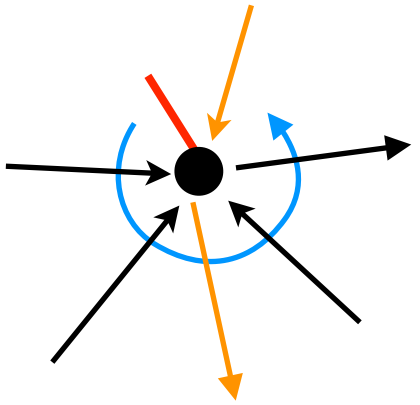

- select one of the ends for each edge $e \in E(\Gamma)$, sets $I_v \subset \{1, ..., |v|\}$
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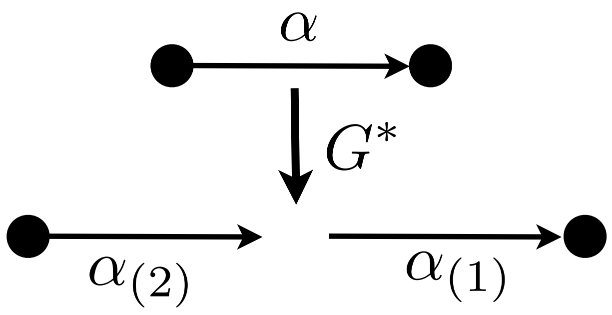

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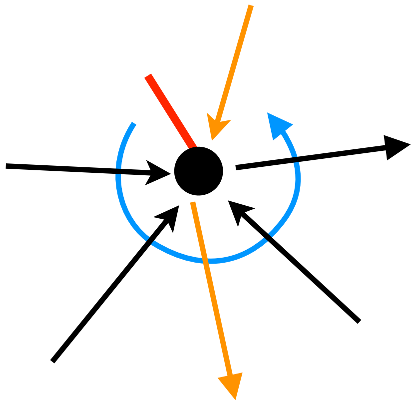

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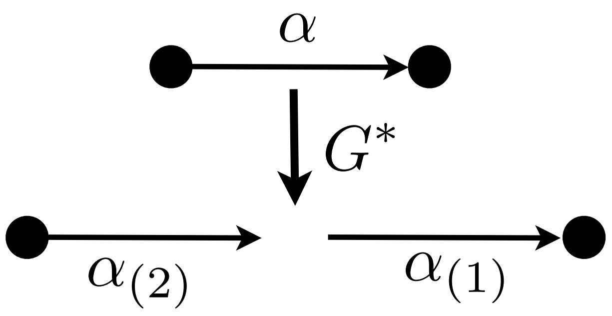

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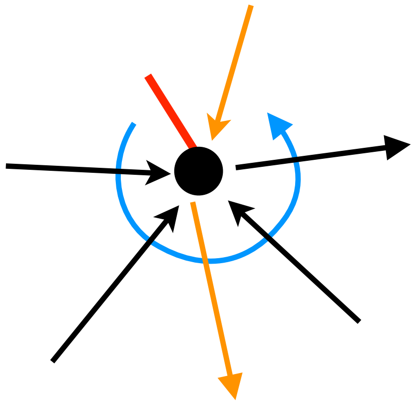

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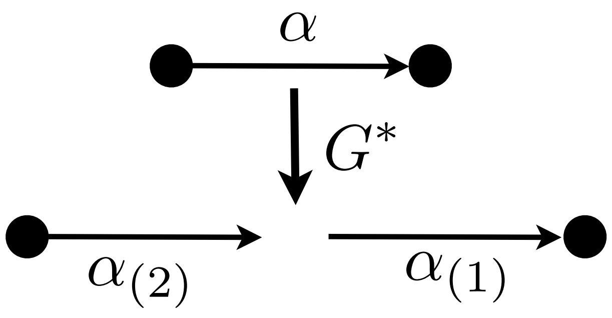

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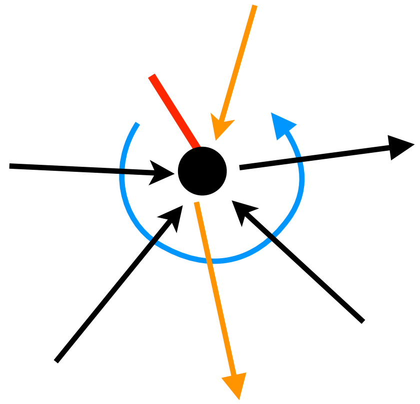

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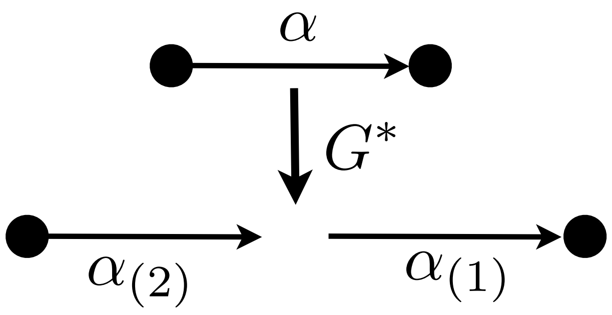

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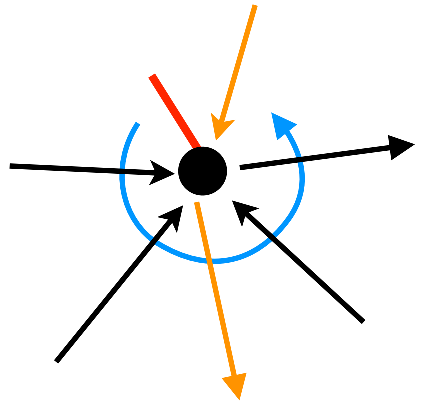

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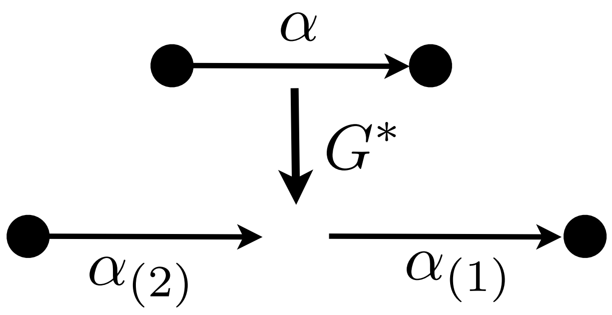

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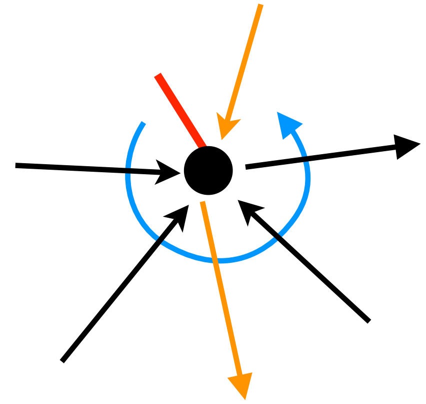

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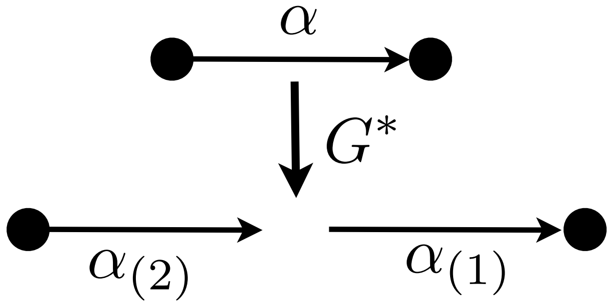

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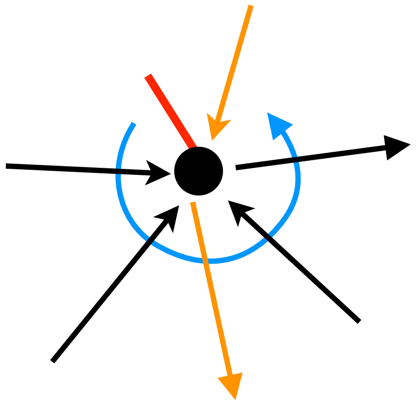

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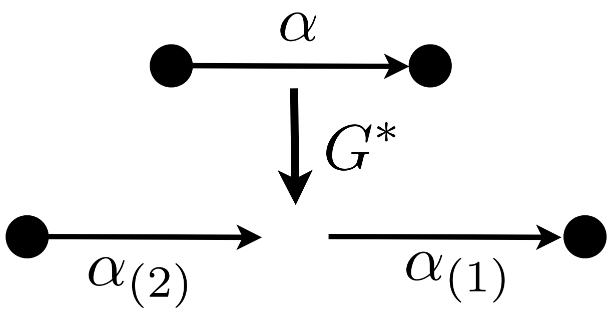

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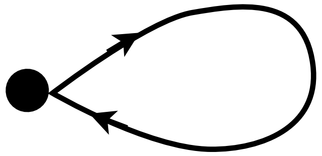

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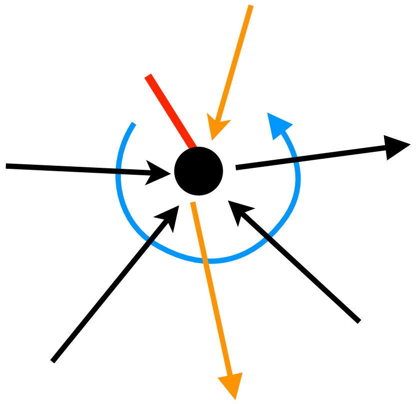

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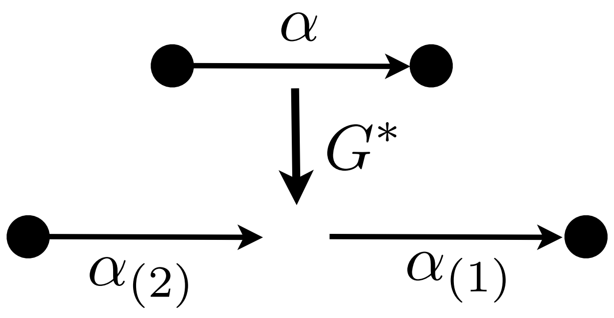

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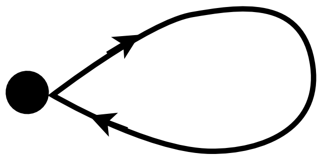

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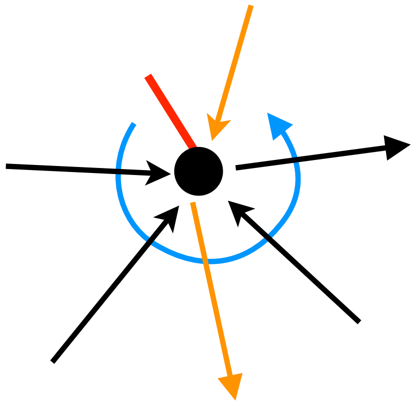

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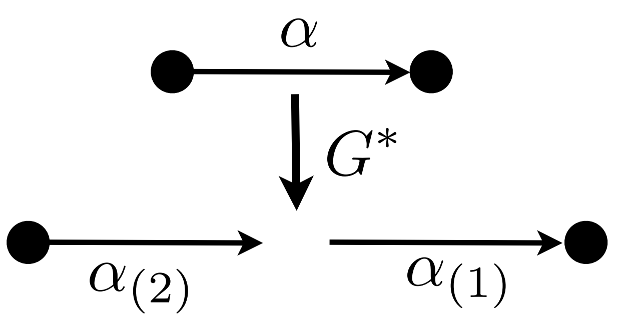

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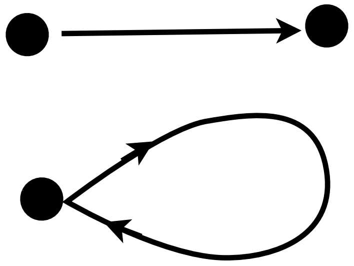

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⇨ lattice algebra from combinatorial quantisation of Chern-Simons theory
[Alekseev, Grosse, Schomerus '94] , [Buffenoir Roche '95]

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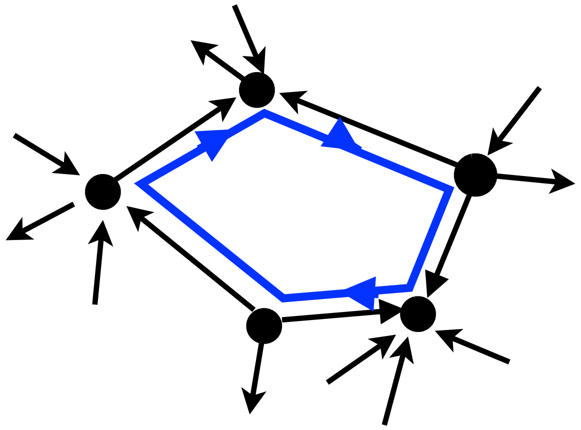

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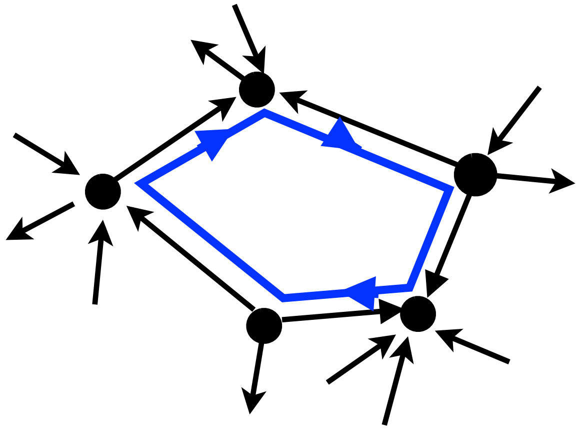

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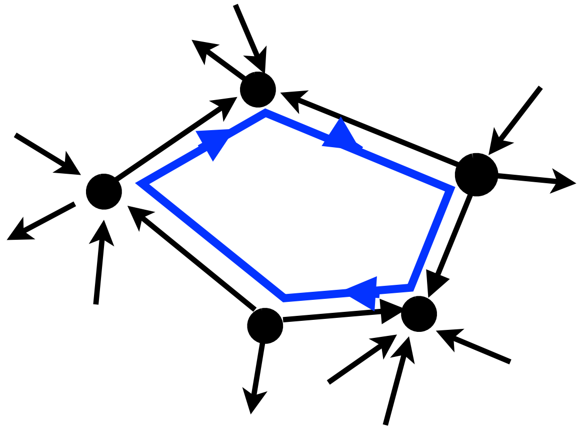

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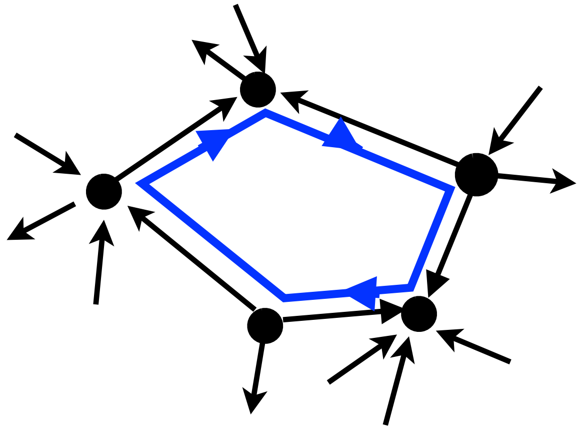

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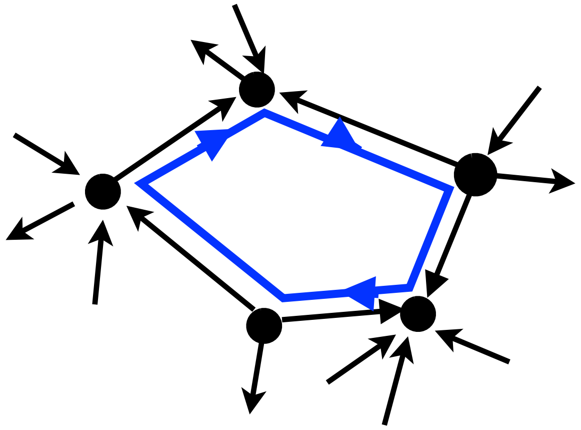

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defined by $\mathrm{Hol}_e = (\alpha)_e$
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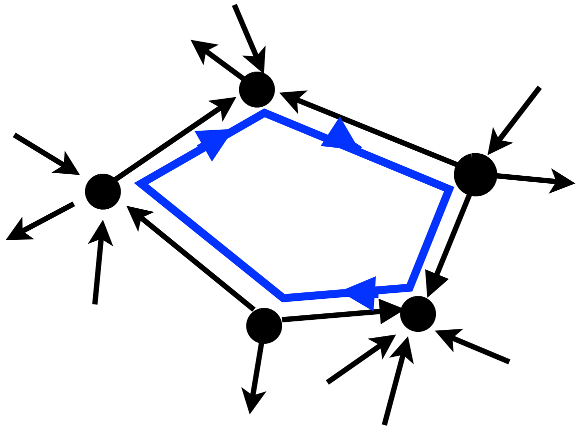

- **curvature:** holonomy of a **face**
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➩ associative algebra

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holonomies and curvature

suppose: K semisimple

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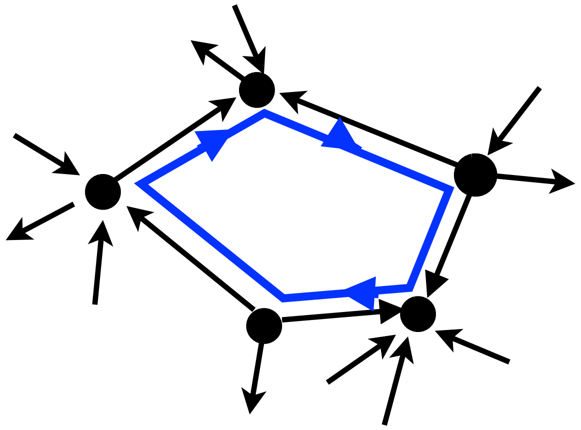

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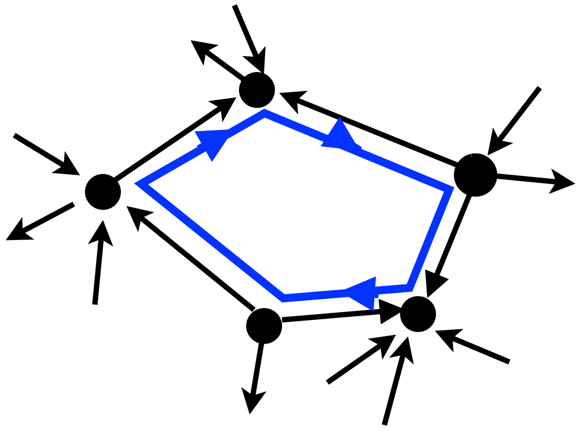

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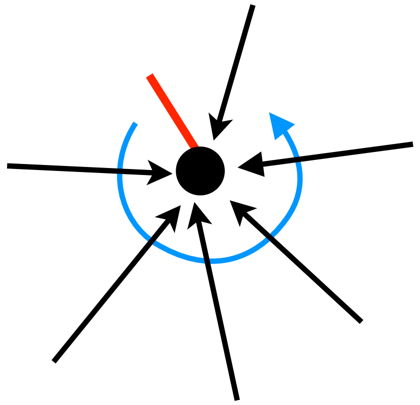

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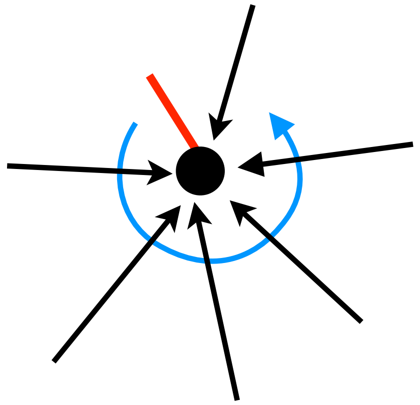

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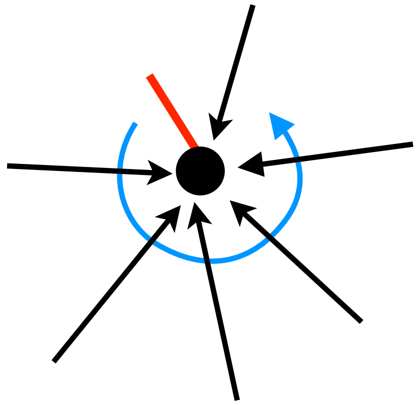

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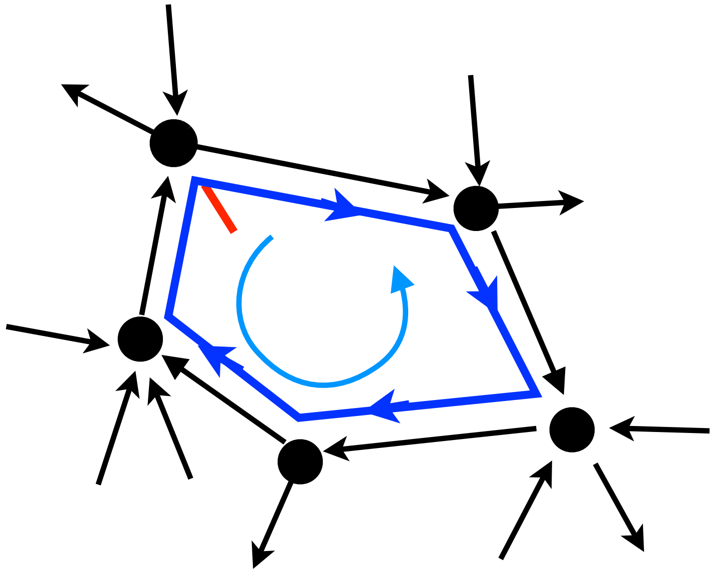

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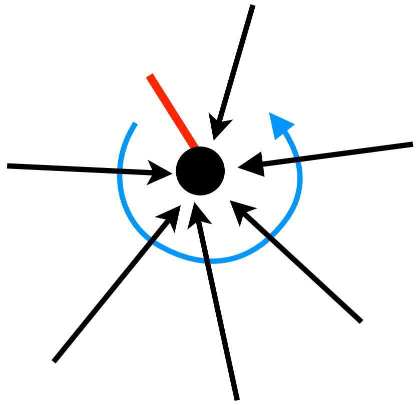

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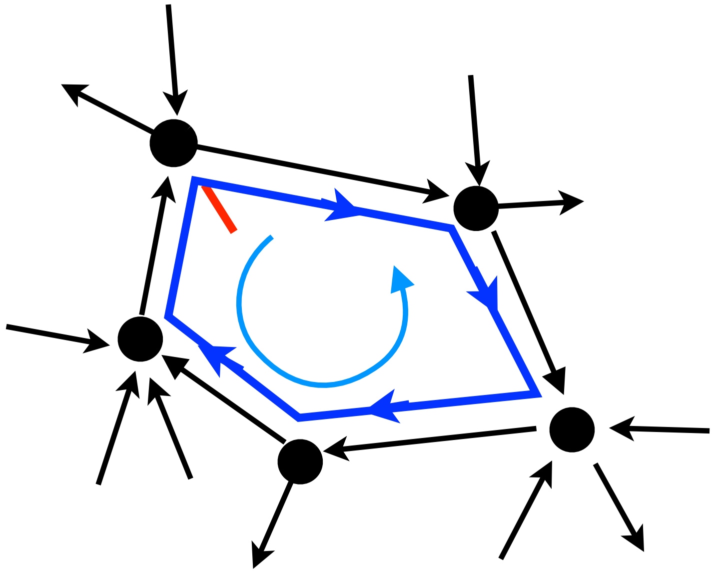

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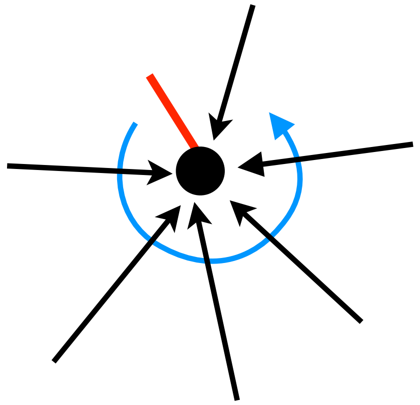

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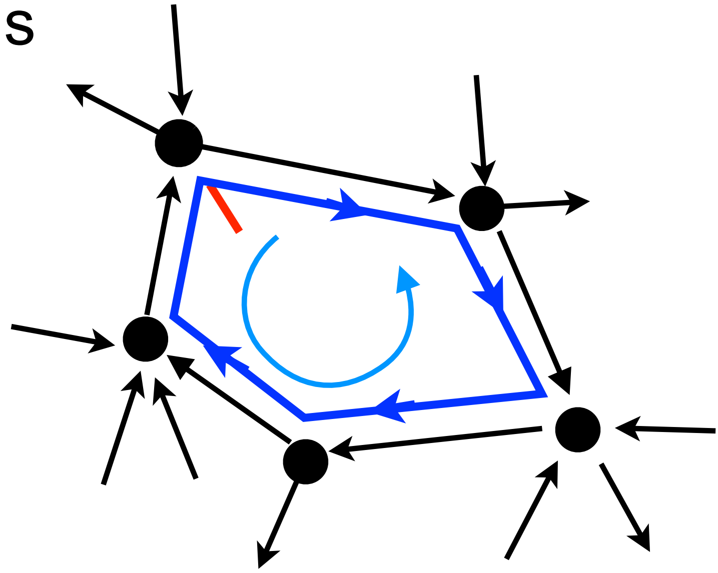

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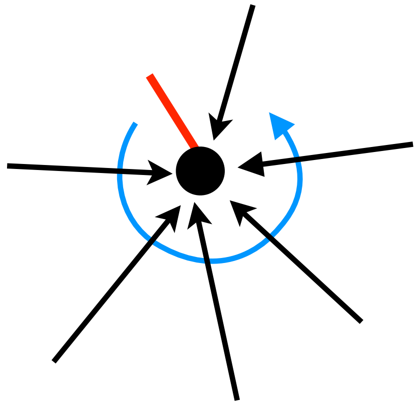

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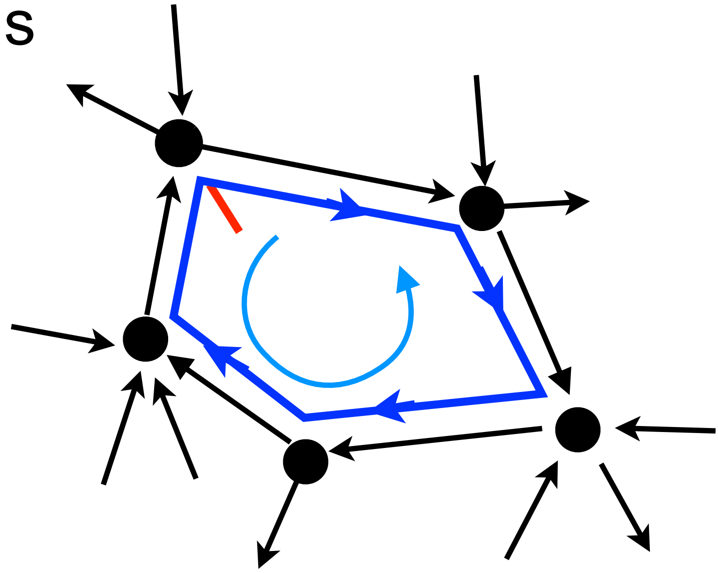

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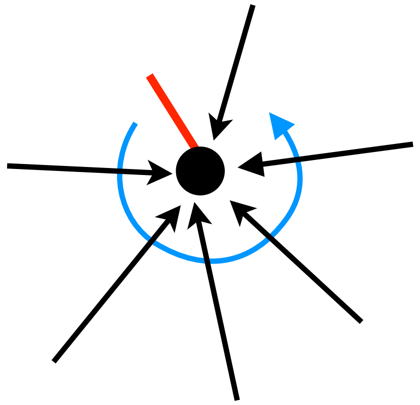

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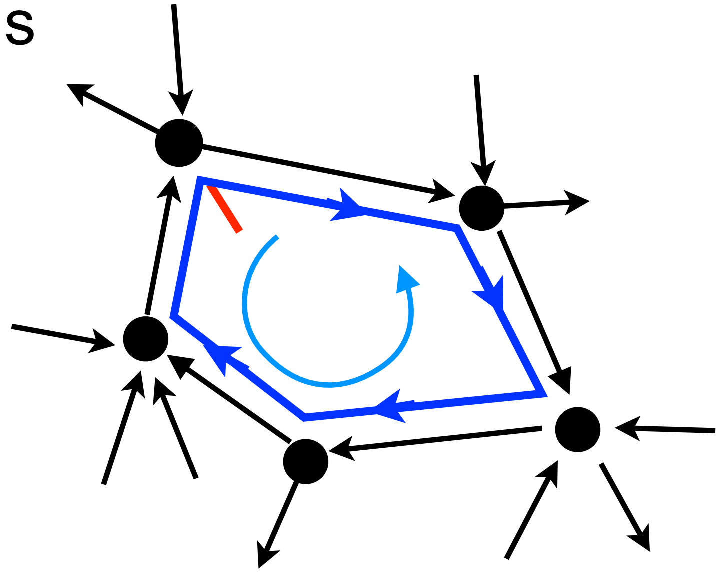

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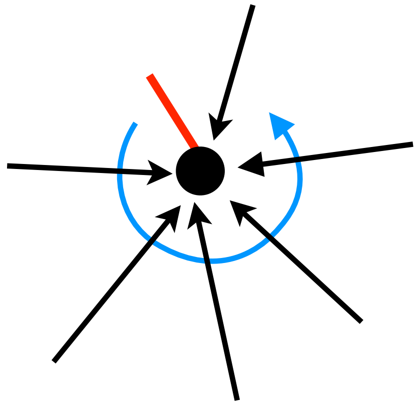

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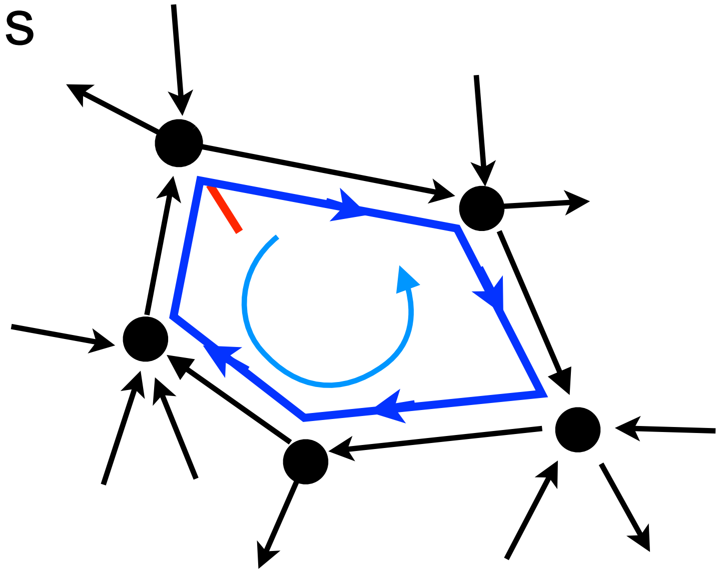

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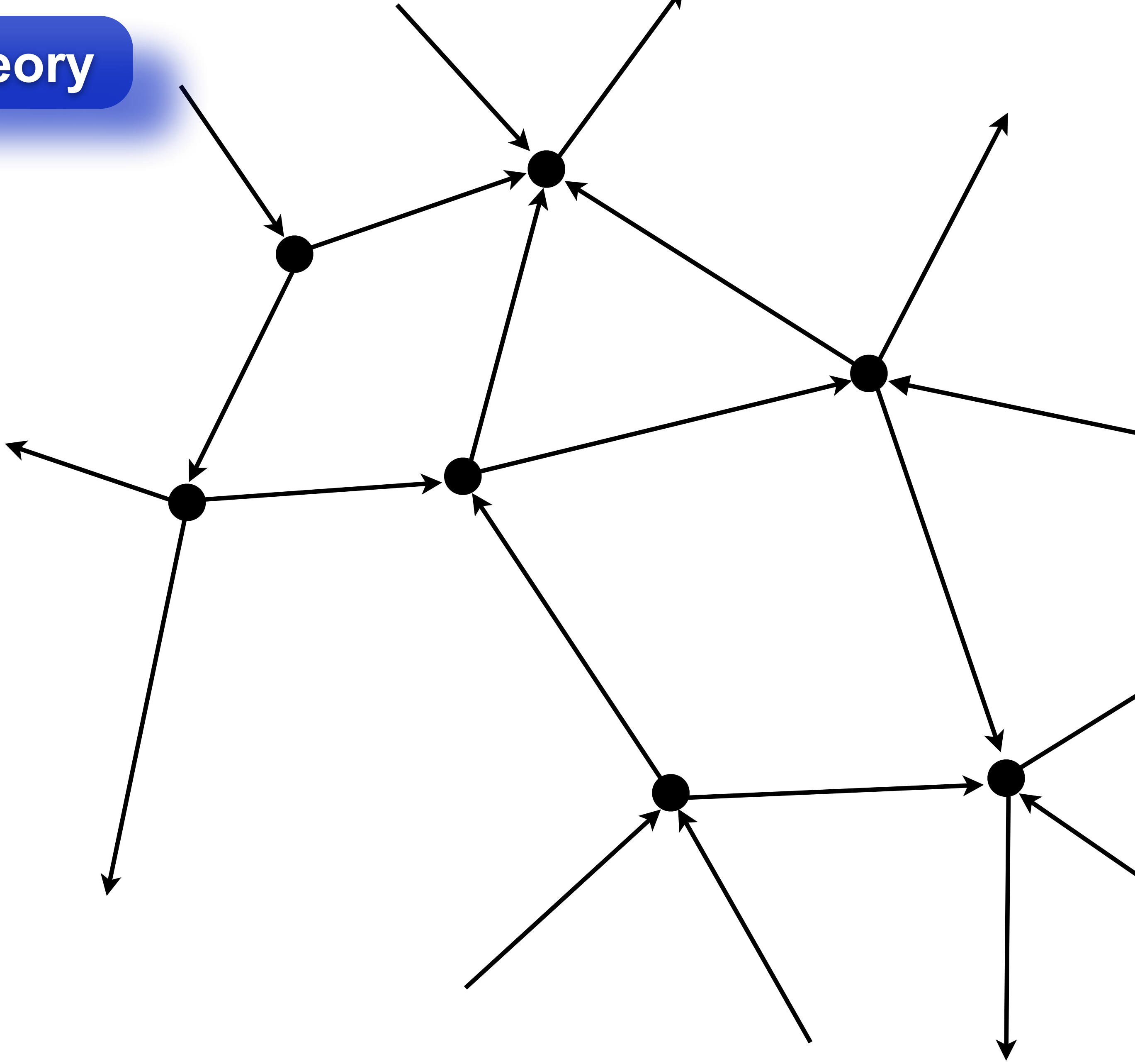

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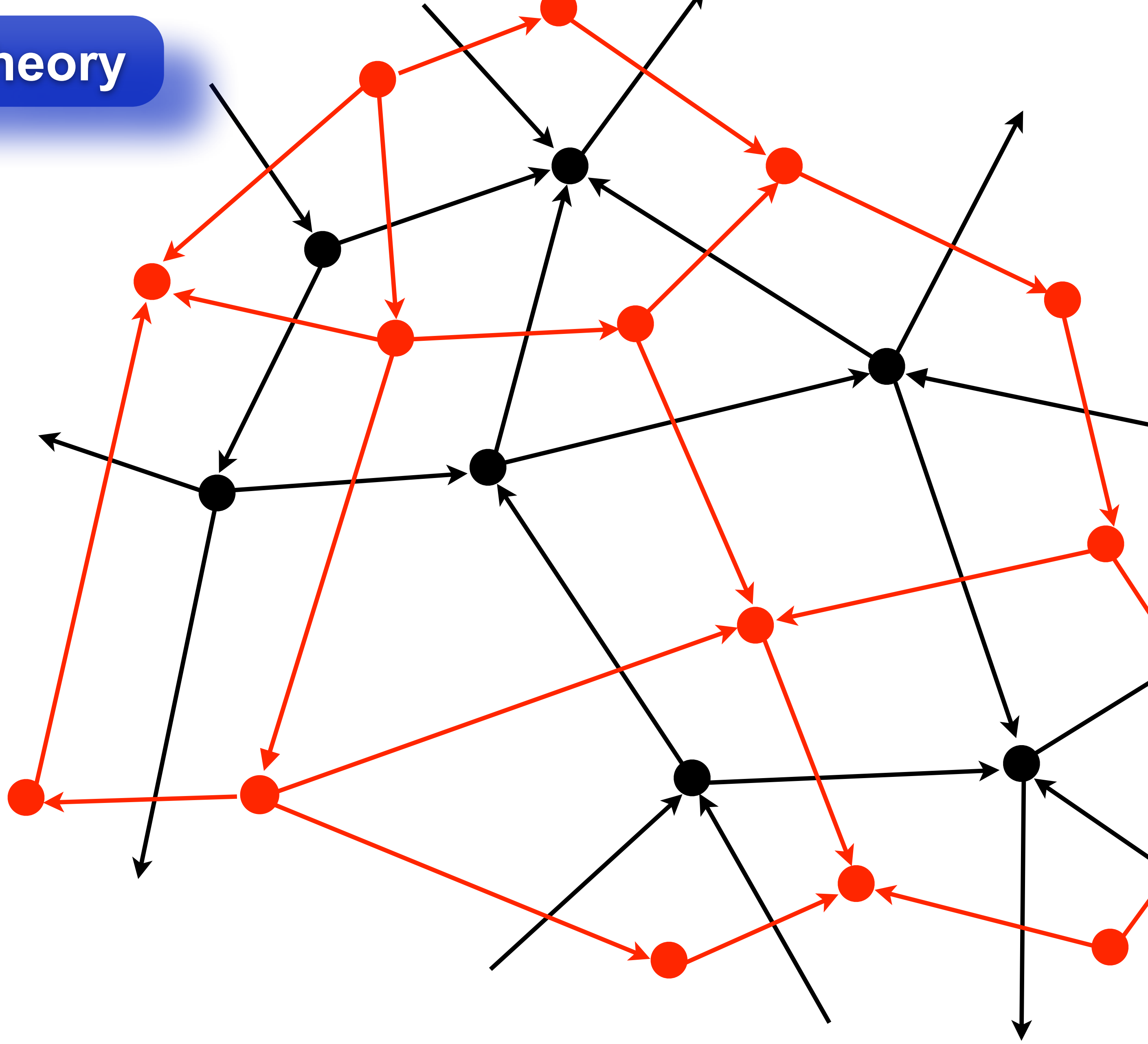

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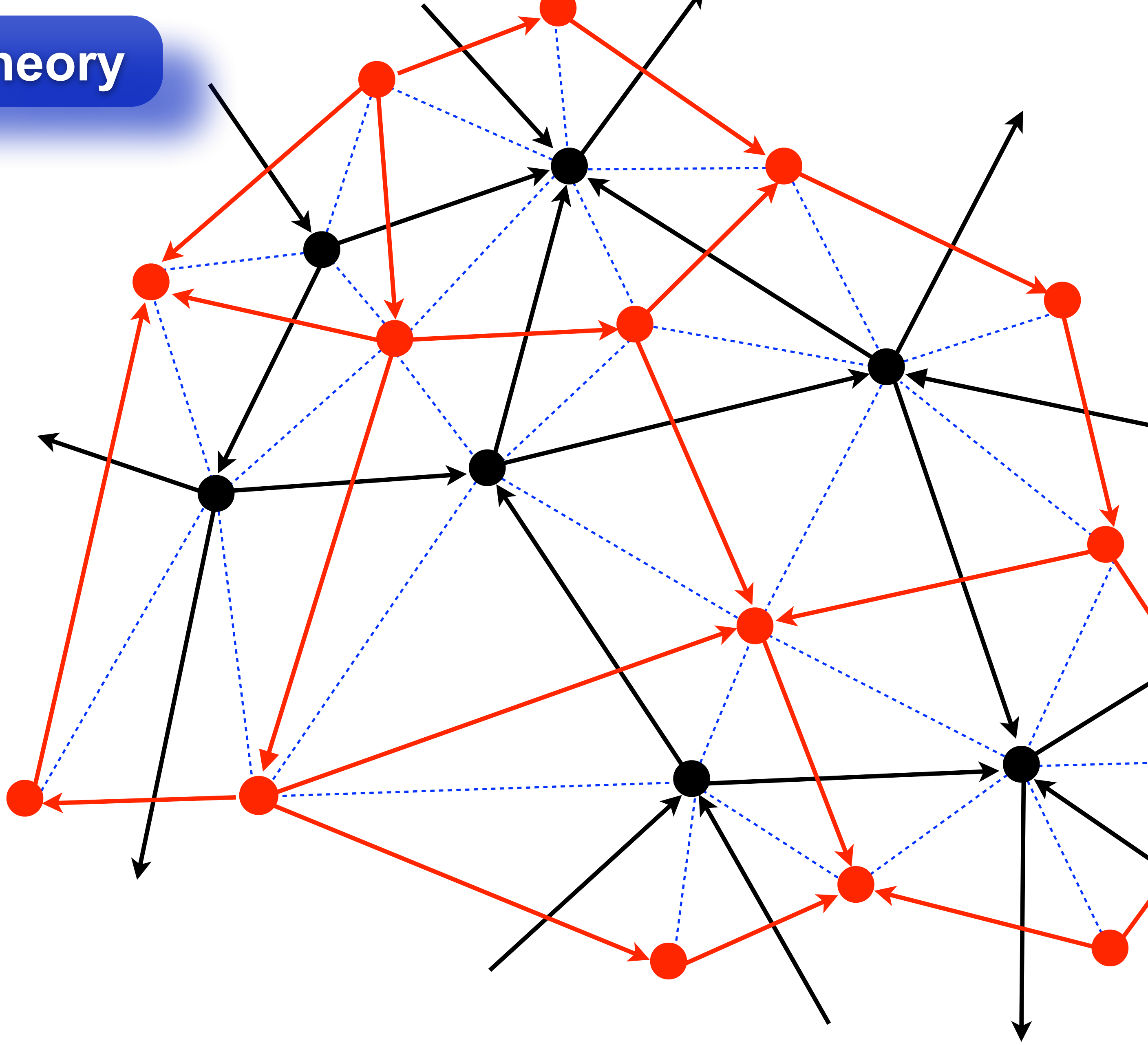

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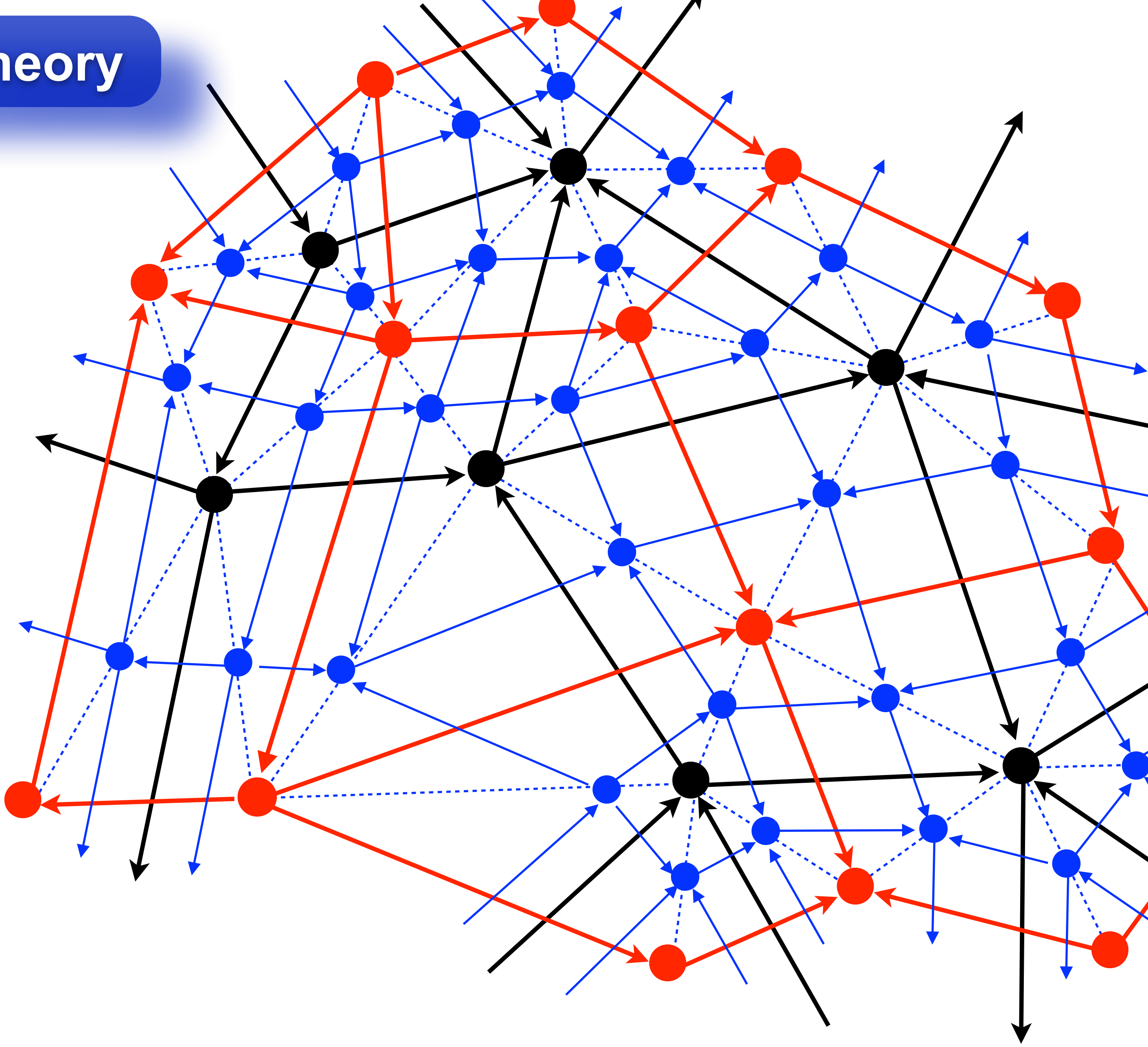
➩ double ribbon graph Γ_D



Kitaev models as a Hopf algebra gauge theory

- combine ribbon graph and its dual

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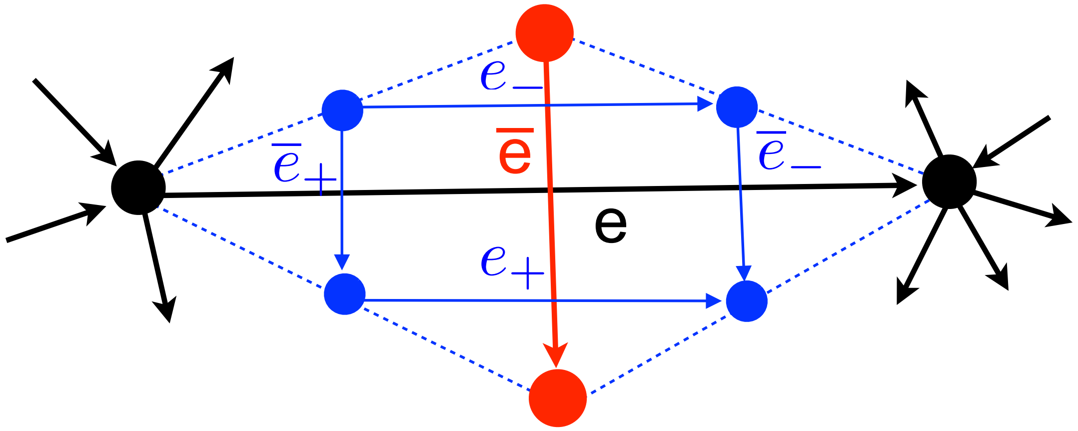
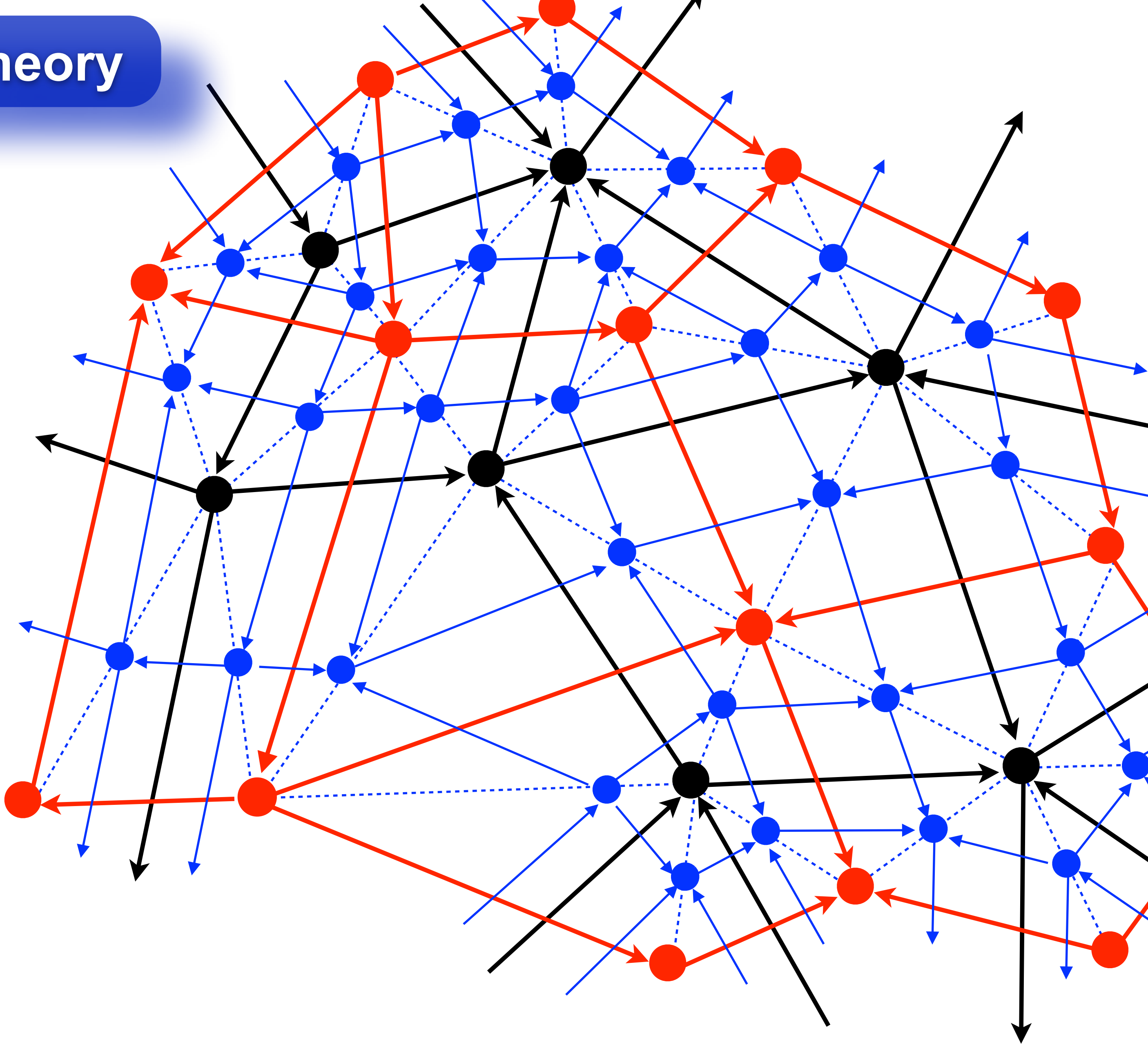
- edge e ➪ 4 triangle operators

$\bar{e}_\pm \to L^h_\pm \quad e_\pm \to T^\alpha_\pm$





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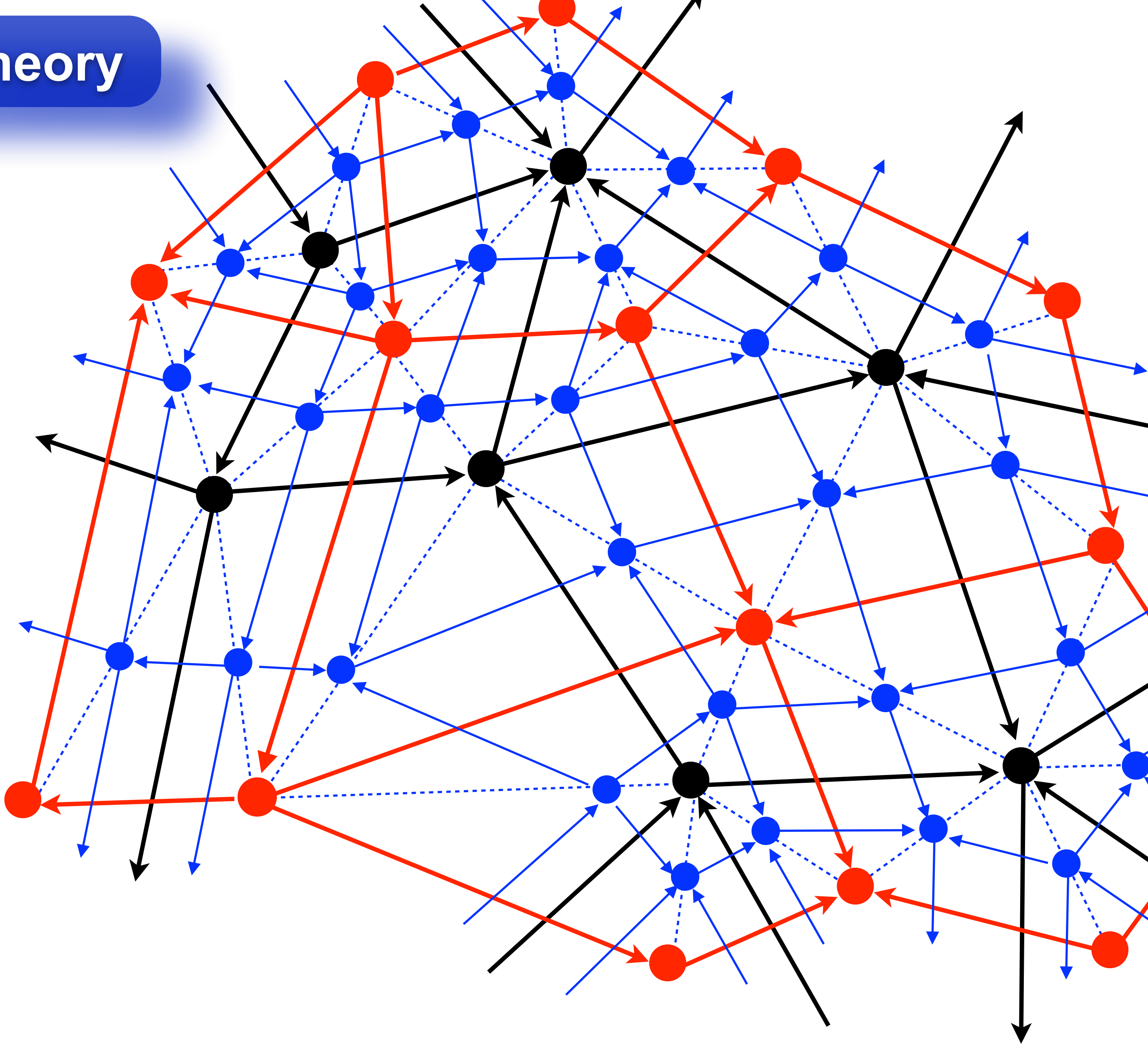

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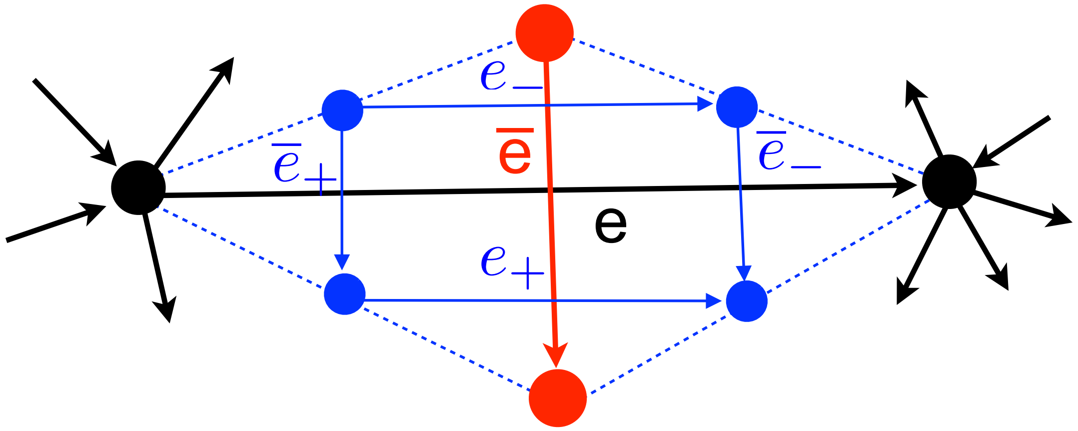

- edge e ⇨ faithful representation of Heisenberg double $H\#H^*$

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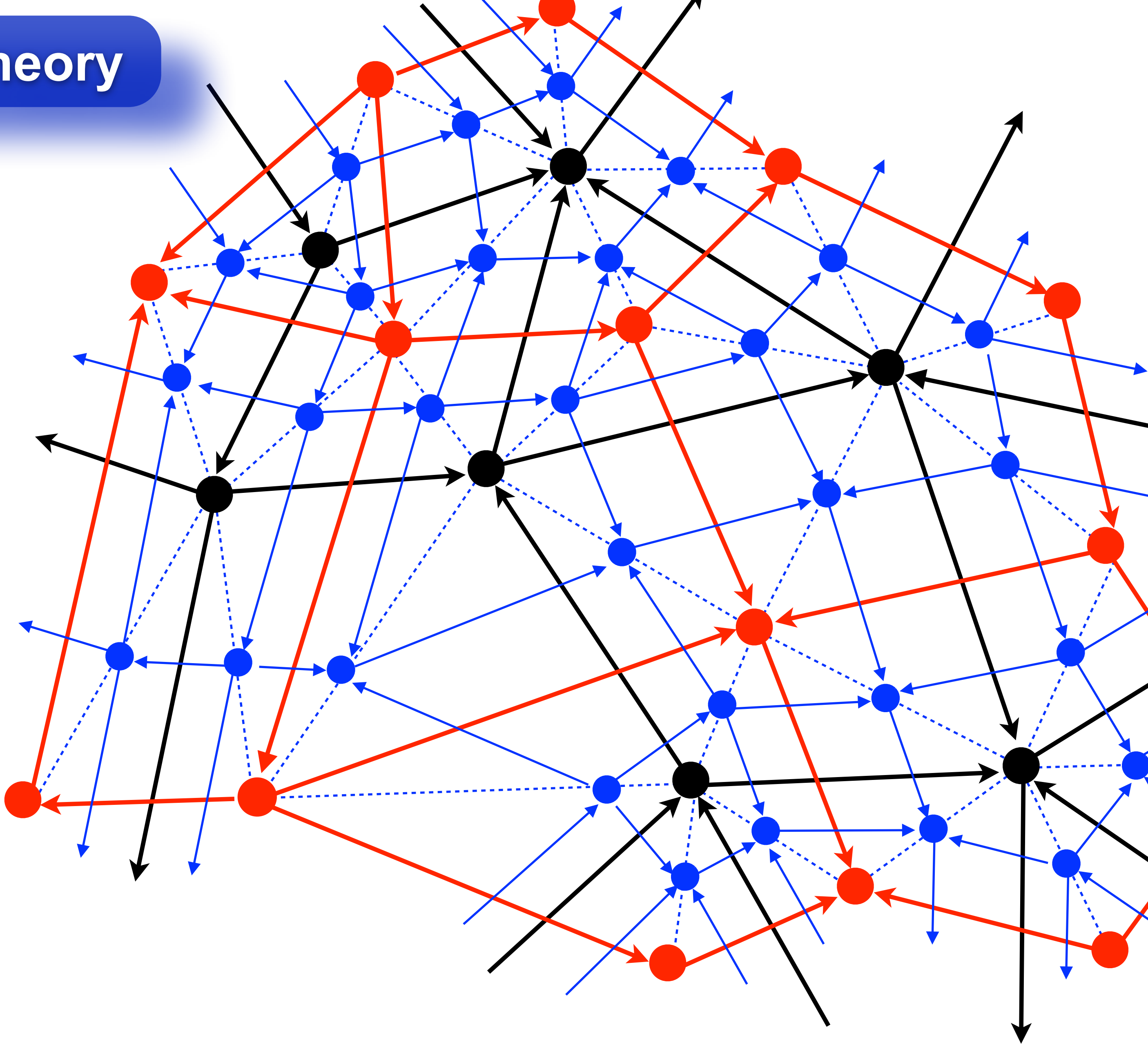

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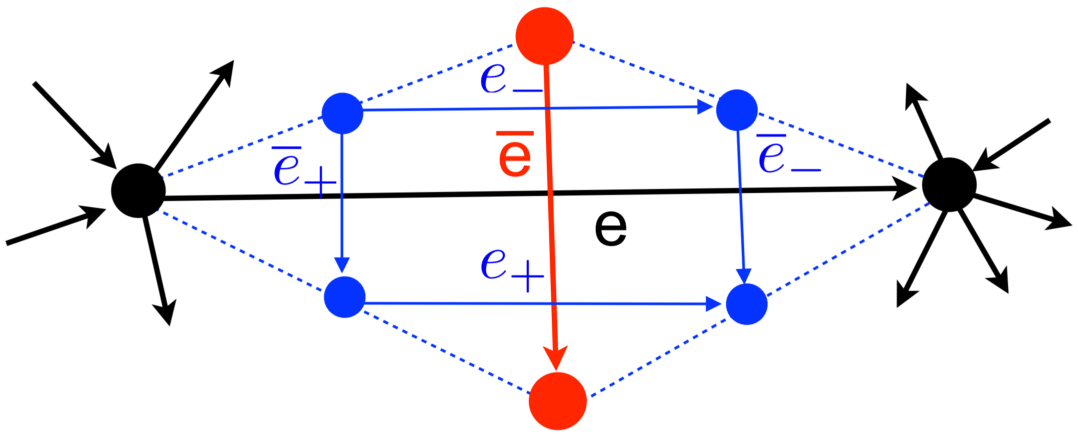

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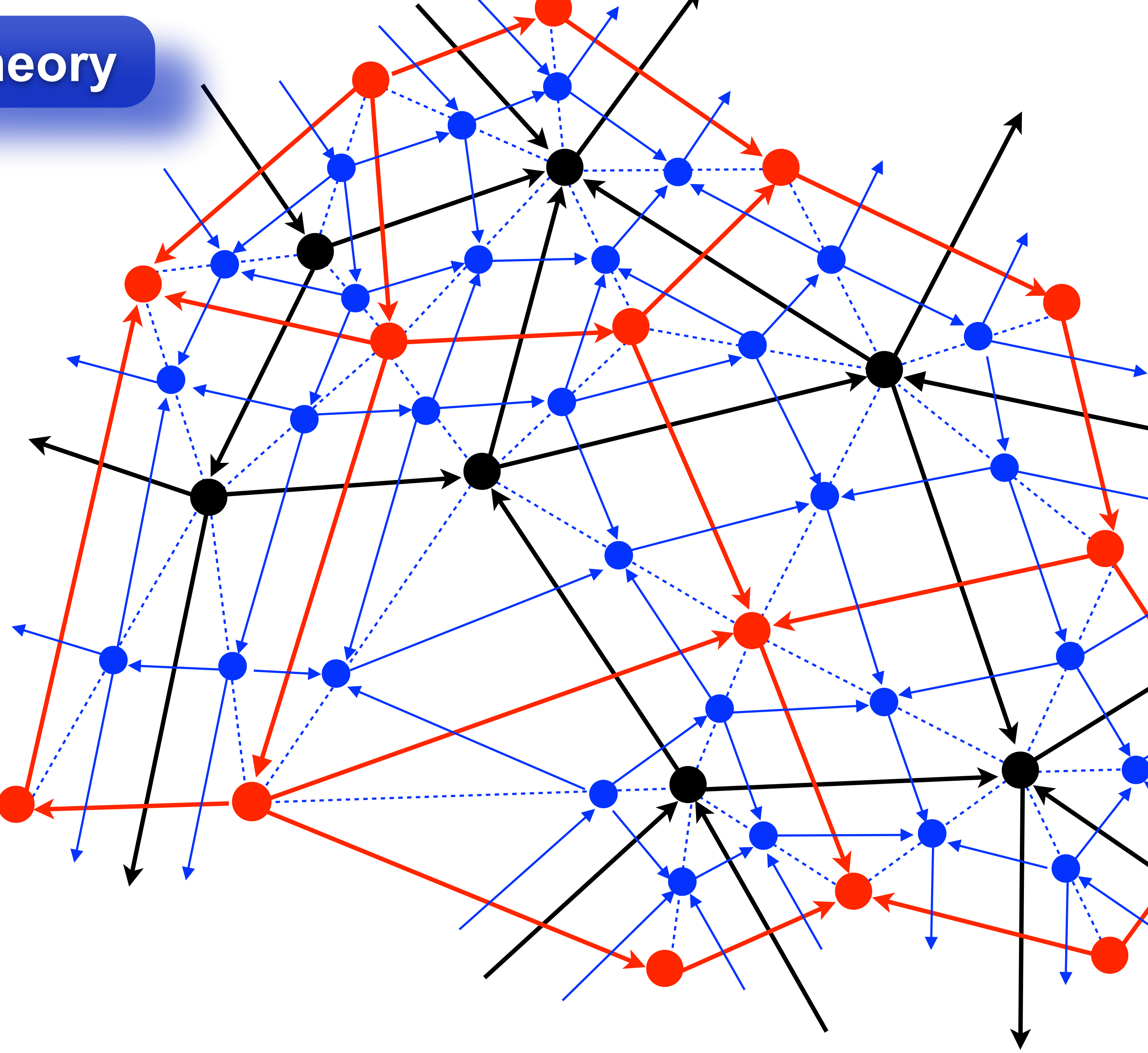

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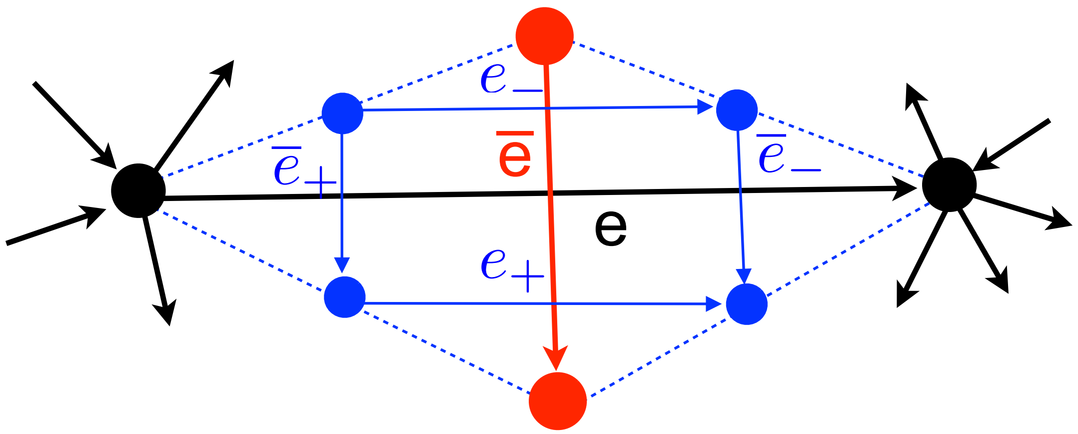

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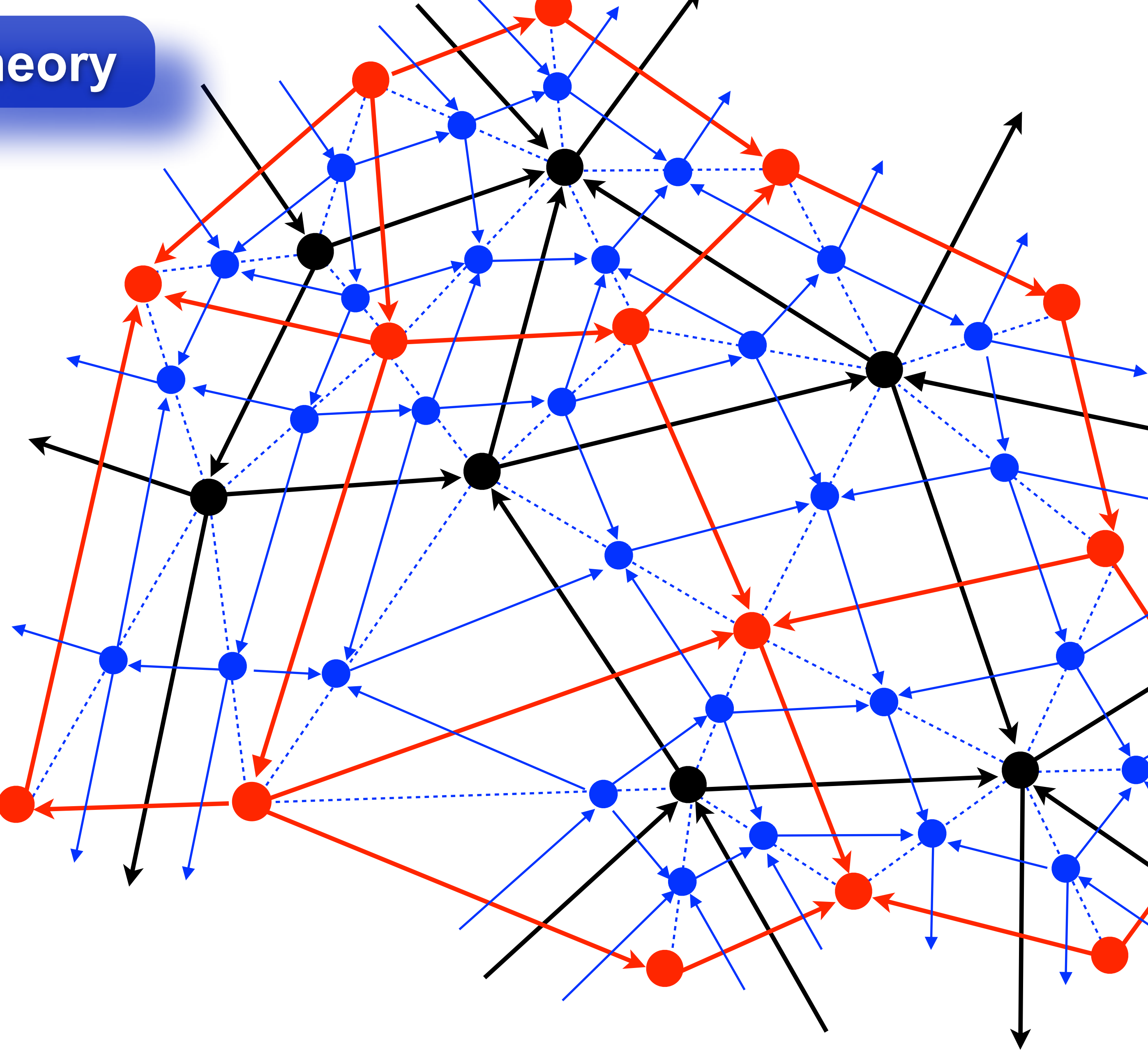

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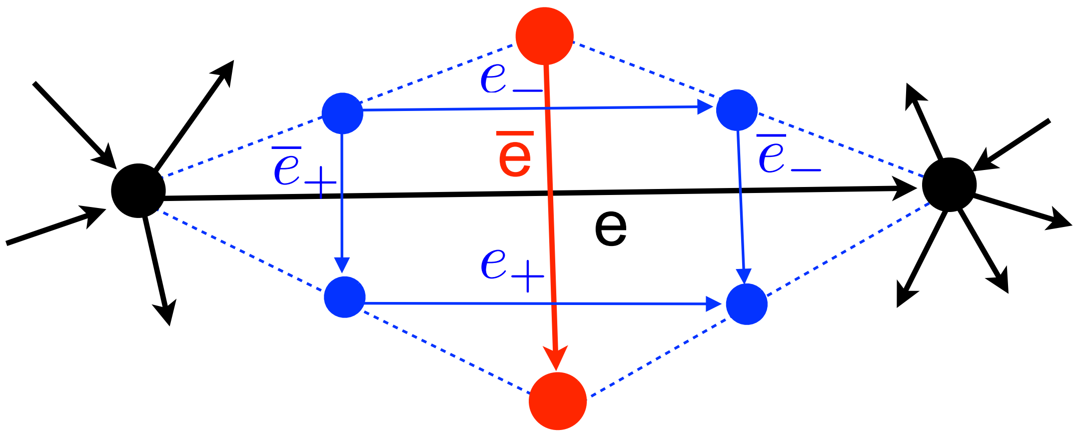

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holonomies

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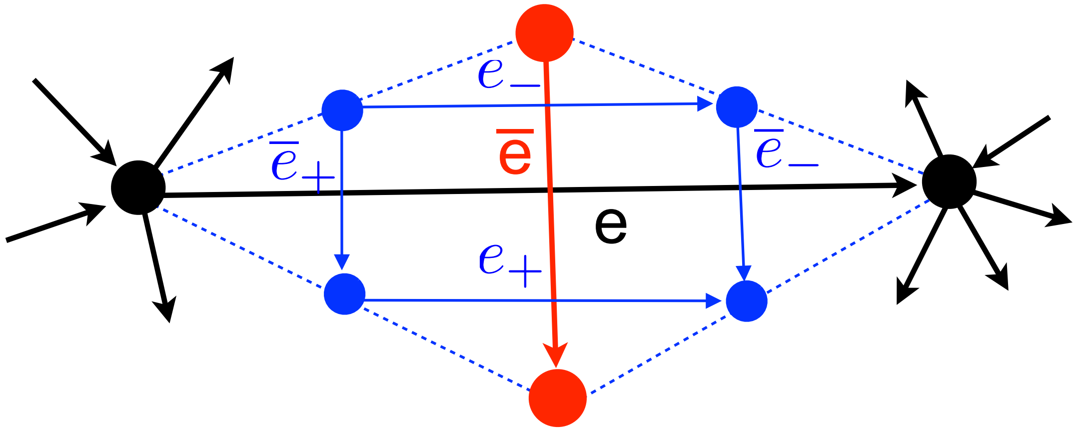

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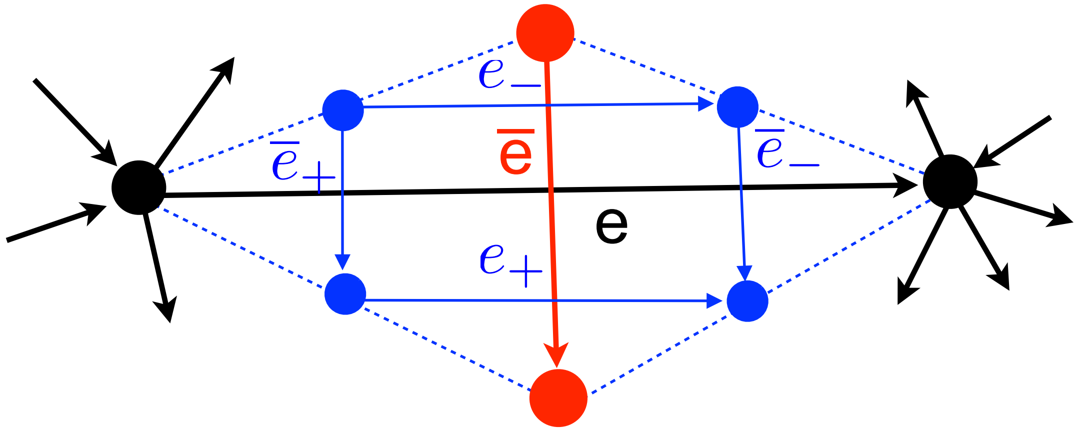

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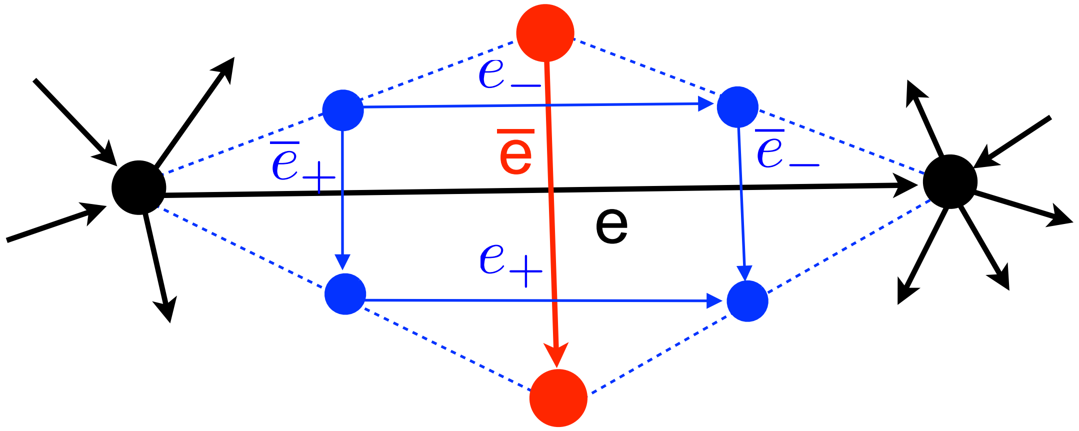

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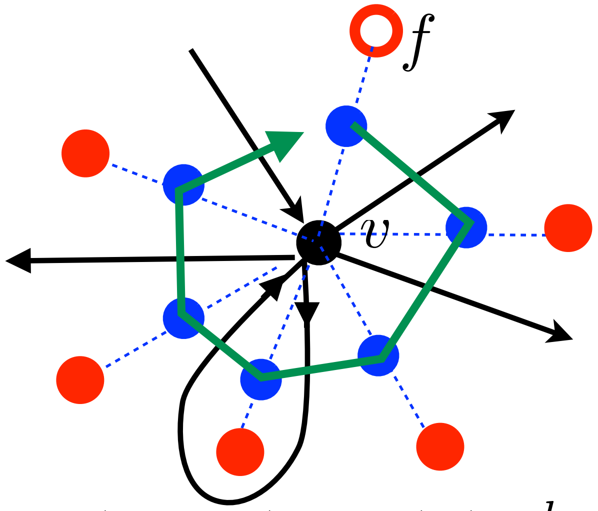

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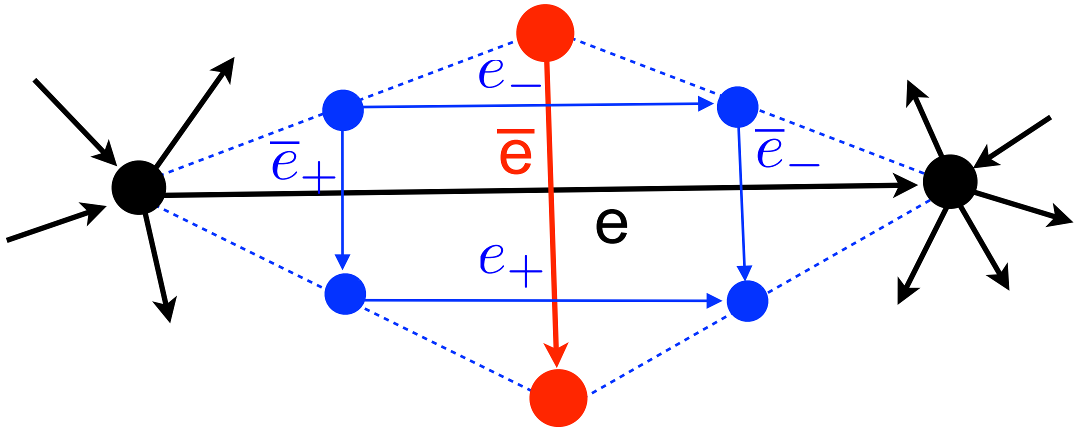

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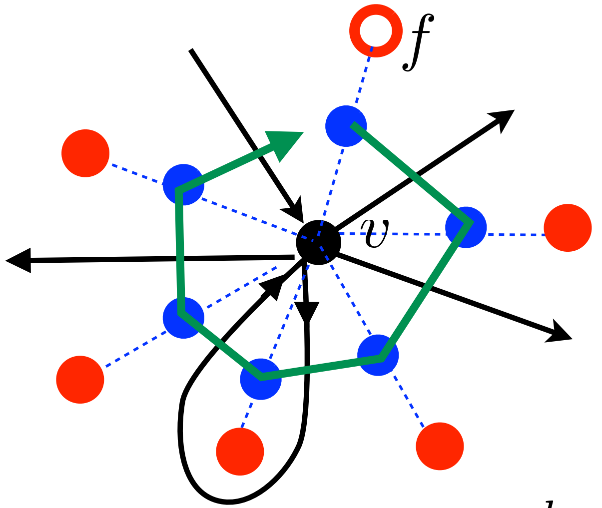

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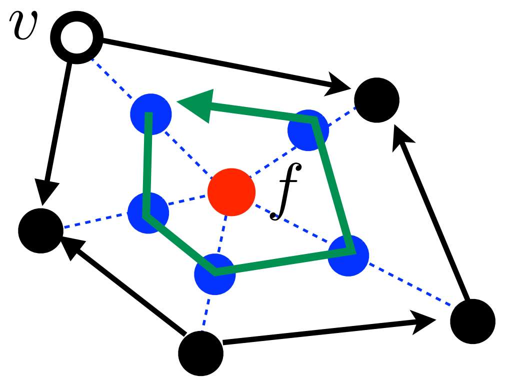

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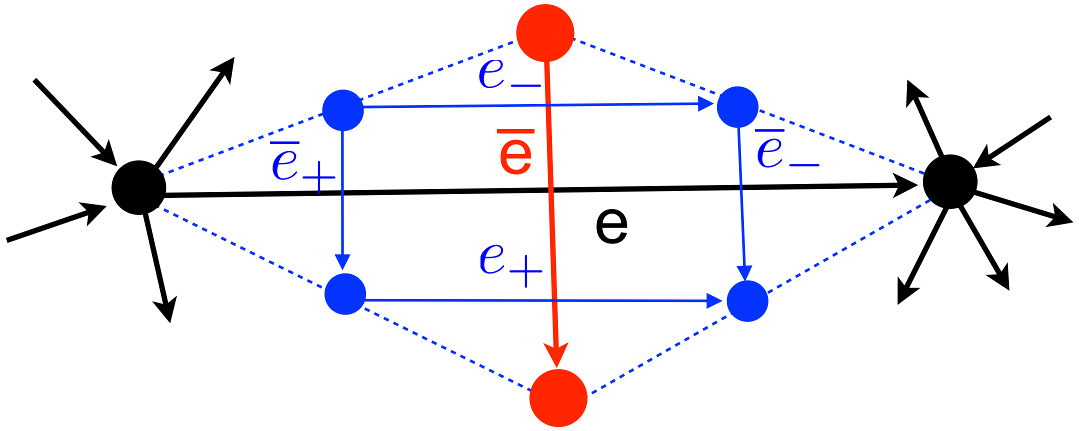

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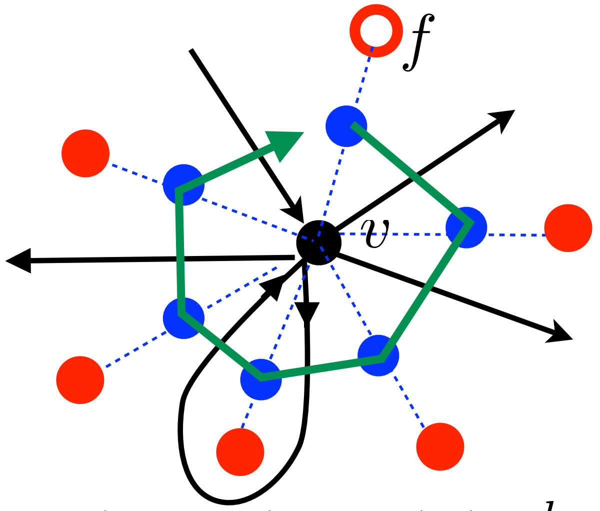

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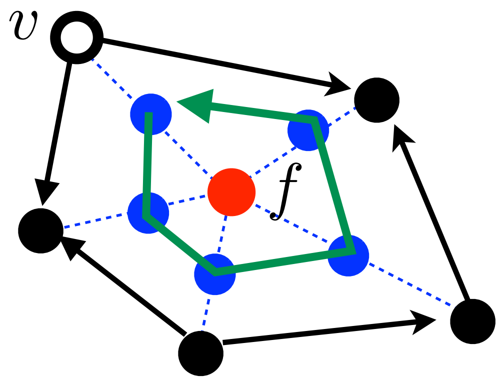

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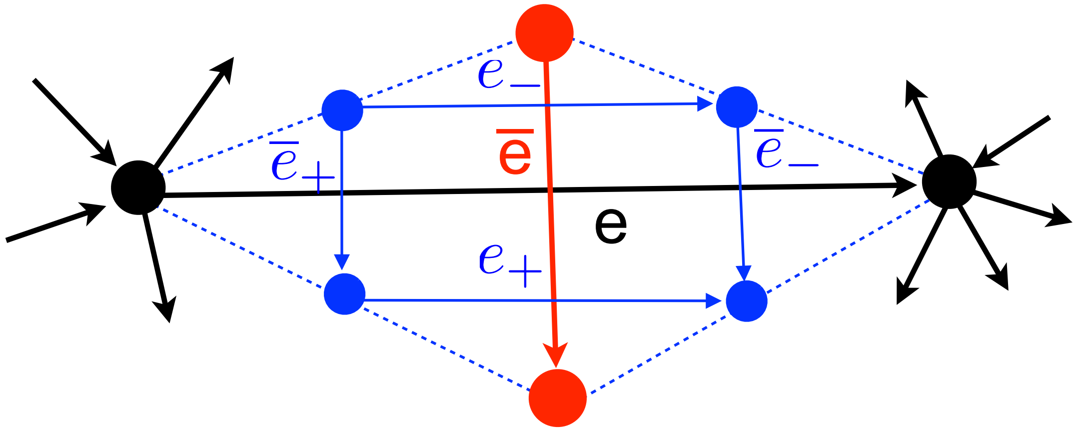

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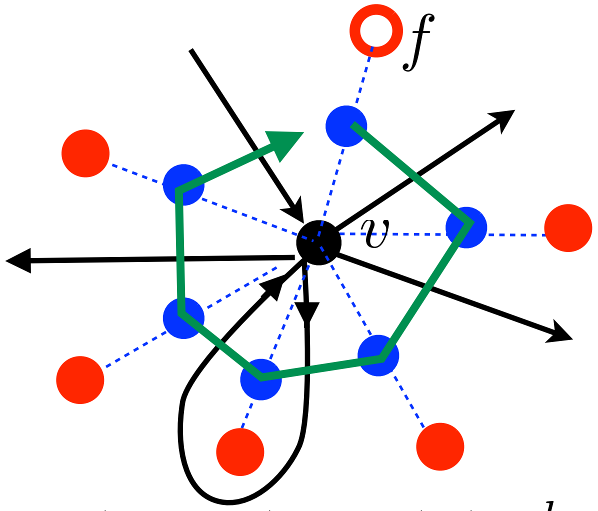

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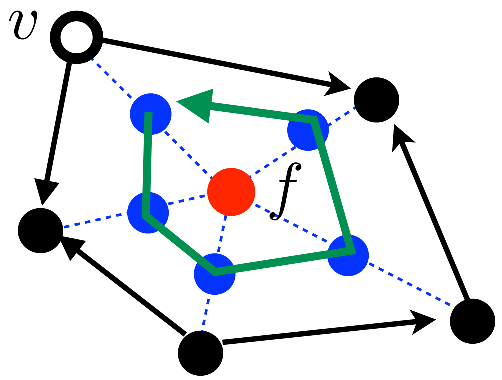

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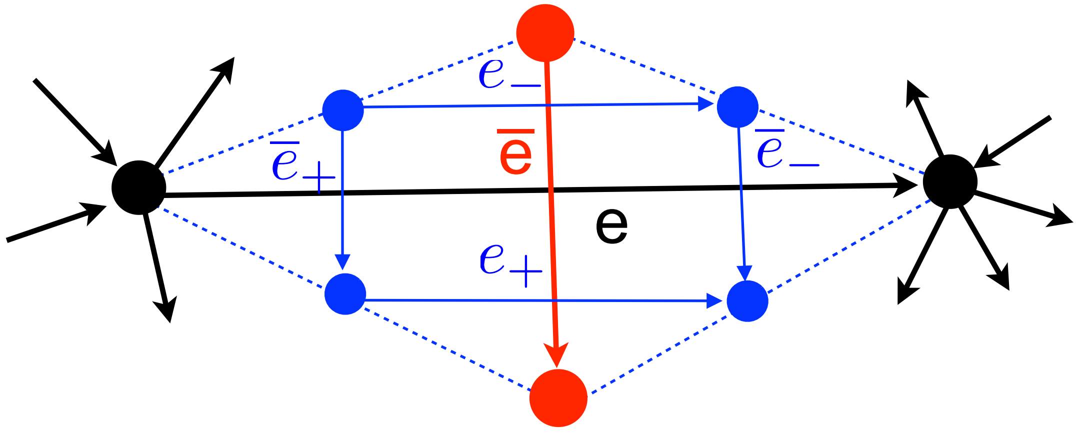

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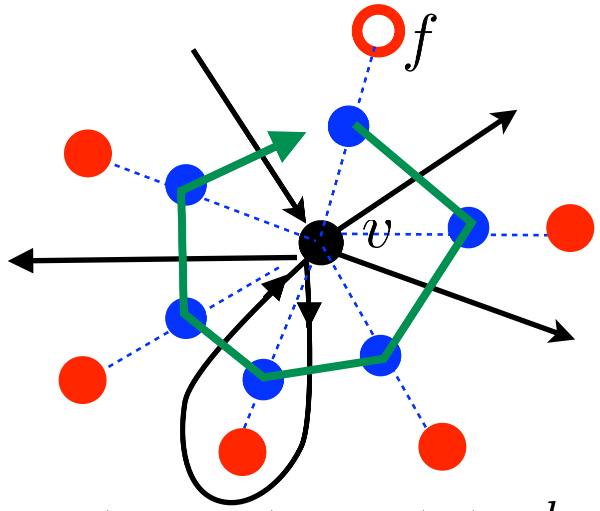

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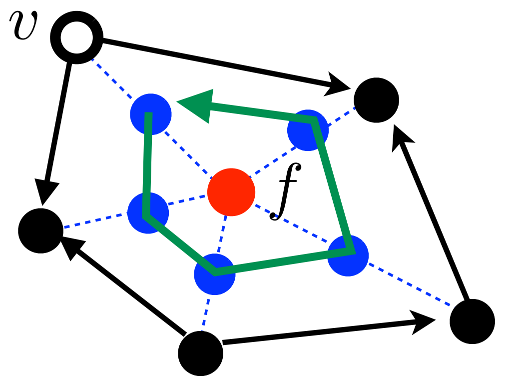

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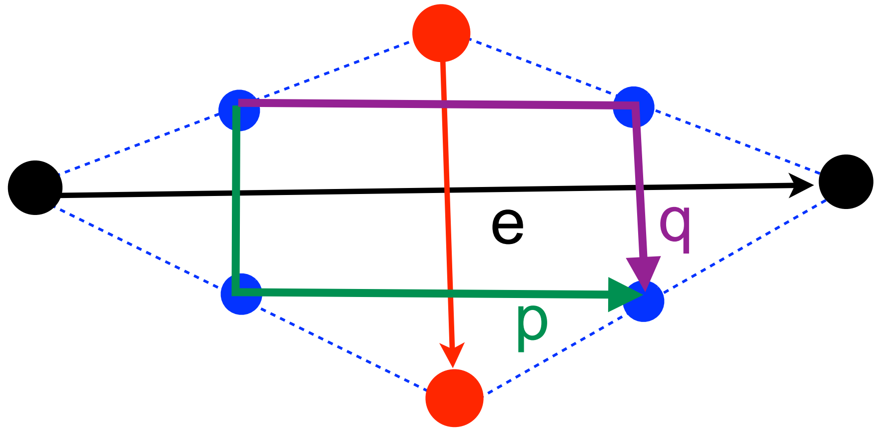

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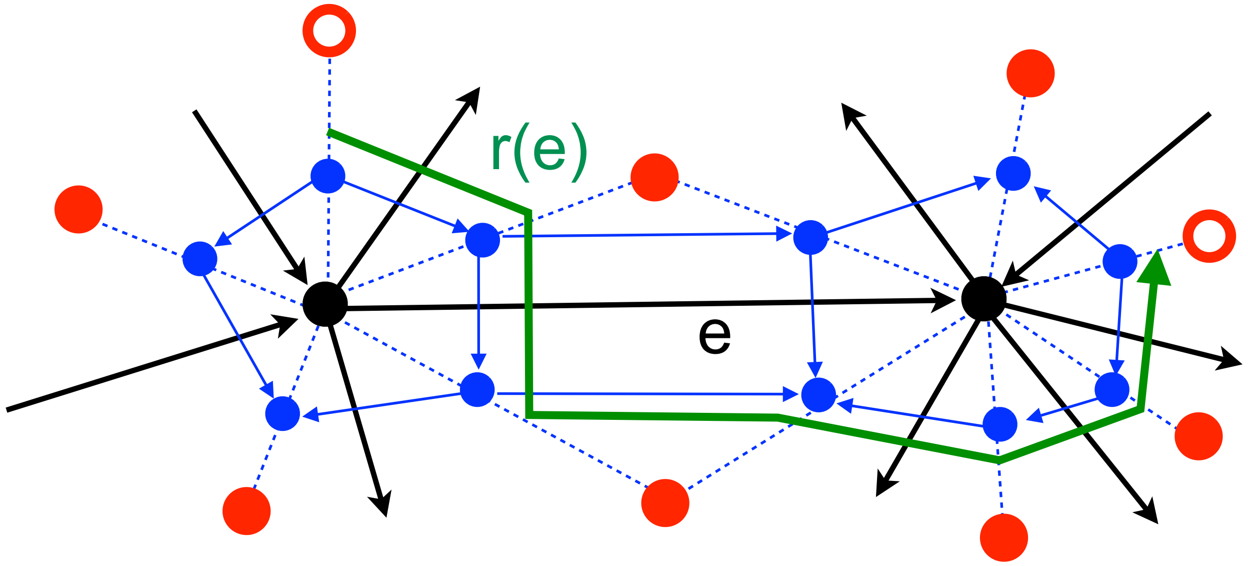

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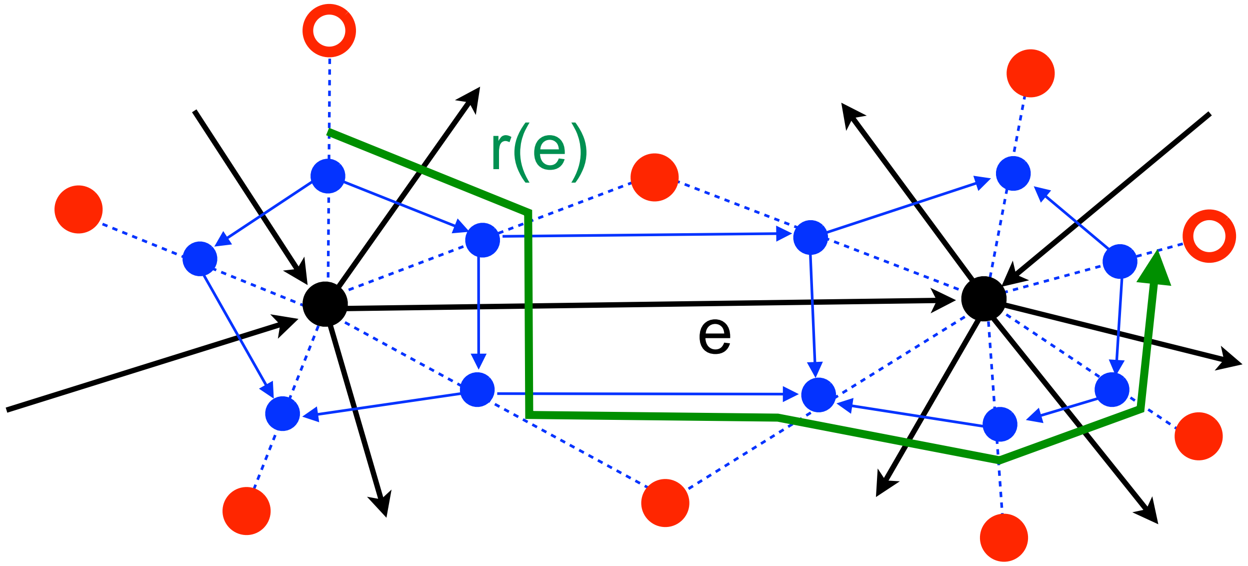

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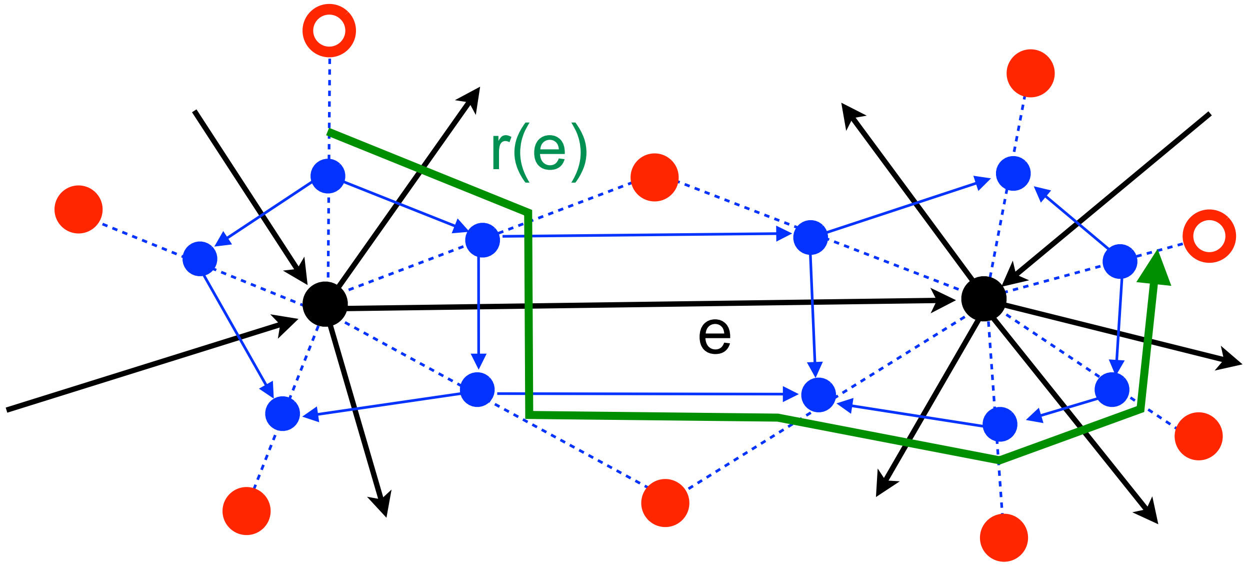

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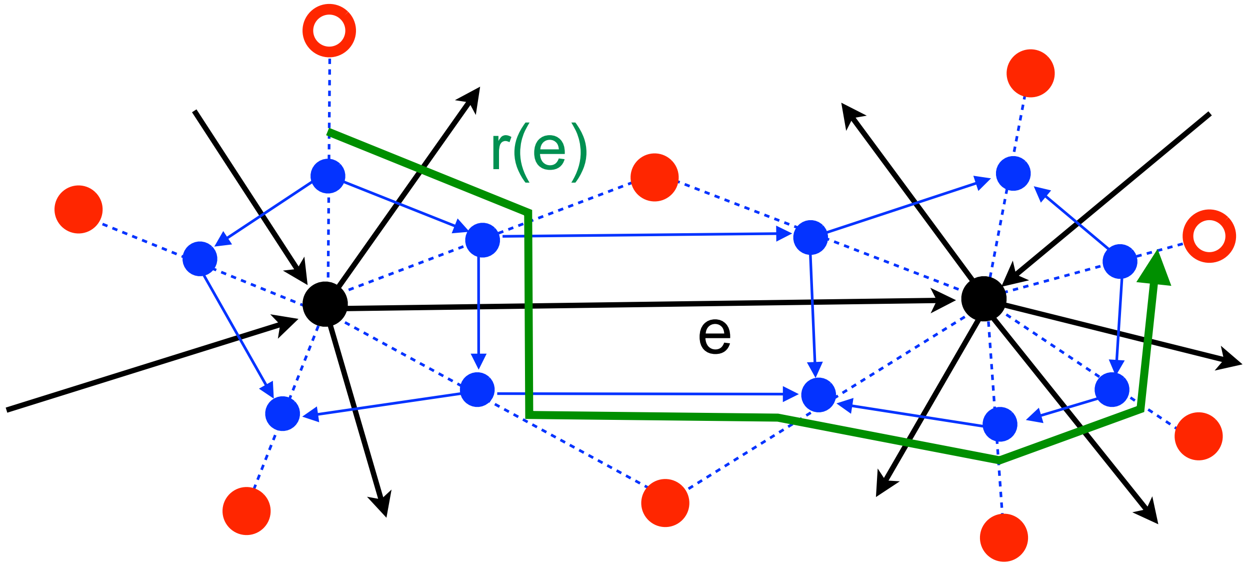

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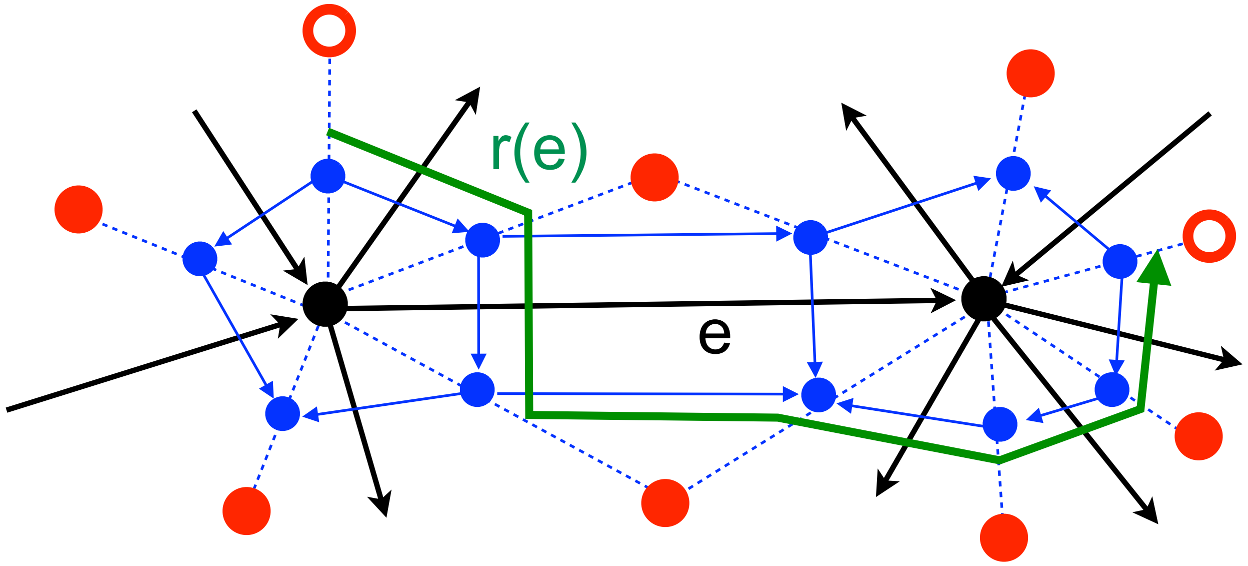

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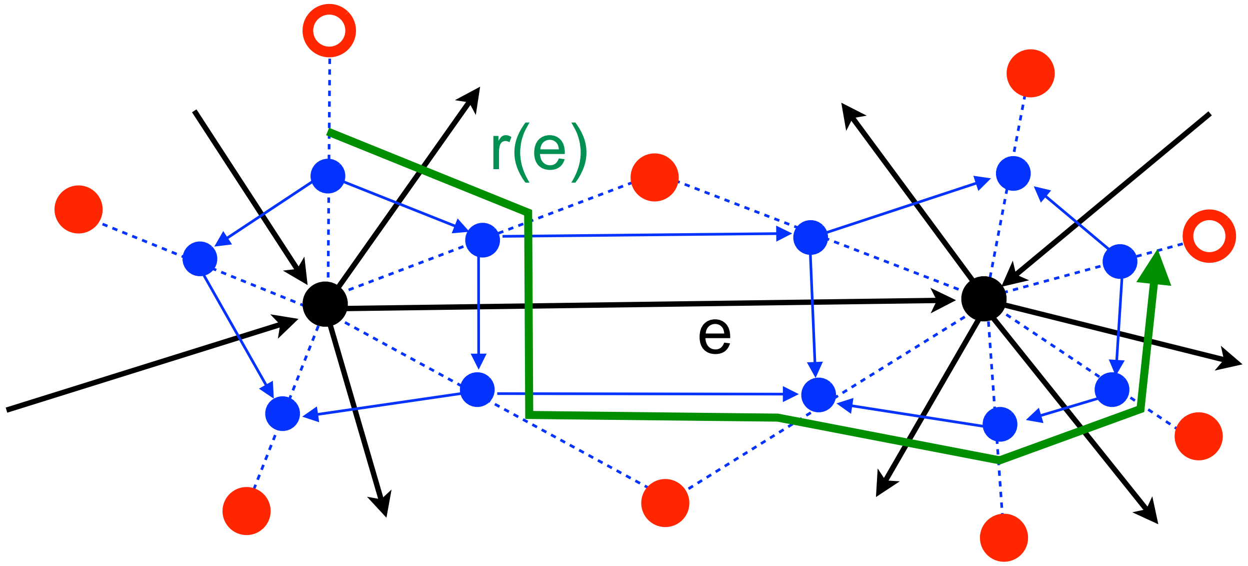

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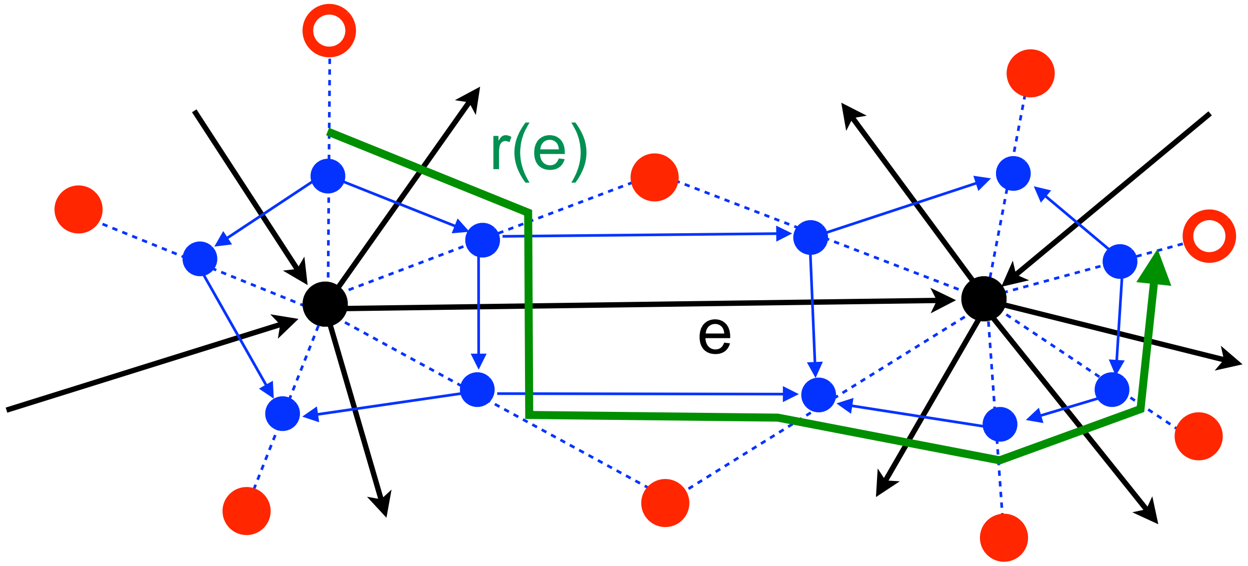
Kitaev models as Hopf algebra gauge theory

- regular ciliated ribbon graph Γ
  - no loops or multiple edges
  - faces traverse each edge at most once
  - every vertex and face: exactly one cilium

- for edge e of Γ: consider path $r(e)$ in Γ_D

  $\mathrm{Hol}_{r(e)} : H \otimes H^* \to (H \otimes H^*)^{\otimes E}$



Theorem [C.M.]

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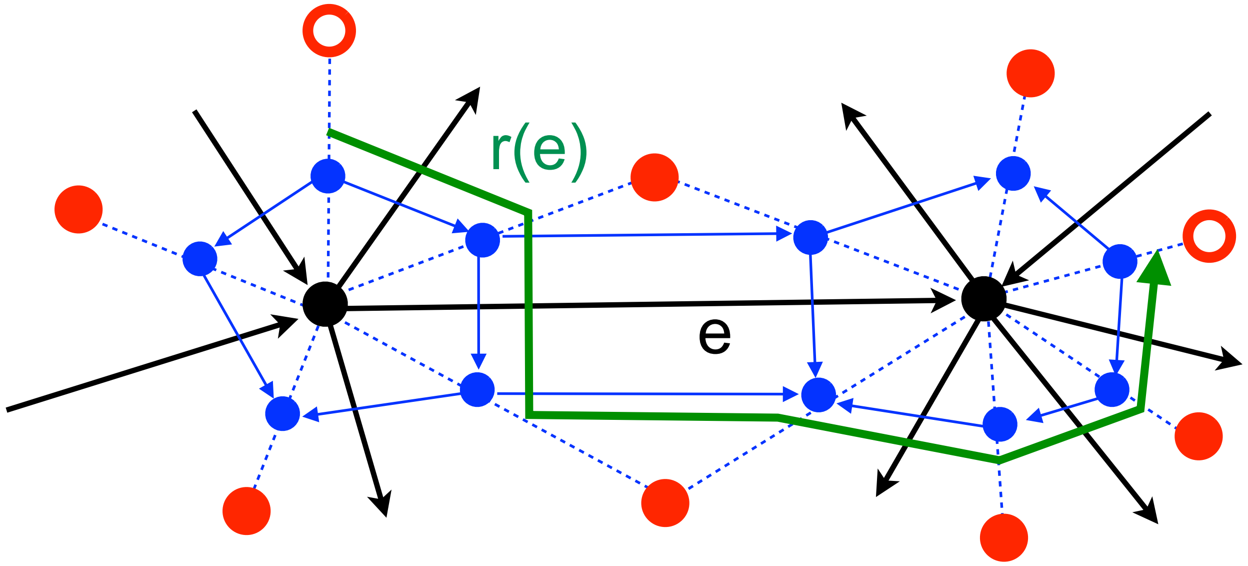

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gauge invariance and flatness

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Kitaev's triangle operators $\xleftrightarrow{\cong}$ **algebra of functions - Hopf algebra gauge theory**

$\mathcal{K} = (H \# H^*)^{\otimes E}$

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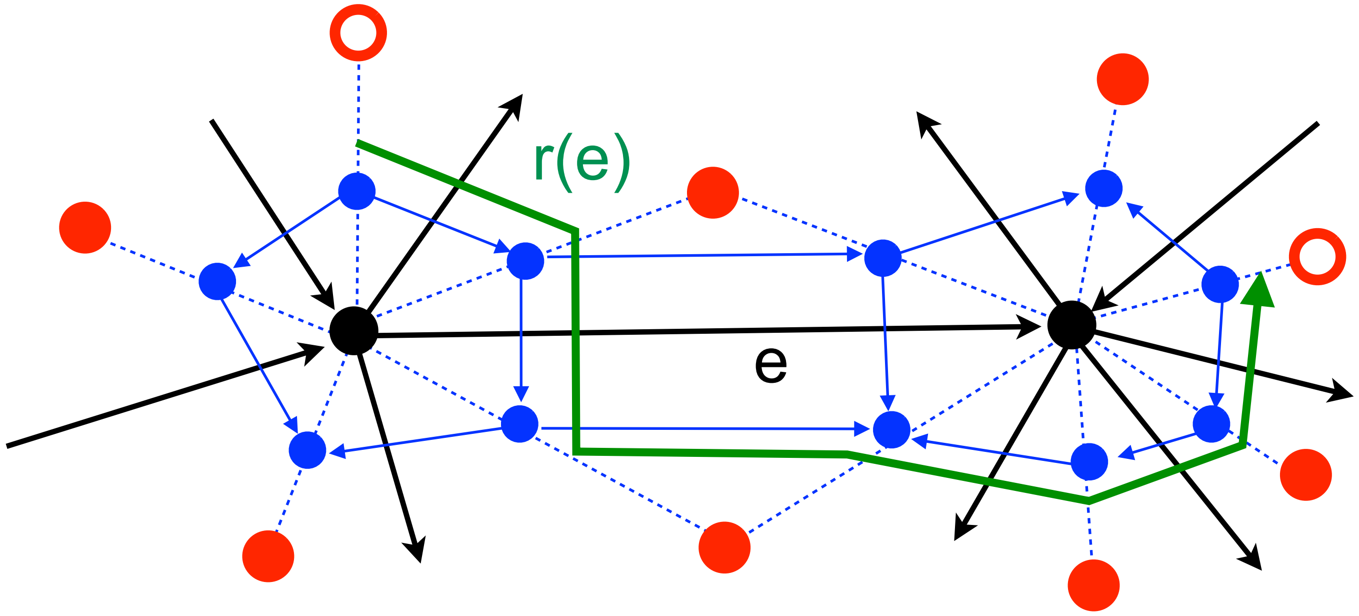

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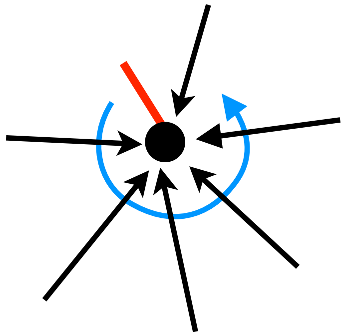

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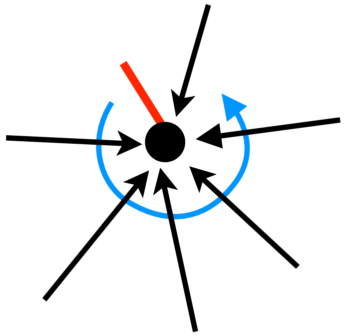

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gauge invariant subalgebra $\overset{\cong}{\longleftrightarrow}$ **gauge invariant subalgebra**

$\Pi_{inv} : X \mapsto X \lhd (\ell\otimes\eta)^{\otimes V}$

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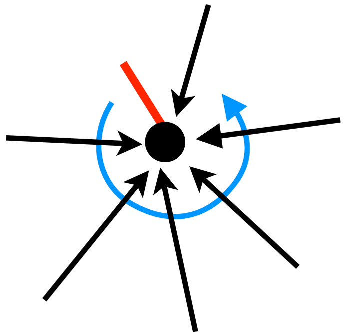

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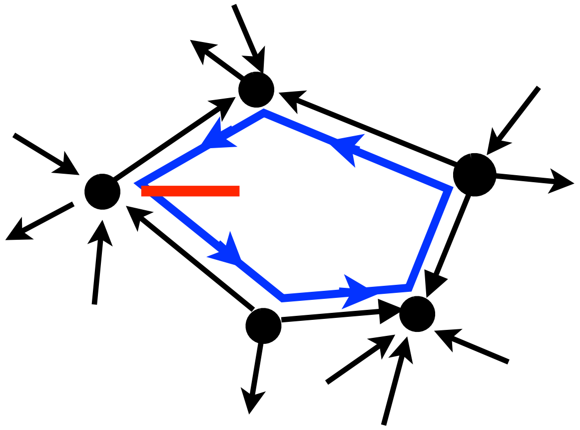

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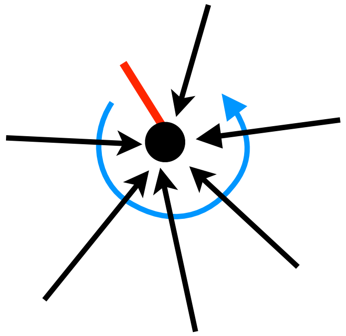

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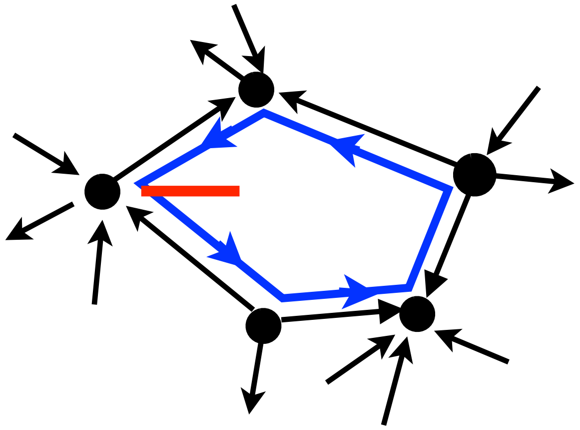

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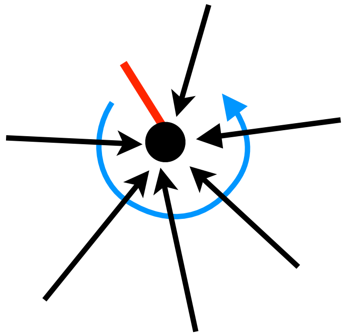

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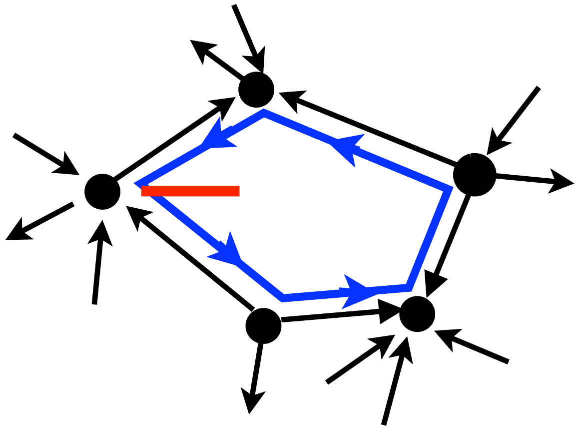

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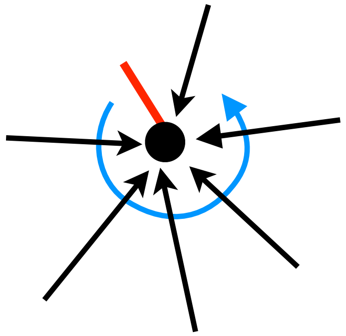

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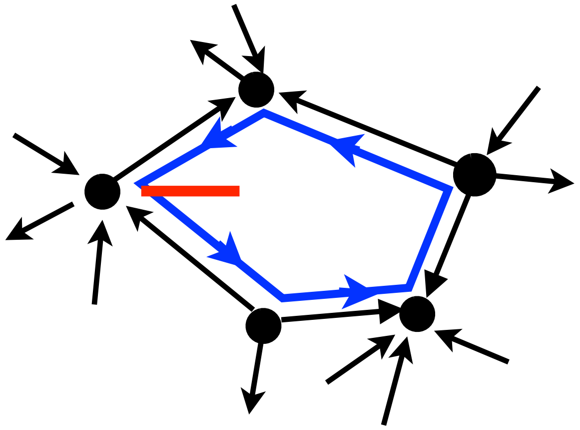

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3. Conclusions

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axiomatic description of Hopf algebra gauge theory

simple axioms for Hopf algebra gauge theory
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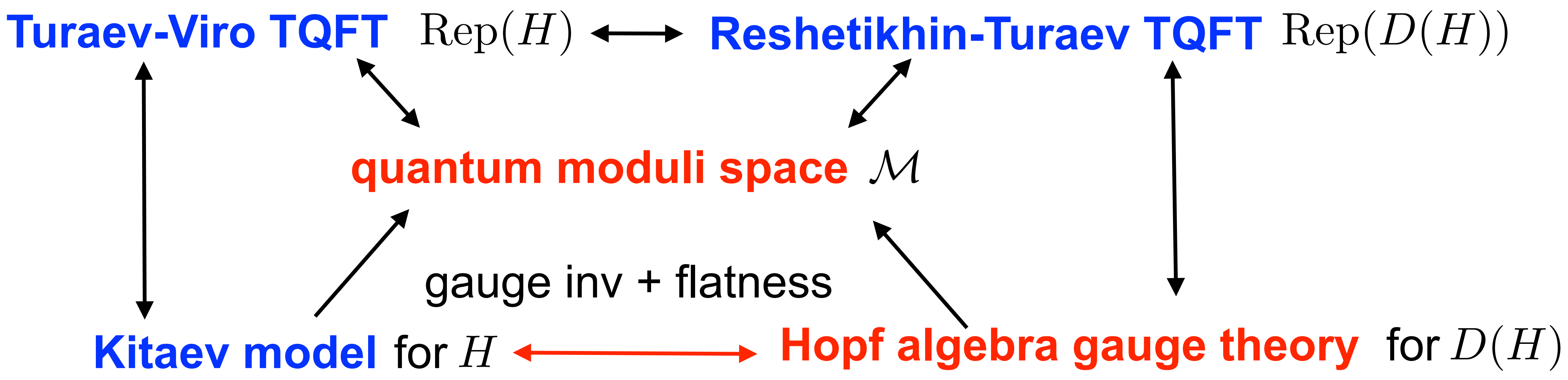

Turaev-Viro TQFT $\mathrm{Rep}(H)$ ⟷ **Reshetikhin-Turaev TQFT** $\mathrm{Rep}(D(H))$

quantum moduli space $\mathcal{M}$

gauge inv + flatness

Kitaev model for H ⟷ **Hopf algebra gauge theory** for $D(H)$