Dual variable representations

for (two) bosonic field theories at nonzero density

Falk Bruckmann
(U Regensburg)

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with C. Gattringer, T. Kloiber, T. Sulejmanpašić; J. Wellnhofer
[1507.04253, 1509.05189, 1607.02457; 1611.05643]

Heisenberg-
Programm

Deutsche
Forschungsgemeinschaft

The setting

- thermodynamic partition function

  $$Z = \mathrm{tr}\exp\left(-\beta\hat{H}[\phi]\right)$$

  as a field theory path integral

  $$Z = \int D\phi\,\exp\left(-\int d\vec{x}\int_0^\beta d\tau\,\mathcal{L}_{\mathrm{Eucl.}}(\phi,\partial_\tau\phi,\partial_{\vec{x}}\phi)\right)$$

  and corresponding lattice formulations = hoppings

  $$\phi(x)^*\phi(x+\hat{\nu})+c.c.$$

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  $$\phi(x)^*\phi(x+\hat{\nu})e^{-\mu}+c.c.e^{\mu} \quad \text{for temporal } \hat{\nu}$$

- ϕ complex and action invariant under U(1) phase rotations

  conserved charge to which a chemical potential μ can be coupled

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and corresponding lattice formulations = hoppings

$$\phi(x)^*\phi(x+\hat{\nu})e^{-\mu} + c.c.e^{\mu} \quad \text{for temporal } \hat{\nu}$$

$\uparrow$ not *c.c.* anymore

- ϕ complex and action invariant under U(1) phase rotations

conserved charge to which a chemical potential μ can be coupled

$\Rightarrow$ generically a sign problem (in the second repr. only)

The models

relativistic bosons on lattices

lattice spacing set to 1

- CP(N-1) models in 1+1d:

$$S = J \sum_{x,\nu} \sum_{f=1}^{N} \Big[\phi_f^*(x) U_\nu(x) \phi_f(x+\hat{\nu}) e^{-\mu_f \delta_{\nu,0}} + \underbrace{\phi_f(x) U_\nu^*(x) \phi_f^*(x+\hat{\nu}) e^{\mu_f \delta_{\nu,0}}}_{\text{not } c.c. \text{ unless imag. } \mu} \Big]$$

$|\phi|^2 = 1$: nontrivial theory, e.g. dynamical mass gap

$U_\nu(x) \in U(1)$: no Maxwell term $\Rightarrow$ auxiliary

integrate out $\Rightarrow$ action quartic in ϕ

global 'flavor' symmetry $U(N) \ni U(1)^N$: conserved charges $\Rightarrow \mu_f$

local (gauge) symmetry $U(1)_{\text{diag}}$: total charge vanishes

- O(N) models similar: invariant under $SO(2) \cong U(1)$

- QCD in 3+1d with scalar quarks: SU(2): electroweak, GUT?!

$$S = \beta \sum_{x,\nu} \sum_{f=1}^{N} \left[\phi_f^\dagger(x) U_\nu(x) \phi_f(x+\hat\nu) e^{-\mu_f \delta_{\nu,0}} + \underbrace{\phi_f(x) U_\nu^\dagger(x) \phi_f^\dagger(x+\hat\nu) e^{\mu_f \delta_{\nu,0}}}_{\text{not } c.c. \text{ unless imag. } \mu} \right]$$

  ϕ integrated with mass term = gaussian measure

  $U_\nu(x) \in SU(3)$: 'gluons', no kinetic term yet = strong coupling

  each $\phi_f(x) \in \mathbb{C}^3$: colored

  again global flavor symmetry $U(N)$

  local (gauge) symmetry $SU(3)$

- other models where dual variables solved sign problems

  C. Gattringer and his group, 2011-last weak

Idea of dual variables

(ii) integrate out original fields

especially the U(1) angles to which μ's couple

(i) at the expense of expanding the weight e^{-S} first

expansion variables = dual variables/occup. numbers: integers

- exact mapping (up to interchanging integrals and infinite sums) to new degrees of freedom

▷ with some luck: new weight positive $\rightsquigarrow$ num. simulations

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- indeed solves the sign problem in CP(N-1) and O(N) models

  almost obvious from O(2) model but other dualizations in the literature

- not so for the *fermionic* sign problem in QCD, even at strong coupl.

  Karsch, Mütter 89, Unger

Dualizing CP(N-1)

(i) expand each term in e^{-S} neglecting $\ldots_\nu^f(x)$

$$e^{J[\phi^* U\phi]e^{-\mu\delta_{\nu,0}}}\, e^{J[\phi U^*\phi^*]e^{\mu\delta_{\nu,0}}} = \sum_{k,\bar{k}=0}^{\infty} \frac{J^{k+\bar{k}}}{k!\bar{k}!} \underbrace{[\phi^* U\phi]^k [\phi U^* \phi^*]^{\bar{k}}}_{r^{k+\bar{k}+\text{shifted}}\, e^{i\varphi(k-\bar{k}-\text{shifted})}\, e^{iA(k-\bar{k})}} e^{-\mu(k-\bar{k})_{\nu=0}}$$

used $\phi = re^{i\varphi}$ and $U = e^{iA}$

(ii) integrate out original fields

- radii: doable (ratio of gamma functions) $\Rightarrow$ positive weight
- angles: $\int_0^{2\pi} d\varphi\, e^{i\varphi X} = 2\pi\delta_{\text{Kronecker}}(X)$

  with $X = \sum_\nu [m_\nu^f(x) - m_\nu^f(x-\hat{\nu})] = \nabla_\nu m_\nu^f(x)$ and $m = k - \bar{k}$

  explicit conservation of symmetry currents m^f $\Rightarrow$ world lines

- μ's couple to the conserved charges: $e^{-\beta\mu_f \sum_{\vec{x}} m_0^f(\vec{x})}$

  as in the energy representation of the grand canonical ensemble

  conserved charge is the net number of *m*-loops winding in time

- μ's do not cause minus signs in the weight just real μ

  $\Rightarrow$ sign problem solved if there was none at vanishing μ

  cancellations in $\int_0^{2\pi} d\varphi\, e^{i\varphi X} \sim \delta(X)$: either positive or zero
  $\uparrow$ neglect

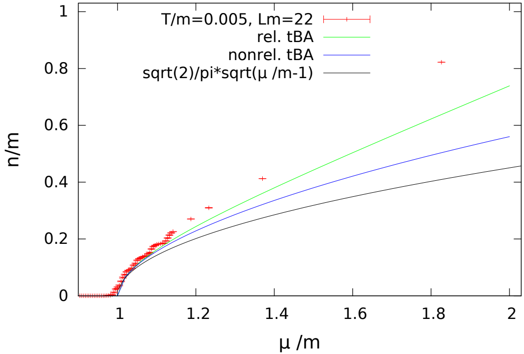
- gauge fields:

  $$\int_0^{2\pi} dA_\nu(x) e^{iA_\nu(x)\sum_f m_\nu^f(x)} = \delta\big(\textstyle\sum_f m_\nu^f(x)\big)$$

  total charge (over all flavors) vanishes

- constraint systems $\Rightarrow$ lattice simulations with worm algorithm

- low T: no particle/charge density until μ reaches the dyn. mass



$\Rightarrow$ quantum phase transition at $\mu = m$ for $T = 0$, else crossover

- for $\mu \gtrsim m$ contact to Bethe ansätze for repulsive 1d bosons
  $\approx$ free 1d fermions — Lieb-Liniger model

- $T \to 0$ & thermod. limit need $N_t, N_s \to \infty$, up to 6400×160 or 320^2

Physics of the O(3)=CP(2) model at nonzero μ II

SS/SD

- susc. of FE wrt. spatially TBC

$$\varphi(x_0, x_1 + L) = \varphi(x_0, x_1) + \alpha \quad \Rightarrow \quad \sigma := L\,\partial_\alpha^2 F\big|_{\alpha=0} \sim \langle w_{\text{spat}}(x)^2 \rangle$$

- FSS

$$\sigma = L^{1-z}\,\text{func}\Big(L^{1/\nu}\big(\frac{\mu}{\mu_c} - 1\big),\, TL^z\Big)$$

$z \ldots$ DCE

Fisher et al. 89

- found a universal σ curve if TL^2 kept constant
  $\Rightarrow z \simeq 2$ (and $\nu \simeq 1/2$) in agreement with free 1d fermions

Physics of the O(3)=CP(2) model at nonzero μ II

Spin Stiffness/Superfluid Density

- susc. of Free Energy wrt. spatially Twisted Boundary Conditions

$$\varphi(x_0, x_1 + L) = \varphi(x_0, x_1) + \alpha \quad \Rightarrow \quad \sigma := L\,\partial_\alpha^2 F\big|_{\alpha=0} \sim \langle w_{\mathrm{spat}}(x)^2 \rangle$$

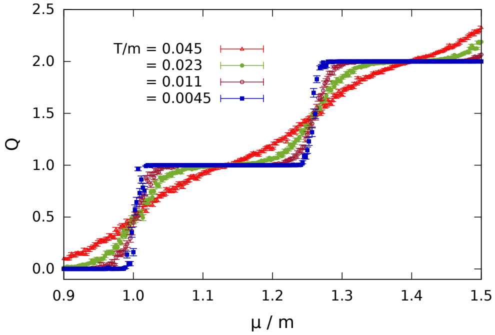
- Finite Size Scaling

$$\sigma = L^{1-z}\,\mathrm{func}\Big(L^{1/\nu}\big(\frac{\mu}{\mu_c} - 1\big), TL^z\Big)$$

$z \ldots$ Dynamical Critical Exponent — Fisher et al. 89

- found a universal σ curve if TL^2 kept constant
  $\Rightarrow z \simeq 2$ (and $\nu \simeq 1/2$) in agreement with free 1d fermions

- low T and small sizes L (but $> 1/m$)



$Lm = 4.4$

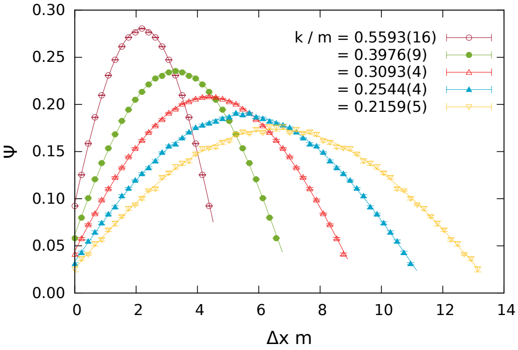
$\Rightarrow$ plateaus and sharp jumps in particle number as function of μ

- $\mu_{c,1} = m$: mass threshold as for large L above

  $\mu_{c,2} = E_{\text{min}}^{Q=2} \Rightarrow$ particle interaction $\Rightarrow$ phase shifts δ a la Lüscher

- on $Q = 2$ plateaus

distribution (histogram) of two unit m_0's $\leftarrow$ particle world lines = probability $|\psi(\Delta x_1)|^2$:



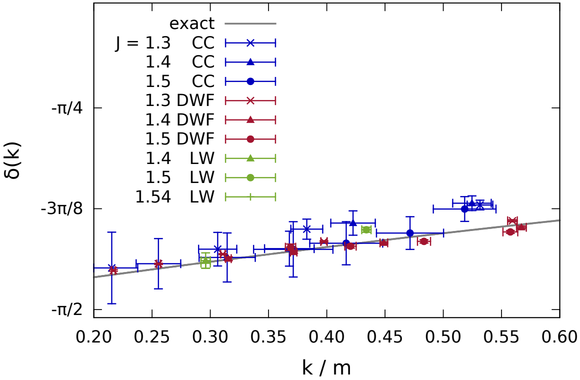
different L's

$\Rightarrow$ perfect standing waves, up to a cusp at the origin

$\Rightarrow$ two particle potential is Dirac-delta like $\Rightarrow$ phase shifts again

- results for phase shifts δ as a function of momentum k

  Charge Condensation and Dual Wave Function method:



- agrees with analytical result from S-matrix: Zamolodchikov[2] 78

$$\delta(k) = -\arctan \frac{\pi}{2\,\mathrm{arsinh}(k/m)}$$

  and lattice spectroscopy data Lüscher, Wolff 90

Dualizing scalar QCD

- action again, for simplicity same μ for all flavors:

$$S = \beta \sum_{x,\nu} \mathrm{tr}\Big[\overbrace{\sum_f \phi_f(x+\hat{\nu})\phi_f(x)^\dagger}^{J_\nu(x)\ \text{(matrix)}} U_\nu(x)\, e^{-\mu\delta_{\nu,0}} + J_\nu(x)^\dagger U_\nu(x)^\dagger e^{\mu\delta_{\nu,0}} \Big]$$

- $U_\nu(x) \in SU(3)$: group integrals not so simple

  fortunately a closed expression exists: Eriksson, Svartholm, Skagerstam 81

$$\int dU \exp\left(\mathrm{tr}\left[JUe^{-\mu} + J^\dagger U^\dagger e^{-\mu}\right]\right) = \sum_{a,b,c,k,\bar{k}=0}^{\infty} \frac{\beta^{2a+..+3\bar{k}}\,\mathrm{pos}(a,..,\bar{k})}{a!..\bar{k}!}$$

$$\times\ (\mathrm{tr}JJ^\dagger)^a \times \mathcal{O}((JJ^\dagger)^2)^b \times (\det JJ^\dagger)^c \times (\det J\, e^{-\mu})^k \times (\det J^\dagger e^{\mu})^{\bar{k}}$$

- upon integrating the link fields (step ii) we have expanded e^{-S} into a five-fold sum (step i) with dual variables/occup. numbers $(a,..,\bar{k})$

Interpreting dualized scalar QCD

$$(\mathrm{tr}JJ^\dagger)^a \times \mathcal{O}((JJ^\dagger)^2)^b \times (\det JJ^\dagger)^c \times e^{-3\mu(k-\bar{k})_{\nu=0}} \times (\det J)^k \times (\det J^\dagger)^{\bar{k}}$$

- first three terms μ-independent: 'mesons' since quark-antiquark positive functions of the positive operator $JJ^\dagger$
- next term: 3μ couples to the charge of the current $k - \bar{k} = m$

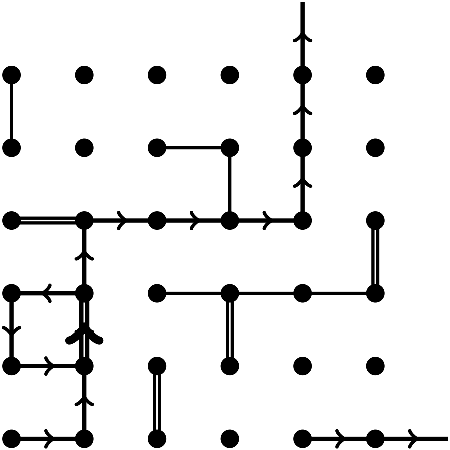
  $\uparrow$ baryon chemical potential ✓

  current conserved? yes, by the remaining integral over ϕ-integral, schematically:

  $$\int_{\mathbb{C}} d\phi\, e^{-\mathrm{mass}^2|\phi|^2}\, \phi^A \phi^{\dagger B} \neq 0 \quad \text{iff } A = B \qquad \text{(angle integration!)}$$

  constrains the last two terms exactly such that m conserved
- last two terms: '(anti)baryons' since three (anti)quarks

- example configuration



- differences to fermionic QCD:

  k, $\bar{k}$ from 0 to $\infty \leftarrow$ bosonic occup. numbers (empty sites possible)

  intersections of mesons and baryons possible

Sign problem in scalar QCD

depends crucially on the number of flavors:

- $N = 1, 2$: μ-independent

  no (anti)baryons: $\det J = \det_{3\times 3}\left(\phi^{\text{shifted}}_{f=1} \otimes \phi^{\dagger}_{f=1} + \phi^{\text{shifted}}_{f=2} \otimes \phi^{\dagger}_{f=2}\right) = 0$

  matrix has at most two indep. rows/columns

  no sign problem

- $N = 3$: μ-dependent

  'scalar baryon needs 3 flavors'

  note that:

$$\det J = \det_{3\times 3}\Big(\sum_{f=1}^{3} \phi_f(x+\hat{\nu}) \otimes \phi_f(x)^{\dagger}\Big) = \det\big(\phi_1|\phi_2|\phi_3\big)_{x+\hat{\nu}} \det\big(\phi_1|\phi_2|\phi_3\big)^{*}_{x}$$

  along a loop $\det(\ldots)^{*}_{x}$ meets $\det(\ldots)_{x}$ from the next (anti)baryon

  positive $\Rightarrow$ sign problem solved

- $N \geq 4$: μ-dependent

  sign problem unsolved

  a similar formula for the decomposition of the determinant exists, but $\det(\ldots)_x^*$ does not only meet $\det(\ldots)_x$

- this case would be interesting for going beyond strong coupling

  if plaquette action linearized to $\exp\left(\mathrm{tr}\left[K_\nu(x)U_\nu(x) + K_\nu^\dagger(x)U_\nu^\dagger(x)\right]\right)$ then U and $U^\dagger$ can again be integrated out

  e.g. via Hubbard-Stratonovich bosons — Vairinhos, de Forcrand 14

  or 'induced QCD' (= bosons for each plaquette) — Budczies, Zirnbauer 03; Brandt, Lohmayer, Wettig 16

Revisit fermionic QCD

- action, to be dualized

$$S = \beta \sum_{x,\nu} \eta_\nu(x) \, \mathrm{tr}\Big[\sum_f \psi_f(x+\hat{\nu})\psi_f(x)^\dagger U_\nu(x)\, e^{-\mu\delta_{\nu,0}} - \ldots e^{\mu\delta_{\nu,0}} \Big]$$

- sources of minus signs:
  – staggered fermion factors $\eta \in \{-1, 1\}$
  – minus in front of second term: Dirac operator is first order
  – reordering Grassmannians in final integration: -1 per quark loop
  – antiperiodic boundary conditions: -1 per winding quark loop

$\Rightarrow$ $\exists$ configurations with negative weights, at $\mu = 0$ already (!) example

- all sources absent for scalar quarks:
  'separated μ-sign problem from fermionic sign problem'

Instead of conclusions

Thanks to Shailesh, Christof and Dean!