Supersymmetric Gauge Theories and Geometric Representation Theory

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Foundational and Structural Aspects of Gauge Theories

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A class of 2d/4d dualities

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(D) There is an infinite-dimensional commutative algebra (integrals of motion) acting in $\mathcal{H}_r$ which is diagonalised in the basis $\{\vec{v}_p\}$

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Conformal blocks over Seiberg–Witten curve Σ

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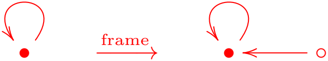
$\widehat{A}_0 =$ 

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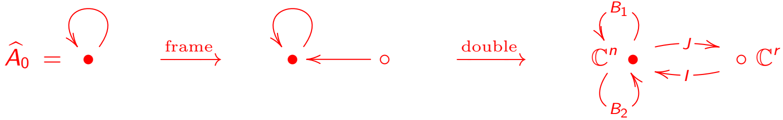

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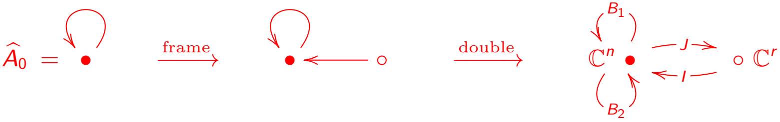

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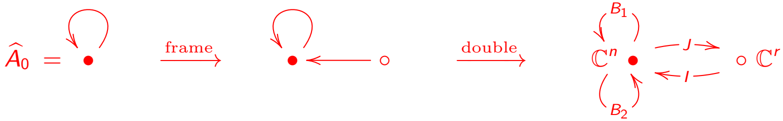

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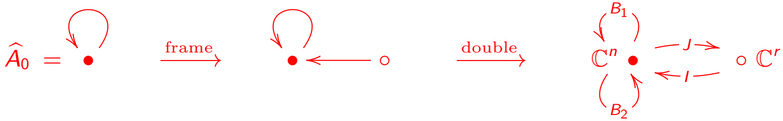

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- $p \in \mathcal{M}_{r,n}^T$ parameterized by Young diagrams $\boldsymbol{Y} = (Y_1, \ldots, Y_r)$ with $|\boldsymbol{Y}| = \sum_i |Y_i| = n$

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- Extension to $\mathcal{N}=2$ quiver gauge theories on $\mathbb{R}^4$: Matter fields represent Euler classes of universal vector bundles on $\mathcal{M}_{r,n}$

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Then $\mathcal{Z}_{\mathbb{R}^4} = \big\langle \psi, q^{L_0}\psi \big\rangle$ where $q^{L_0} = q^n$ on $H_T^*\big(\mathcal{M}_{r,n}\big)$.

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- $\mathcal{H}_r \simeq$ symmetric functions: Fixed point basis $\vec{v}_{\boldsymbol{Y}} = [\boldsymbol{Y}] \Longleftrightarrow$ generalised Jack symmetric functions $J_{\boldsymbol{Y}}$, eigenfunctions of quantum Calogero–Moser–Sutherland model
  (Nakajima '96; Vasserot '01; Alba, Fateev, Litvinov & Tarnopolsky '10; Smirnov '14)

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- For $r = 2$: Matrix elements correspond to primary field insertions in conformal blocks (Fateev & Litvinov '10; Hadasz, Jaskolski & Suchanek '10)

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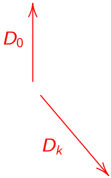

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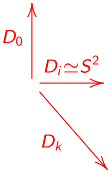

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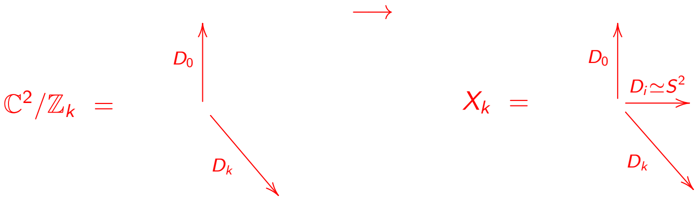

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- ALE space $X \simeq X_k$ with ALE Kähler metric, i.e. Euclidean at ∞

Quiver varieties

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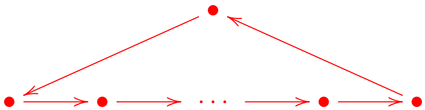

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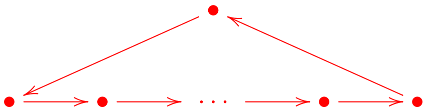

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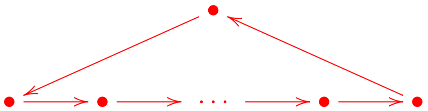

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- Two distinguished chambers: C_0 and C_∞

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- For $\xi_0 \in C_0$, $\mathcal{M}_{\xi_0}(\vec{v}, \vec{w}\,) \simeq$ moduli space of (framed) $\mathbb{Z}_k$-equivariant noncommutative instantons on $\mathbb{R}^4$, holonomy $(\rho_1^{w_1}, \dots, \rho_k^{w_k})$ at ∞ (flat gauge fields at ∞: $\pi_1(S^3/\mathbb{Z}_k) \simeq \mathbb{Z}_k$)

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- Nekrasov partition functions as weighted sums over fractional instantons with chemical potentials $\vec{\xi} = (\xi_1,\dots,\xi_{k-1})$ are computed by localization theorem (Fucito, Morales & Poghossian '04; Fujii & Minabe '05):

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- $\mathcal{A}_{1,k} = \widehat{\mathfrak{gl}}(k)_1$, $\mathcal{A}_{2,k} = \widehat{\mathfrak{gl}}(k)_2 \oplus \mathcal{NSR}$ ($\mathbb{Z}_k$-parafermionic A_{r-1} Toda field theory, SUSY Liouville theory for $k = 2 = r$)

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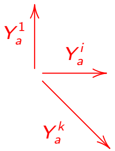

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- Seiberg–Witten geometry: $\displaystyle\lim_{\varepsilon_1,\varepsilon_2\to 0} -k\,\varepsilon_1\,\varepsilon_2\,\log\mathcal{Z}_{X_k} = \frac{1}{k}\,\mathcal{F}_{\mathbb{R}^4}$

Chamber dependence

(Fujii & Minabe '05; Bonelli, Maruyoshi & Tanzini '11; Wyllard '11; Belavin *et al* '13; Alfimov, Belavin & Tarnopolsky '13; Ito, Maruyoshi & Okuda '13; Bruzzo, Sala & RS '14; RS '15; . . .)

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- AGT relations on X_k for $k=2$

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For $r = 1$ and $j \in \{0, 1, \ldots, k-1\}$:

C_0	C_∞
$\mathcal{M}_{\xi_0}(\vec{v}, \vec{e}_j)$	$\mathcal{M}_{\xi_\infty}(\vec{v}, \vec{e}_j) \simeq \mathcal{M}_{\vec{u},n,\vec{e}_j}$
$Y \in \mathcal{M}_{\xi_0}(\vec{v}, \vec{e}_j)^T$ j-coloured with k colours	$\vec{Y} \in \mathcal{M}^T_{\vec{u},n,\vec{e}_j}$ $\vec{Y} = (Y^1, \ldots, Y^k)$
$\mathcal{H}_{0,j} = \bigoplus_{\vec{v}} H^*_T\big(\mathcal{M}_{\xi_0}(\vec{v}, \vec{e}_j)\big)$	$\mathcal{H}_{\infty,j} = \bigoplus_{\vec{u},n} H^*_T\big(\mathcal{M}_{\vec{u},n,\vec{e}_j}\big)$

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  $J_{Y^i}\big(\beta^{(i)}\big) =$ Jack symmetric function with parameter $\beta^{(i)}$ given by fixed point $p_i \in X_k^T$
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Integrability

(Belavin, Bershtein & Tarnopolsky '13; Pedrini, Sala & RS '14)

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- 6d $\mathcal{N}=(1,1)$ gauge theories/elliptic deformation of 2d CFT
  (Iqbal, Kozcaz & Yau '15; Nieri '15)