Intro to TNs (2): PEPS & MERA

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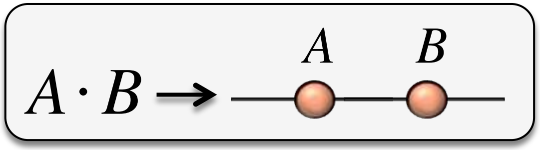
Tensor Networks

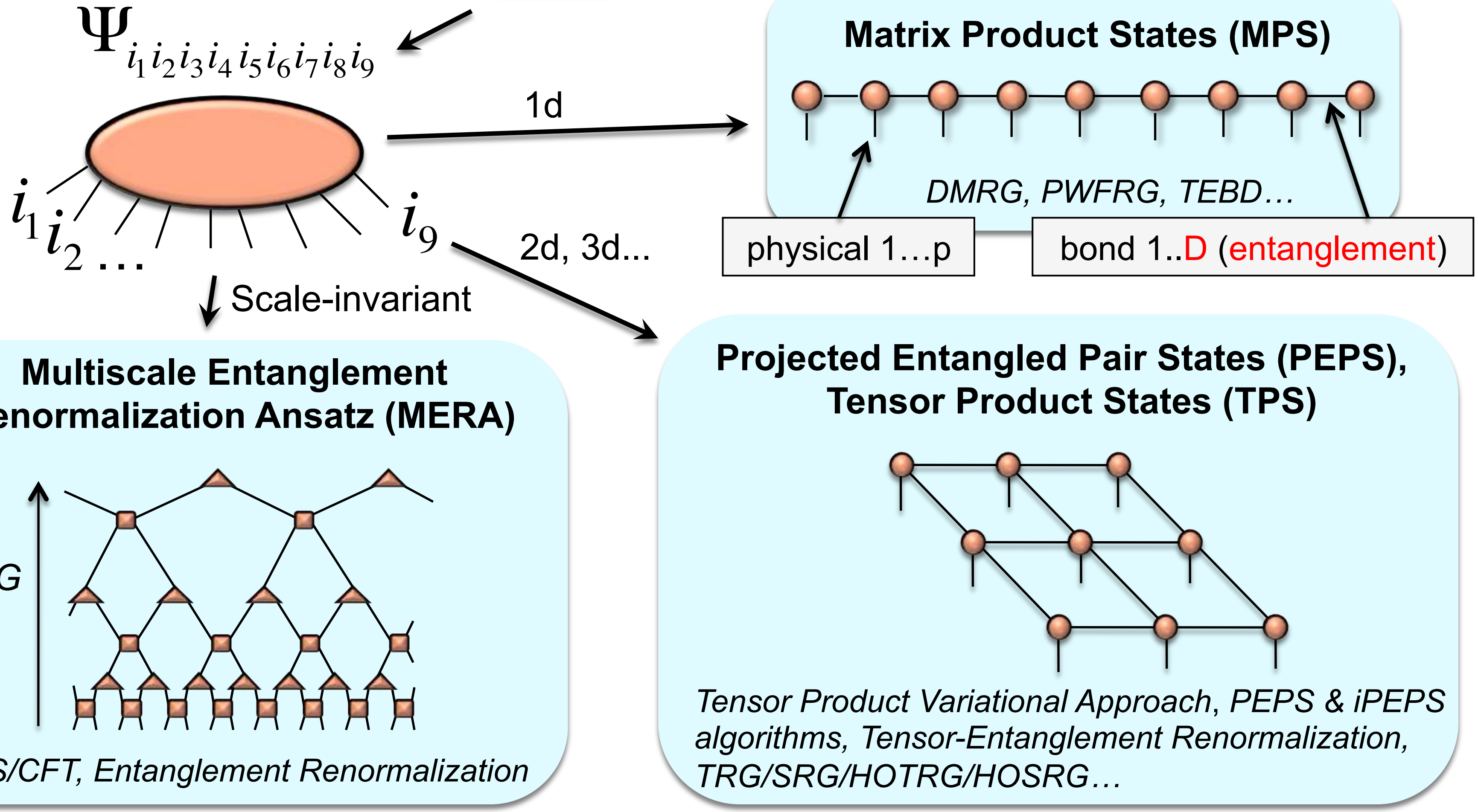
e.g. RO, Annals of Physics ***349*** *(2014) 117–158*



$$A \cdot B \rightarrow$$

$$|\Psi\rangle = \sum_{i's} \boxed{\Psi_{i_1 i_2 \dots i_N}} |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle$$

p-level systems



$$\Psi_{i_1 i_2 i_3 i_4 i_5 i_6 i_7 i_8 i_9}$$

Matrix Product States (MPS)

DMRG, PWFRG, TEBD…

physical 1…p

bond 1..D (entanglement)

Multiscale Entanglement Renormalization Ansatz (MERA)

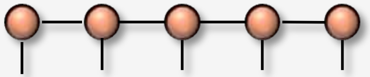
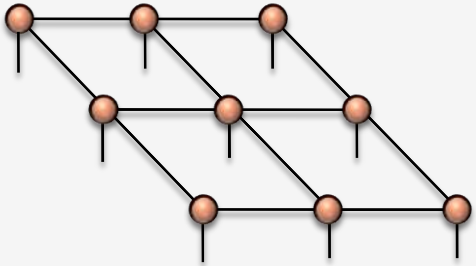
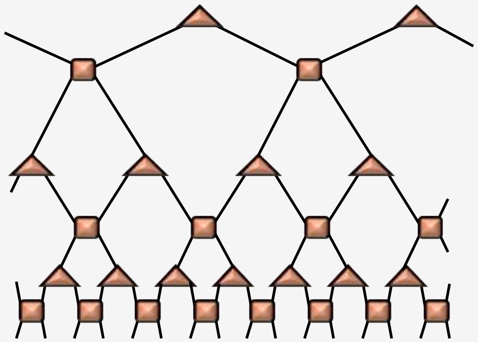
AdS/CFT, Entanglement Renormalization

Projected Entangled Pair States (PEPS), Tensor Product States (TPS)

Tensor Product Variational Approach, PEPS & iPEPS algorithms, Tensor-Entanglement Renormalization, TRG/SRG/HOTRG/HOSRG…

Efficient O(poly(N)), satisfy area-law, low-energy eigenstates of local Hamiltonians

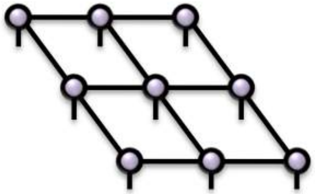
Comparison

	MPS in 1d 	*PEPS in 2d* 	*MERA in 1d* 
Ent. entropy	$S(L)=O(1)$	$S(L)=O(L)$	$S(L)=O(\log L)$
Exact contraction	efficient	inefficient	efficient
Corr. length	finite	finite & infinite	finite & infinite
To/from	1d Ham.	2d Ham.	1d Ham.
Tensors	arbitrary	arbitrary	constrained

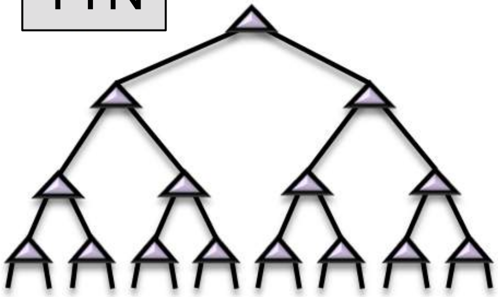
Increasing complexity...

MPS

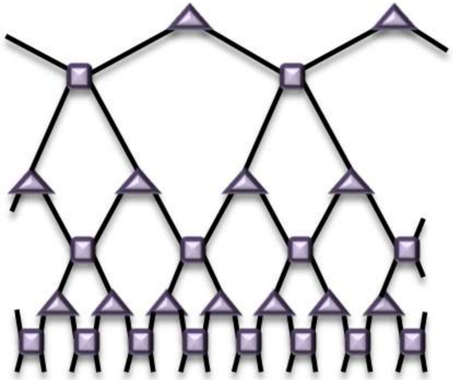
2d PEPS



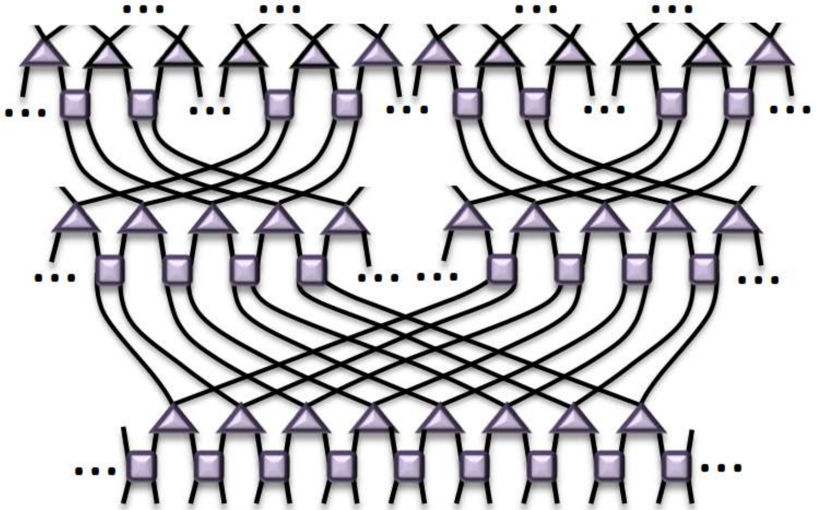
TTN



1d MERA



1d branching MERA



Exact in many cases
Variational ansatz for numerical simulations (e.g. DMRG)

Tensor Networks as an ansatz

Variational optimization

(e.g. DMRG)

Real/Imaginary time evolution

(e.g. TEBD)

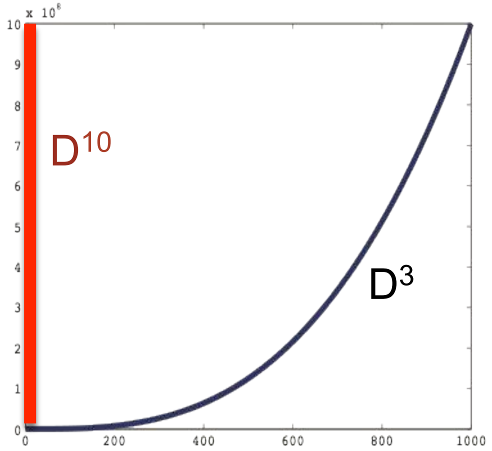
$$\min_{|\Psi\rangle\in TN} \langle\Psi|H|\Psi\rangle/\langle\Psi|\Psi\rangle$$

ground states

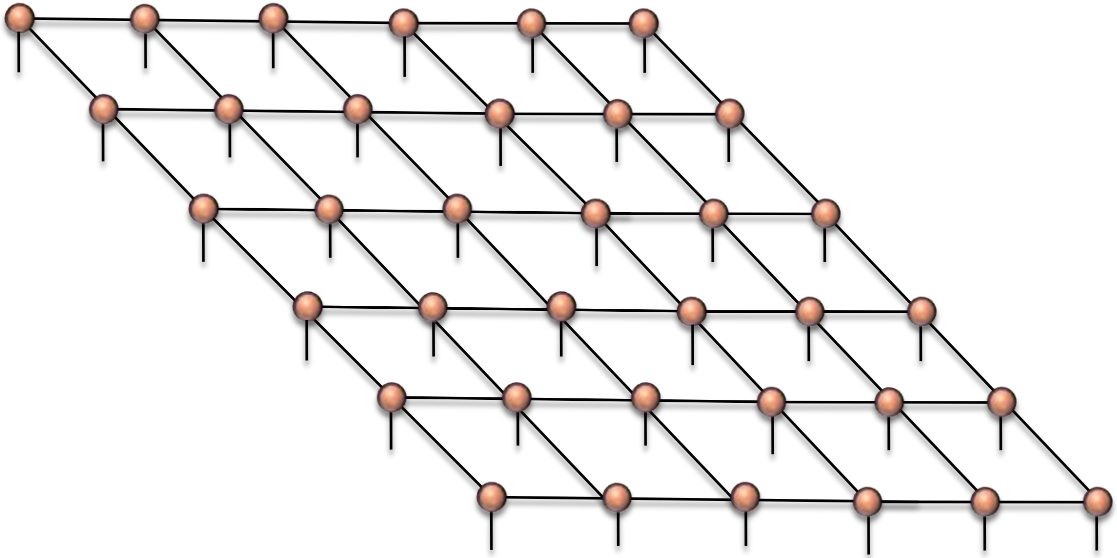
$$e^{-iHt}|\Psi\rangle \qquad e^{-Ht}|\Psi\rangle$$

dynamics ***ground states***

- MPS methods for **1d are very efficient** (e.g. DMRG obc → D^3)
- **2d PEPS** ~ D^{10}. **But low D expected** because of high connectivity, or entanglement monogamy**. D=2 can be critical.**
- **Infinite lattices** for translation-invariant systems (thermodynamic limit)
- **Internal symmetries, gauge symmetries, fermionic systems, continuum limit**
  (cMPS → quantum field theories)
- **Limitation:** amount and structure of **entanglement**



Projected Entangled Pair States (PEPS)



Two exact examples

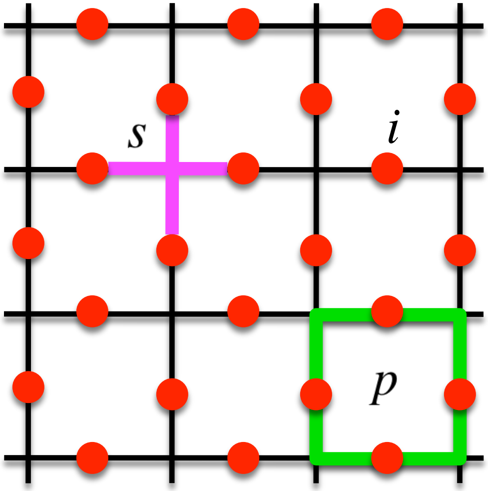
An exact example: Kitaev's Toric Code

Kitaev, 1997

$$H = -J\sum_s A_s - J\sum_p B_p$$

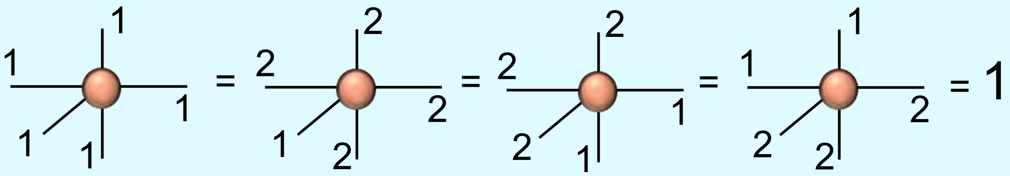
$$A_s = \prod_{i\in s}\sigma_i^x$$ *star operator*

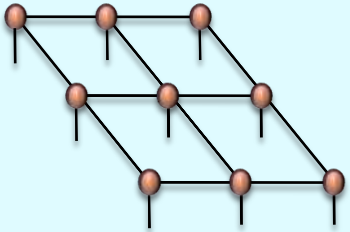
$$B_p = \prod_{i\in p}\sigma_i^z$$ *plaquette operator*



Simplest known model with "topological order"

Ground state (and in fact all eigenstates) are PEPS with D=2

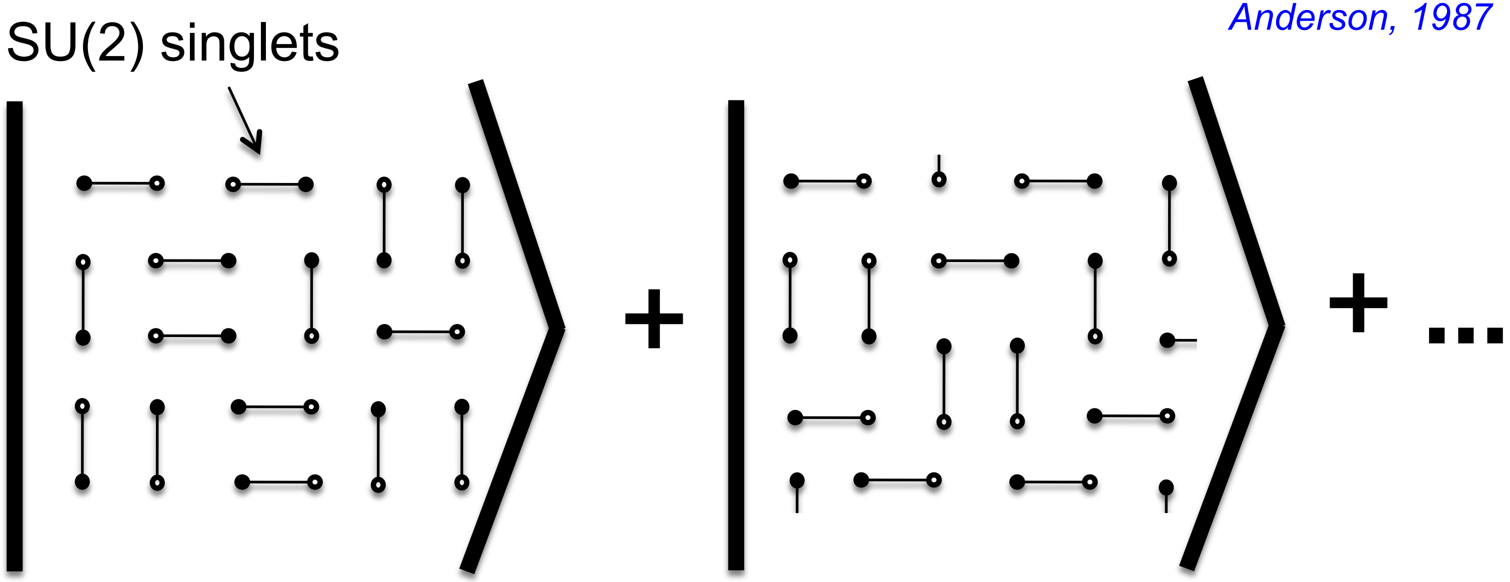




And another tensor rotated 90°

Resonating Valence Bond State

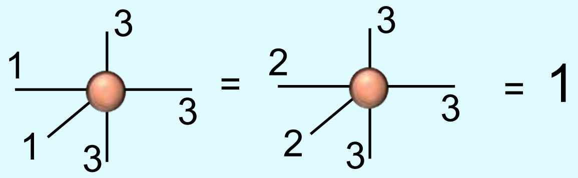
Anderson, 1987



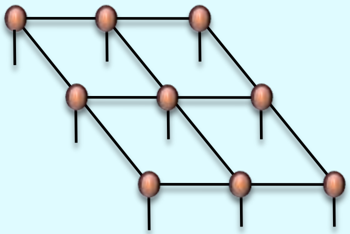
Equal superposition of all possible nearest-neighbor singlet coverings of a lattice (spin liquid)

Proposed to understand high-T_C superconductivity

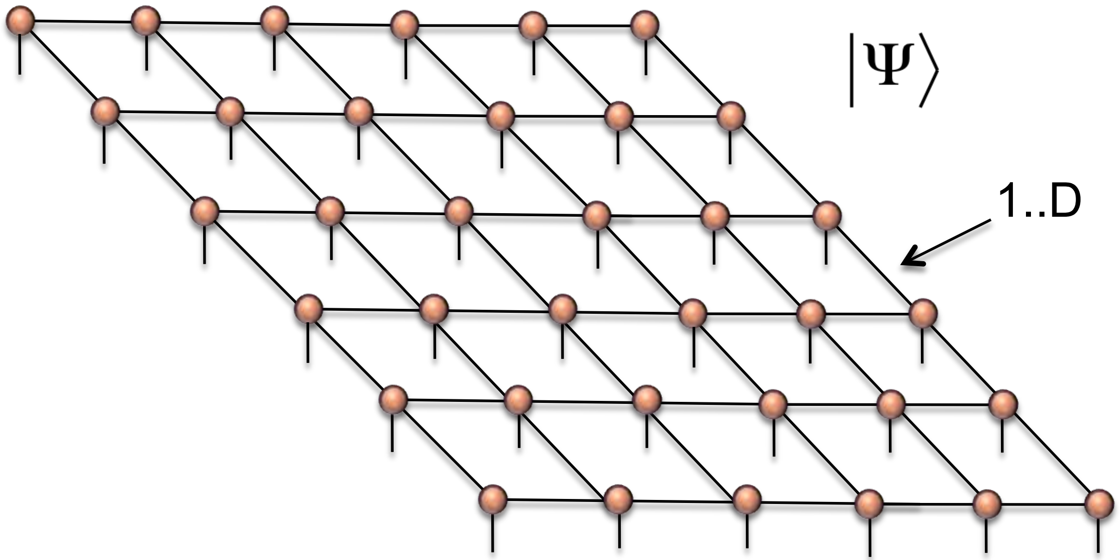
It is a PEPS with D=3



And rotations

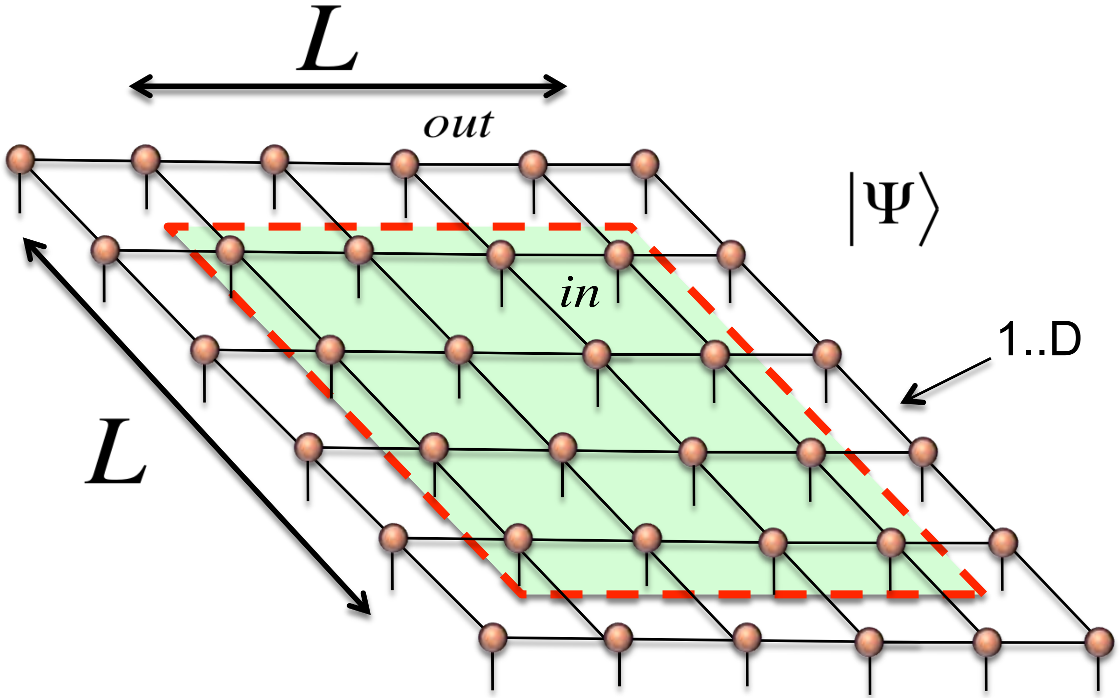


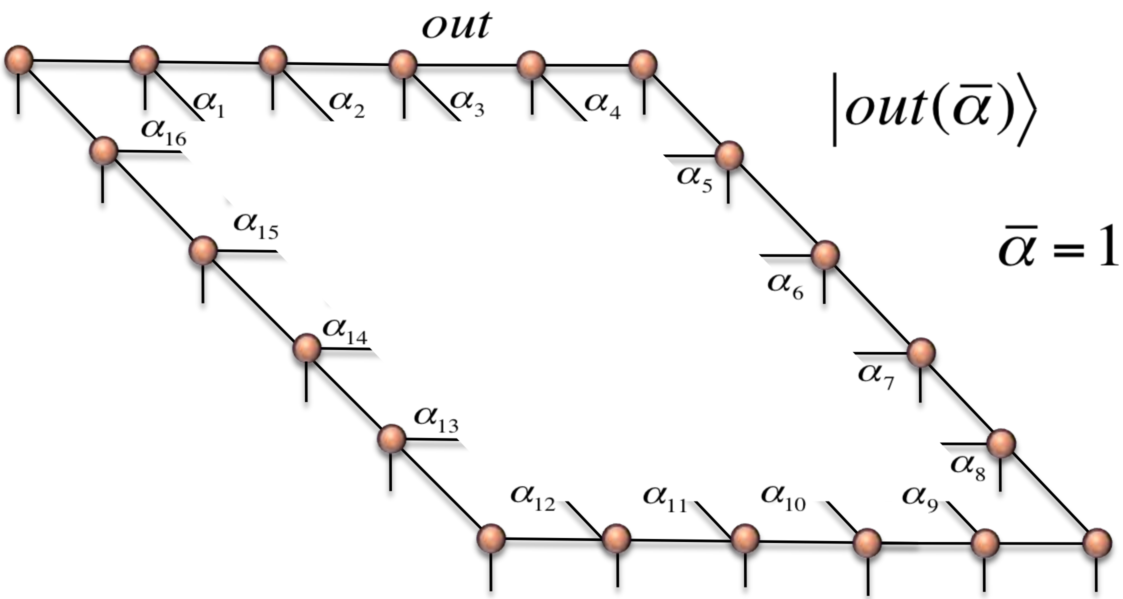
PEPS obey 2d area-law



$|\Psi\rangle$

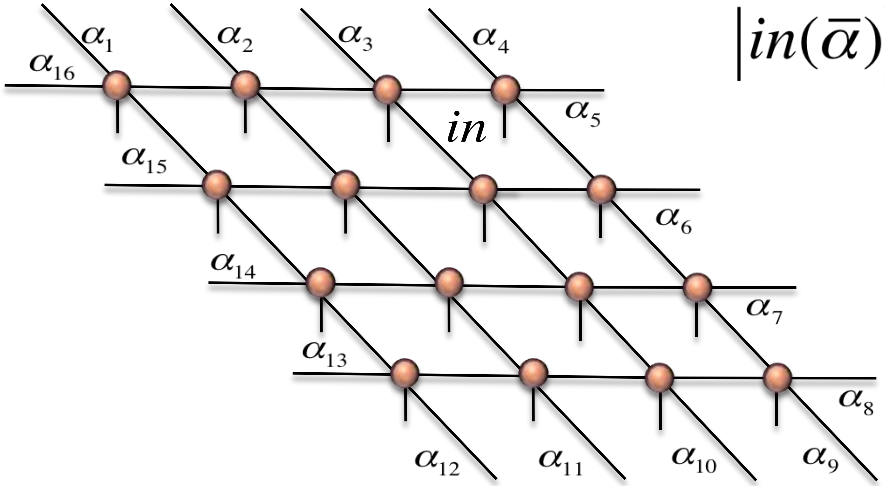
1..D


L
out
$|\Psi\rangle$
in
1..D
L



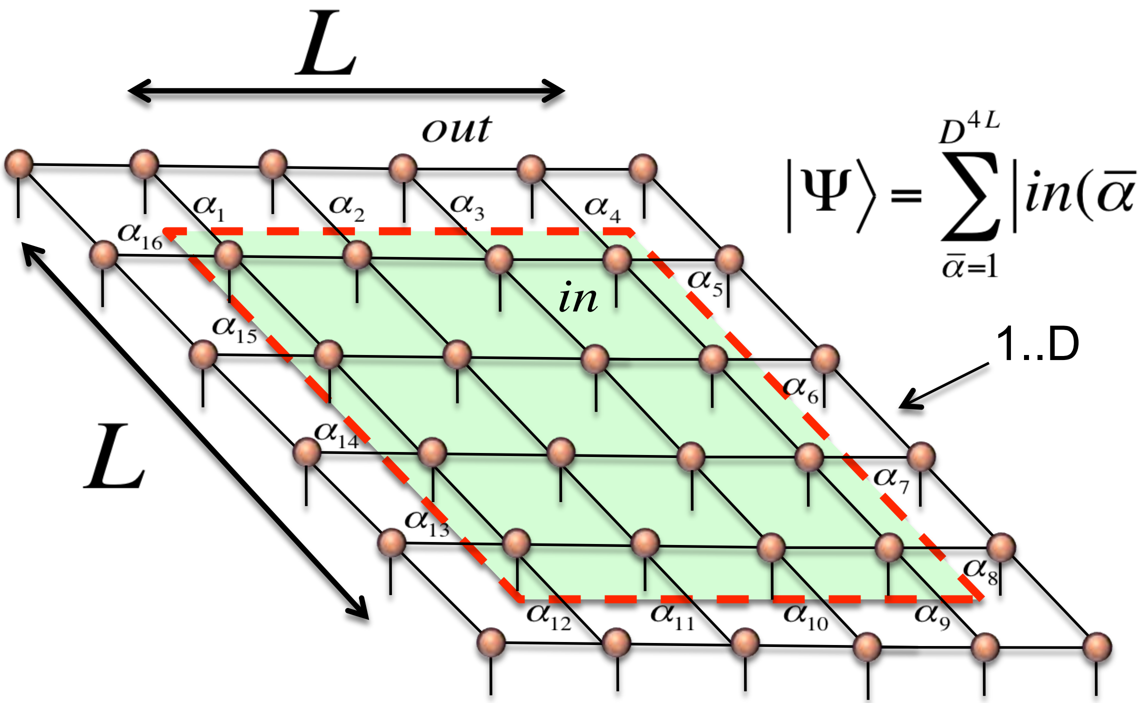
$$|out(\bar{\alpha})\rangle$$

$$\bar{\alpha} = 1, 2, \ldots, D^{4L}$$



$$|in(\bar{\alpha})\rangle$$

$$\bar{\alpha} = 1, 2, \ldots, D^{4L}$$



$$|\Psi\rangle = \sum_{\bar{\alpha}=1}^{D^{4L}} |in(\bar{\alpha})\rangle |out(\bar{\alpha})\rangle$$

$$\rho_{in} = \mathrm{tr}_{out}\left(|\Psi\rangle\langle\Psi|\right) = \sum_{\bar{\alpha},\bar{\alpha}'} X_{\bar{\alpha},\bar{\alpha}'} |in(\bar{\alpha})\rangle\langle in(\bar{\alpha}')|$$

$$X_{\bar{\alpha},\bar{\alpha}'} = \langle out(\bar{\alpha}')|out(\bar{\alpha})\rangle$$

$$rank(\rho_{in}) \leq D^{4L}$$

$$S(L) = -\mathrm{tr}\left(\rho_{in}\log\rho_{in}\right) \leq \log(D)\boxed{4L}$$

prefactor

size of the boundary

PEPS as ansatz:
variational optimization

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$|\Psi\rangle \xrightarrow{\min\limits_{A^1}} |\Psi^1\rangle \xrightarrow{\min\limits_{A^2}} \cdots \xrightarrow{\min\limits_{A^N}} |\Psi^N\rangle \xrightarrow{\min\limits_{A^{N-1}}} |\Psi^{N+1}\rangle \xrightarrow{\min\limits_{A^{N-2}}} \cdots$$

$$E \quad \geq \quad E^1 \quad \geq \quad \cdots \quad \geq \quad E^N \quad \geq \quad E^{N+1} \quad \geq \quad \cdots$$

$$\frac{\partial}{\partial A^{*i}}\left(\overset{A^{*i}}{\langle\Psi|} H \underset{A^i}{|\Psi\rangle} - \lambda\overset{A^{*i}}{\langle\Psi|}\underset{A^i}{\Psi\rangle}\right) = 0$$

Minimization of quadratic function

$$\mathbf{H}^i_{eff}\vec{A}^i = \lambda\mathbf{N}^i\vec{A}^i$$

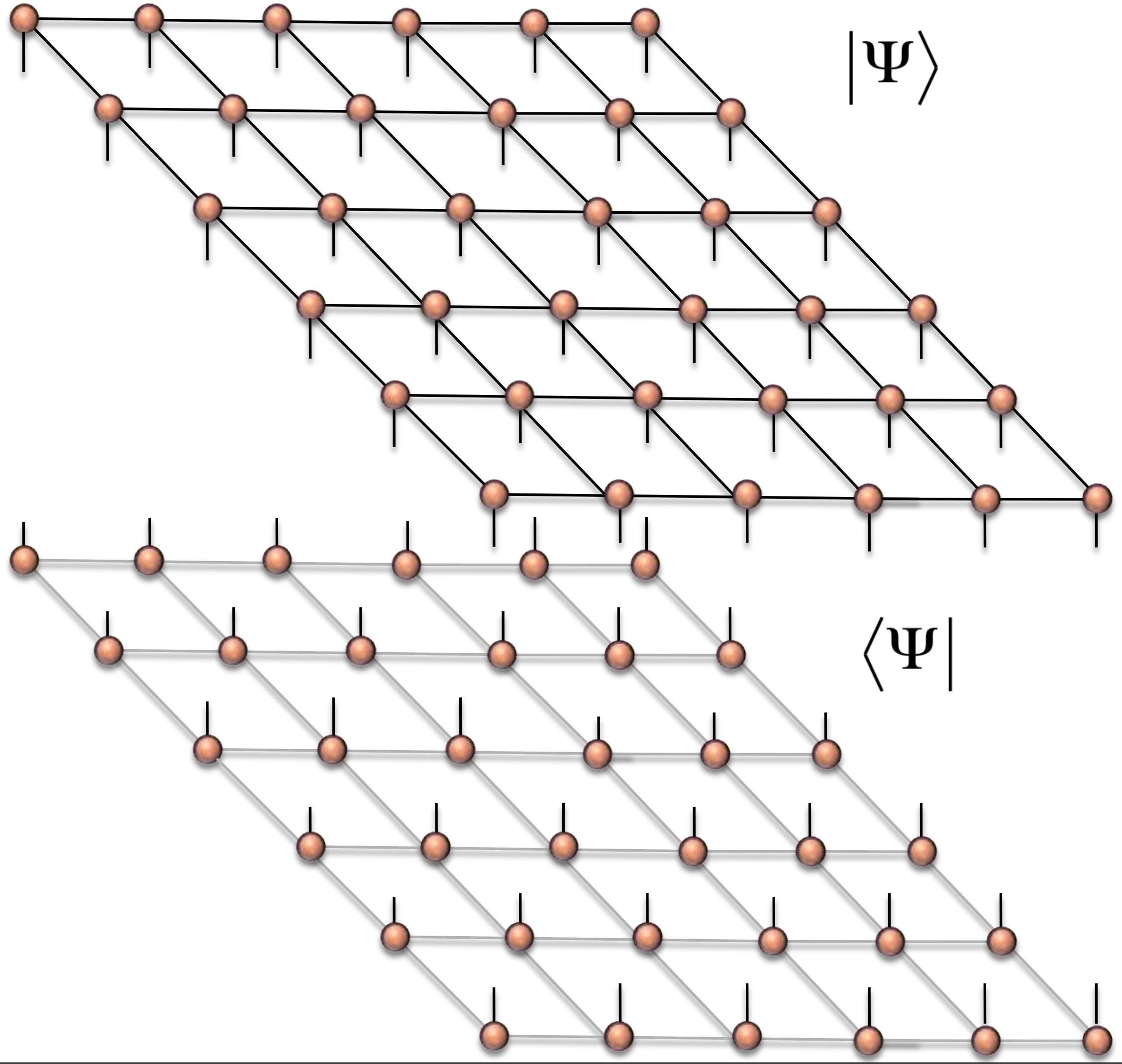
Generalized eigenvalue problem

Once $\mathbf{H}^i_{eff}$ and $\mathbf{N}^i$ are known, we can solve this problem efficiently

Approximate calculation of $\mathbf{H}^i_{eff}$ and $\mathbf{N}^i$

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda\mathbf{N}^i\vec{A}^i$$



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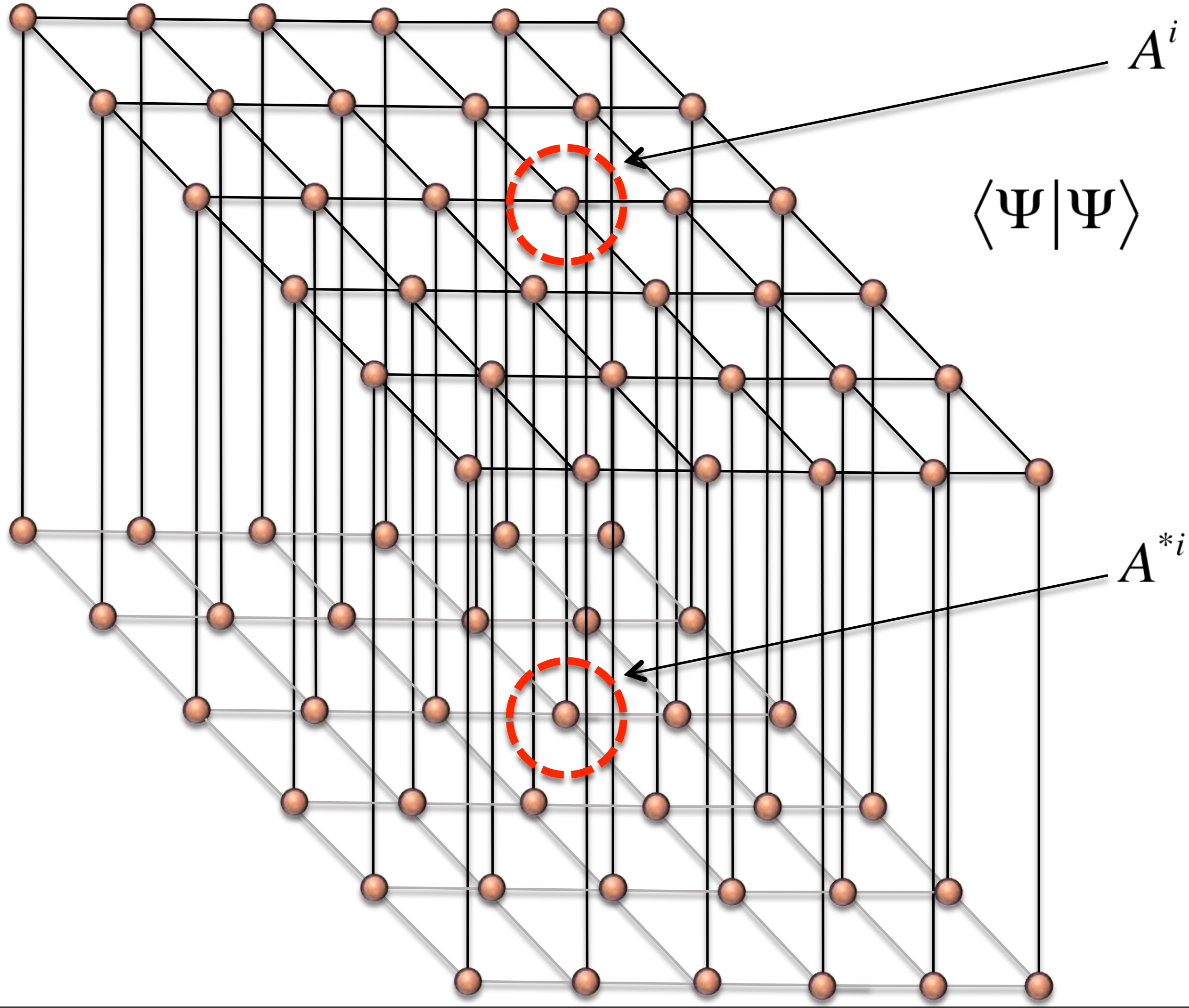

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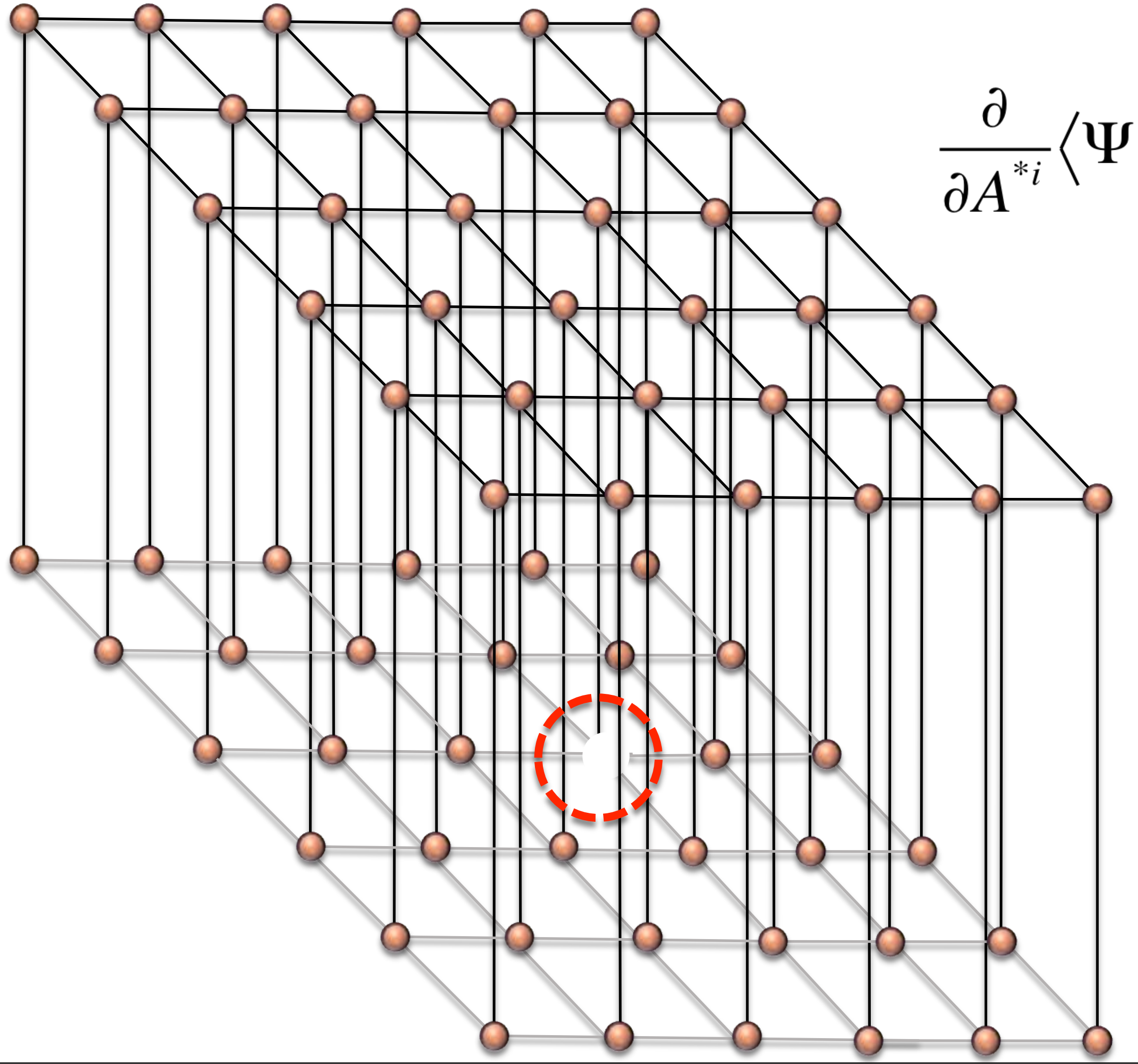

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle=\mathbf{N}^i\vec{A}^i$$

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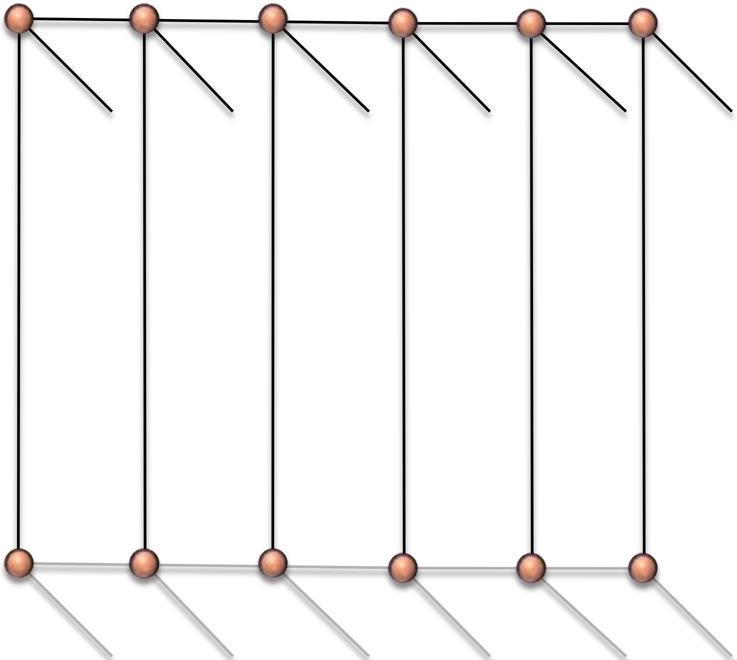

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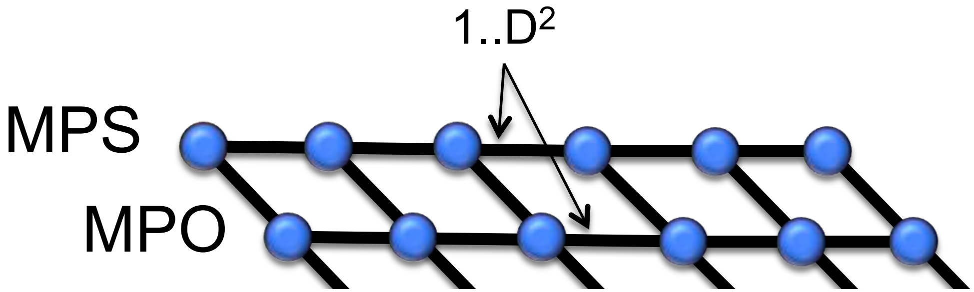
MPS

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i\vec{A}^i$$

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda\mathbf{N}^i\vec{A}^i$$



$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i\vec{A}^i$$

1d problem: use a 1d method for MPS (e.g., DMRG or TEBD)

e.g. calculation of $\mathbf{N}^i\vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda\mathbf{N}^i\vec{A}^i$$

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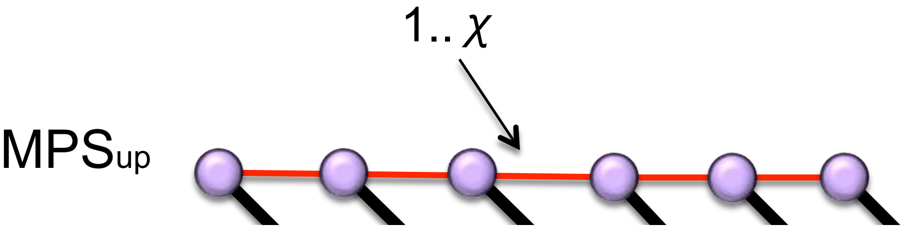

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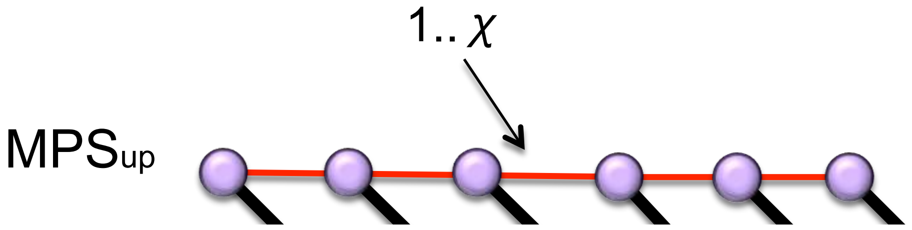

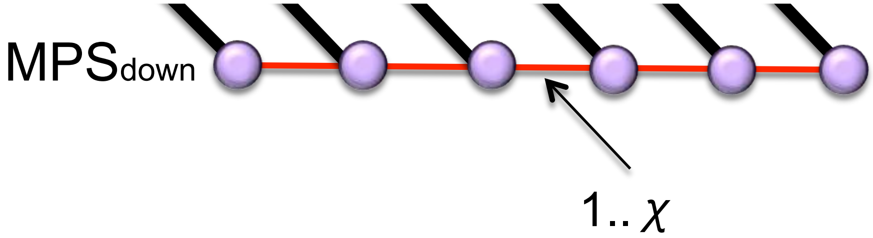

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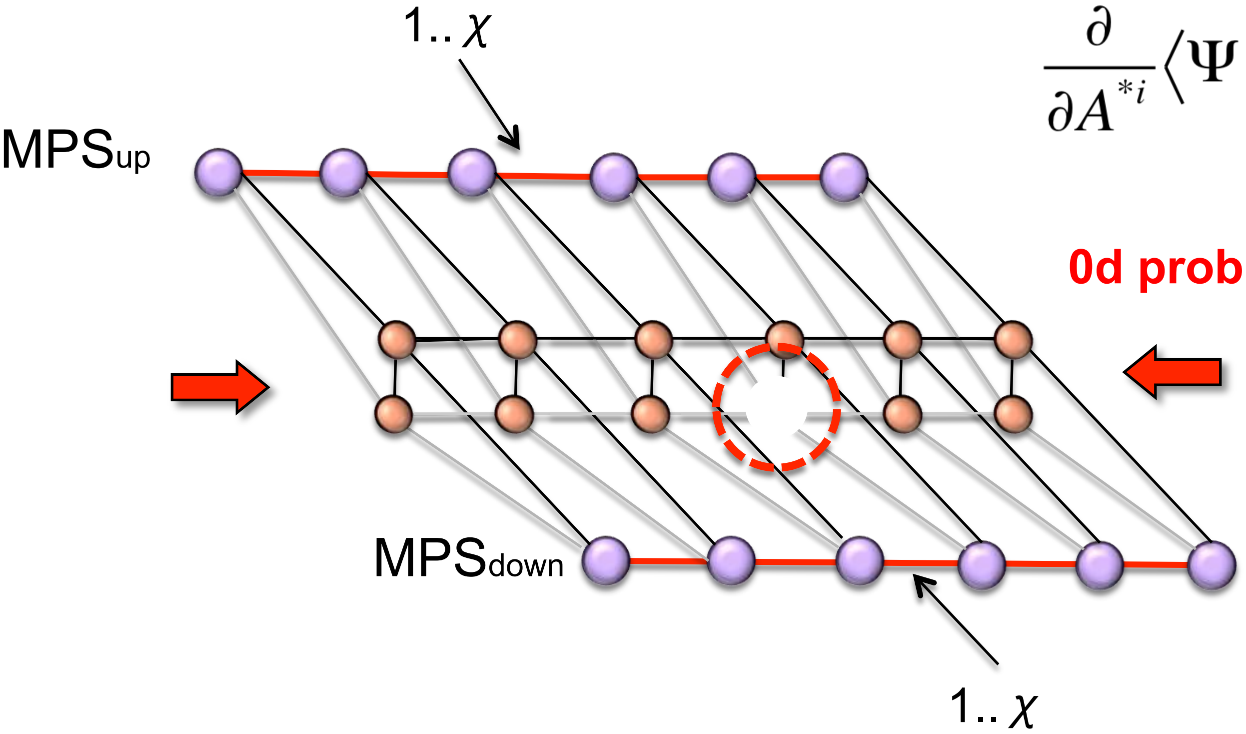


0d problem: exact!

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$

Dimensional reduction

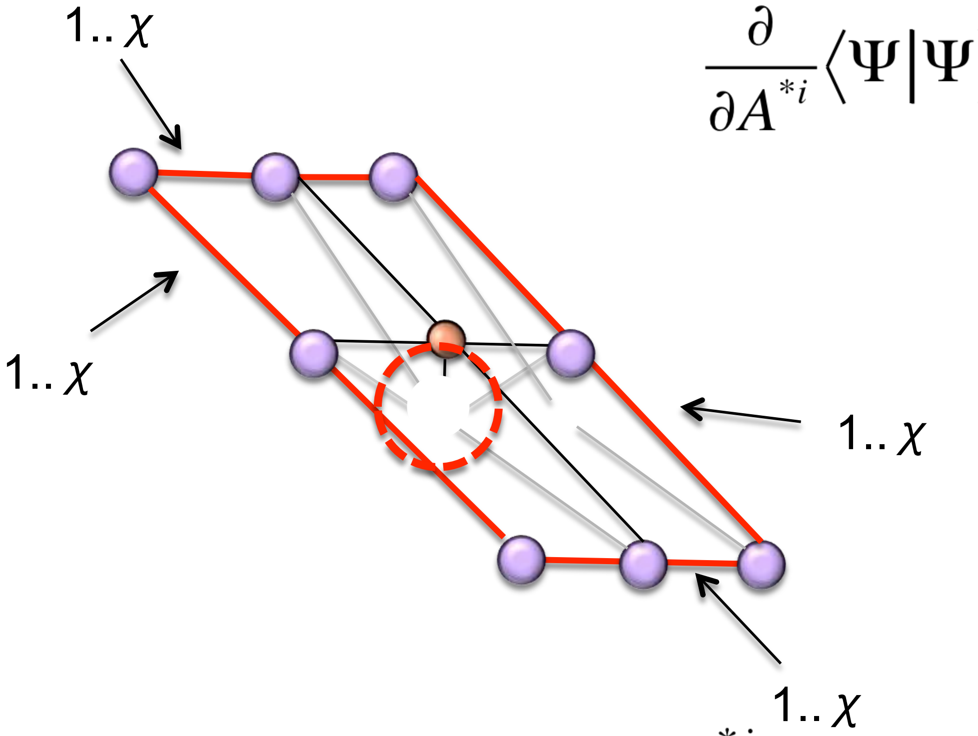
2d problem

↓

1d problem: use DMRG!

↓

0d problem: exact!



$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i\vec{A}^i$$

$\mathbf{N}^i\vec{A}^i$ is the ***environment*** of tensor A^{*i}

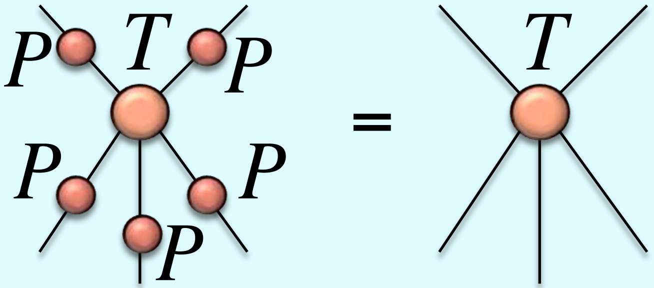
$\mathbf{H}^i_{eff}\vec{A}^i$ is computed similarly, but sandwiching with the Hamiltonian

Valid also for any expectation value

Tensor Networks + Fermions

e.g., P. Corboz, R. Orús, B. Bauer, G. Vidal, PRB 81, 165104 (2010)

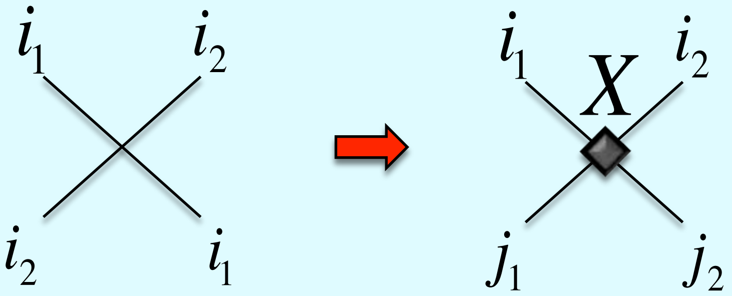
Tensor Network "fermionization" rules



*Use **parity-preserving** tensors*

$$T_{i_1 i_2 \ldots i_M} = 0 \ \text{ if } \ P(i_1)P(i_2)\ldots P(i_M) \neq 1$$

Symmetry of the Hamiltonian



Replace crossings by

fermionic swap gates

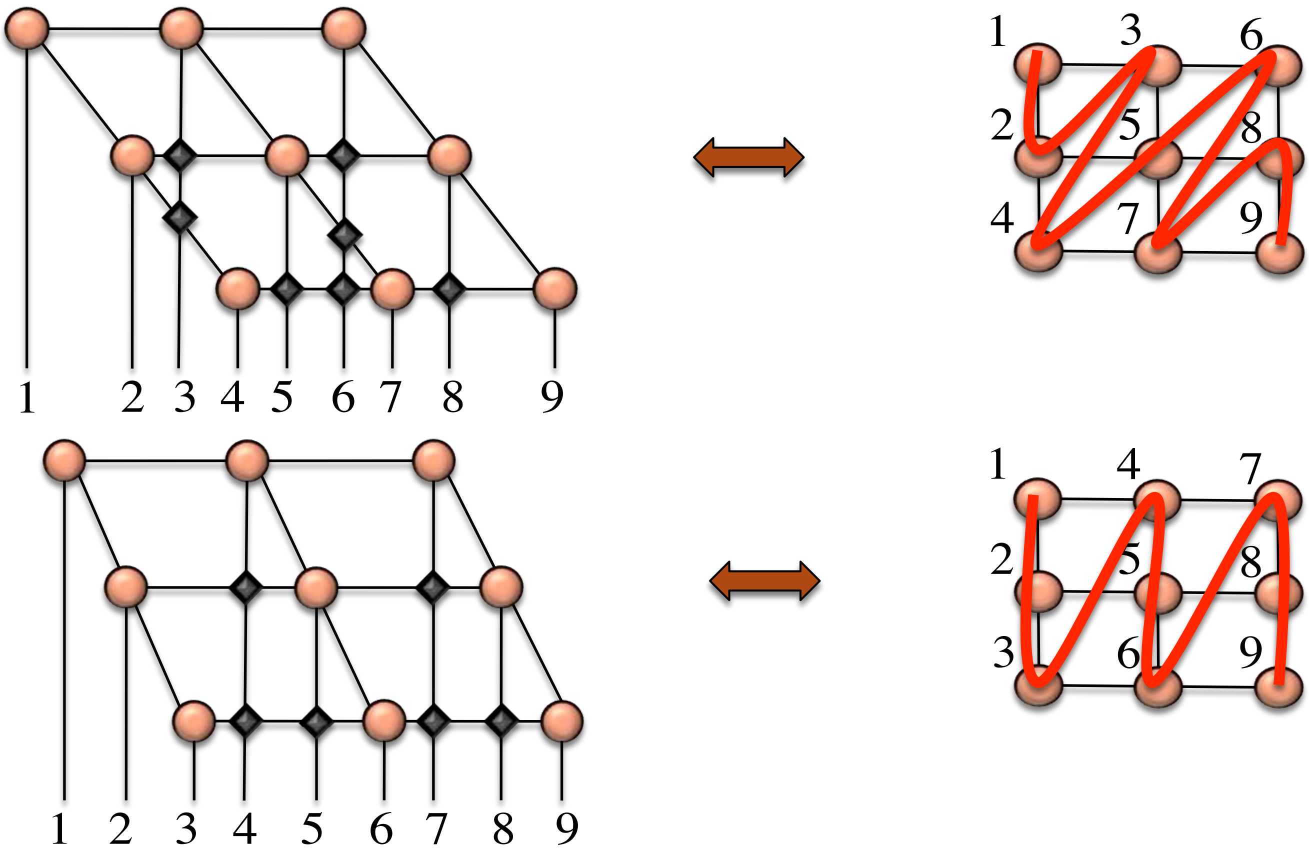
$$X_{i_2 i_1 j_1 j_2} = \delta_{i_1 j_1}\delta_{i_2 j_2} S(P(i_1),P(i_2))$$

$$S(P(i_1),P(i_2)) = \begin{cases} -1 & \text{if } \ P(i_1) = P(i_2) = -1 \\ +1 & \text{otherwise} \end{cases}$$

*Fermionic operators **anti**commute*

The leading order of the computational cost is the same as in the bosonic case

fermionic order ~ graphical projection of a PEPS



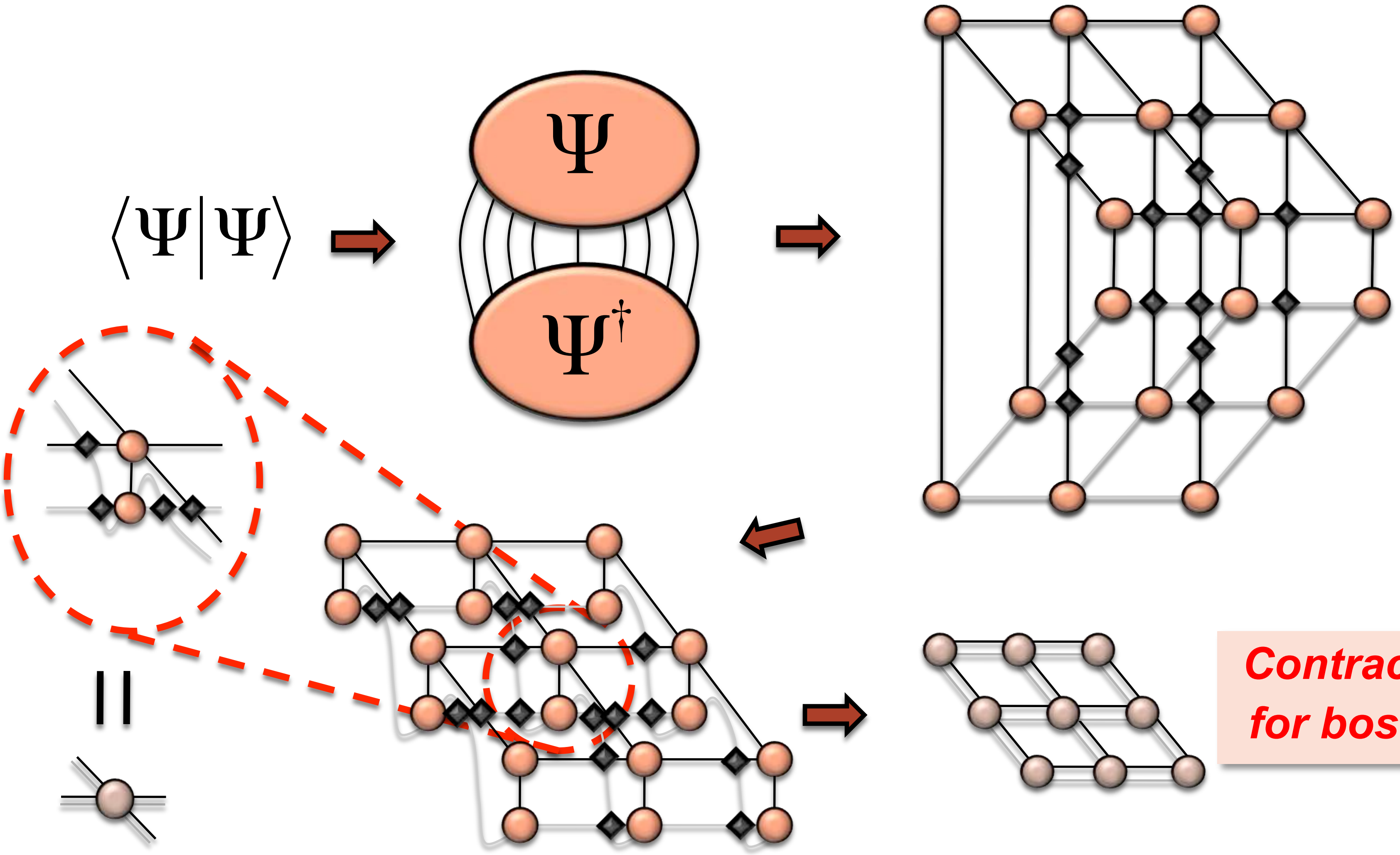
physics is independent of the order

↕

physics is independent of graphical projection

(different choices of Jordan-Wigner transformation, if mapping to a spin system)

Example: *scalar product of 3x3 PEPS*



Contract as for bosons!

But... does it work?

,,Tensor networks provide today the best variational energies for the Hubbard model in the strong coupling limit. iPEPS has really made it“.

Matthias Troyer (at the Korrelationstage 2015)

*J. Jordan, RO, G. Vidal, F. Verstraete, I. Cirac, PRL **101** 250602 (2008)*
*P. Corboz, RO, B. Bauer, G. Vidal, PRB **81** 165104 (2010)*

YES, it does

*P. Corboz, PRB **93** 045116 (2016)*

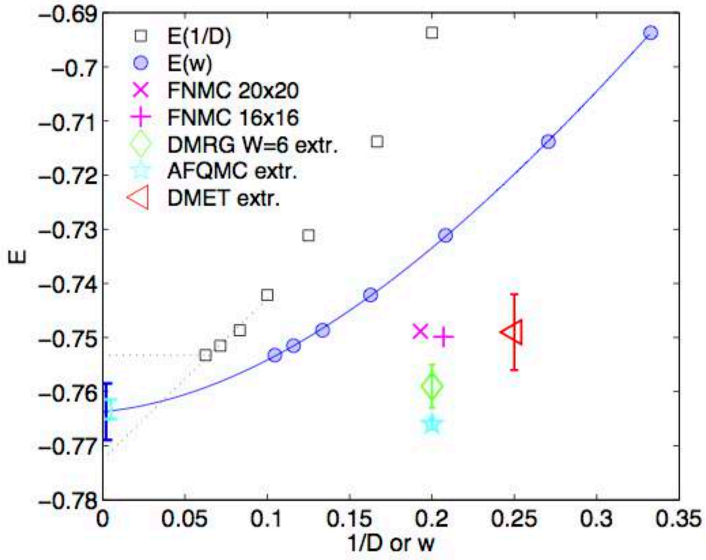
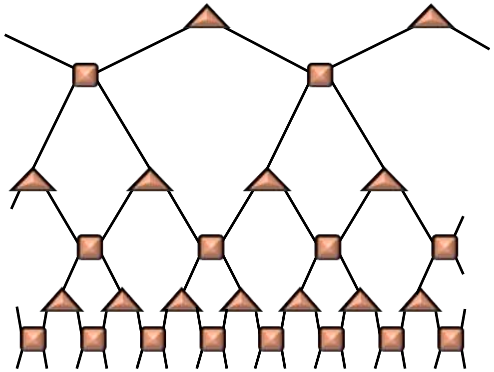
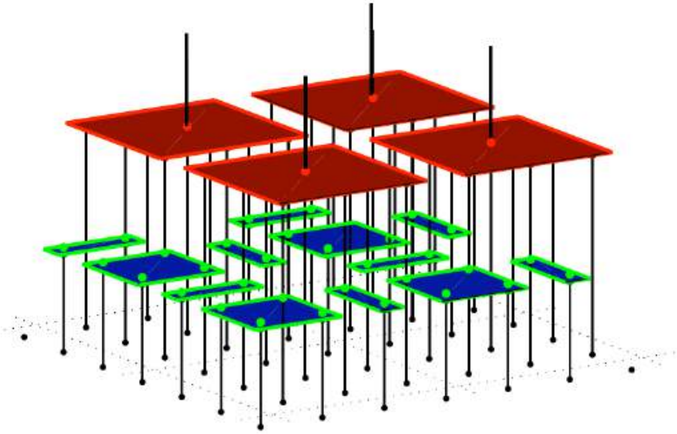


FIG. 4. (Color online) iPEPS energy of a period-5 stripe in the doped case in the strongly correlated regime ($U/t = 8$, $n = 0.875$) in comparison with other methods.

Multiscale Entanglement Renormalization Ansatz (MERA)

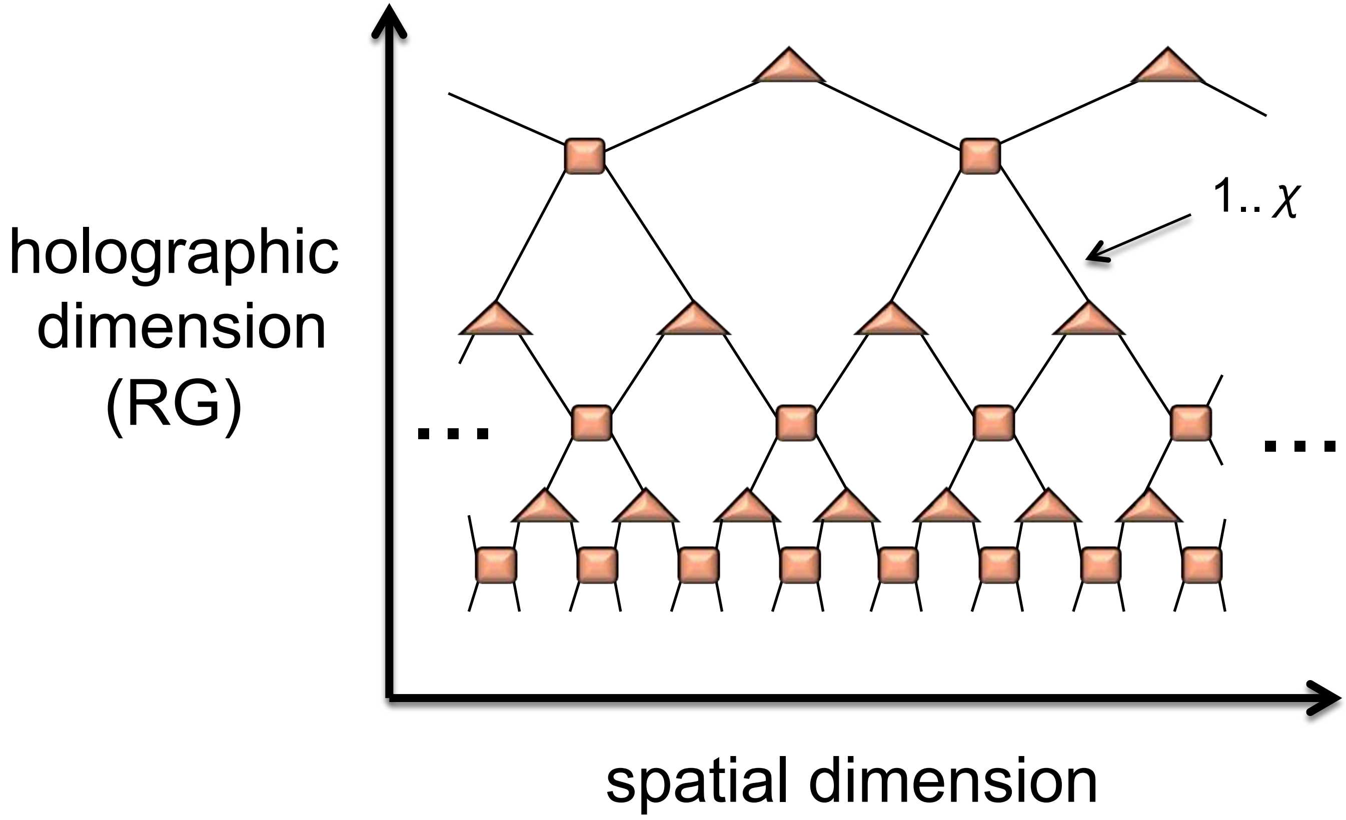


1d systems

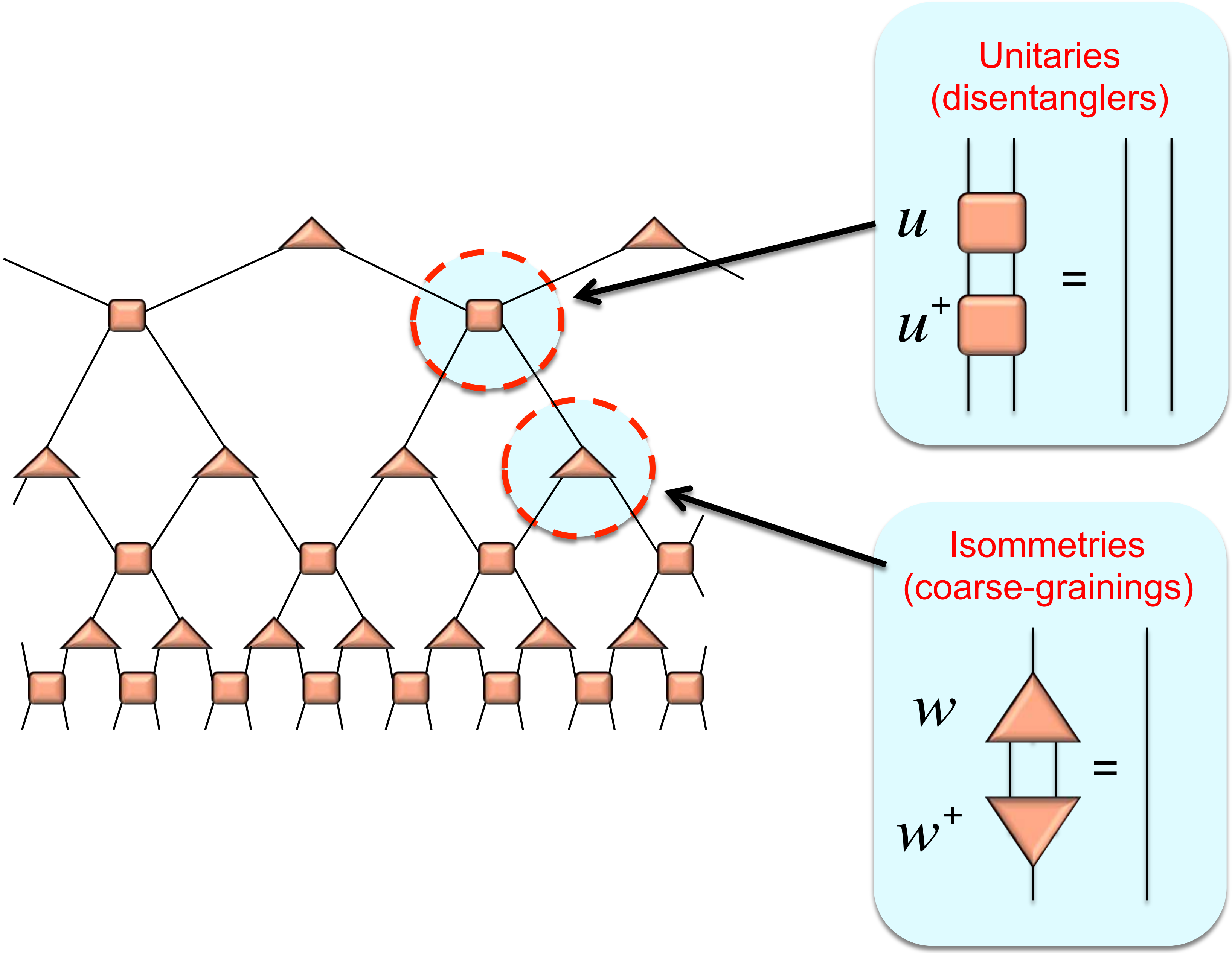


2d systems

1d MERA



Tensors obey constraints

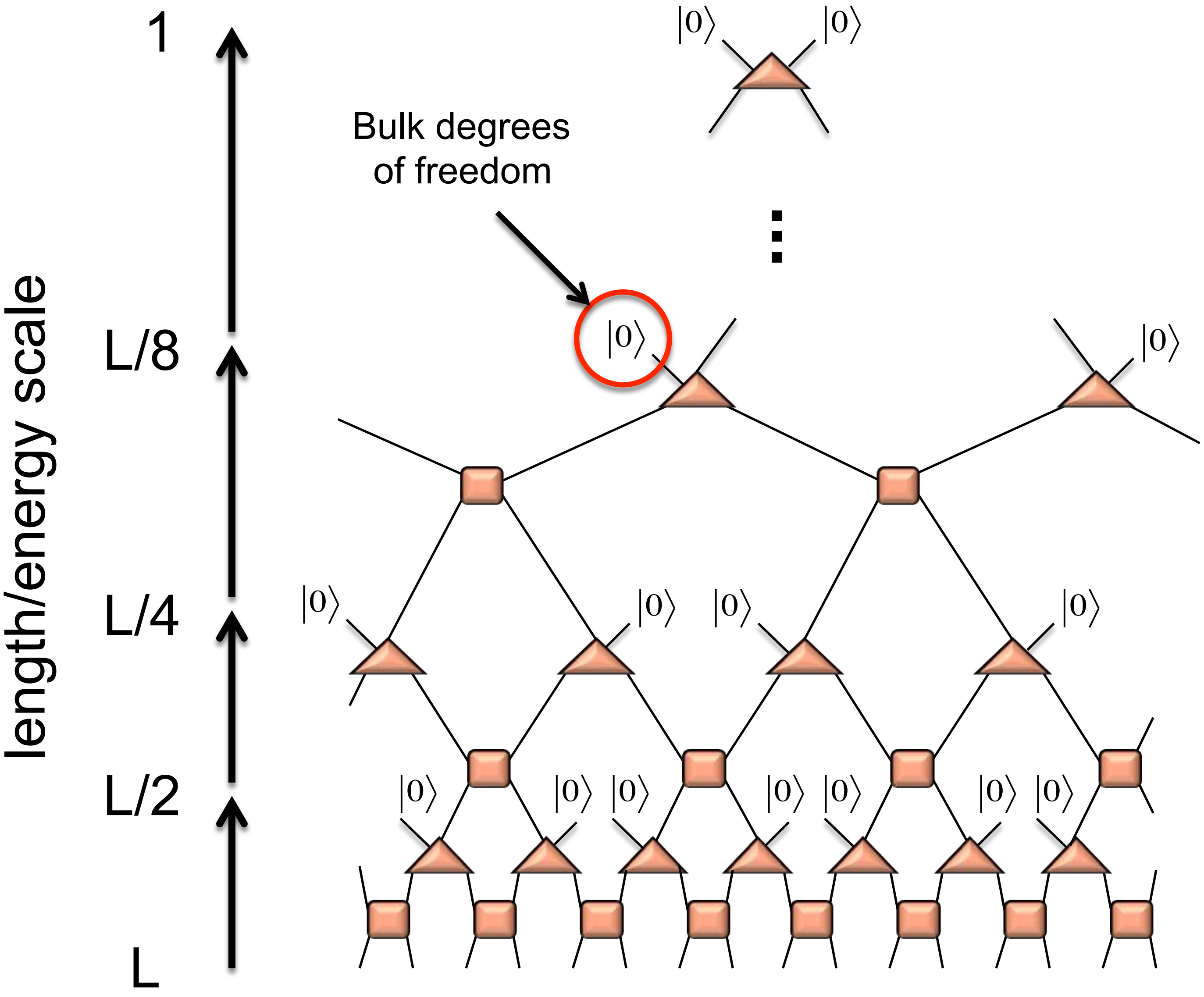


Unitaries (disentanglers)

Isommetries (coarse-grainings)

Reason:

entanglement is built locally
at all length scales



Bulk degrees of freedom

top tensor (for finite system)

coarse-grain

entangle locally

coarse-grain

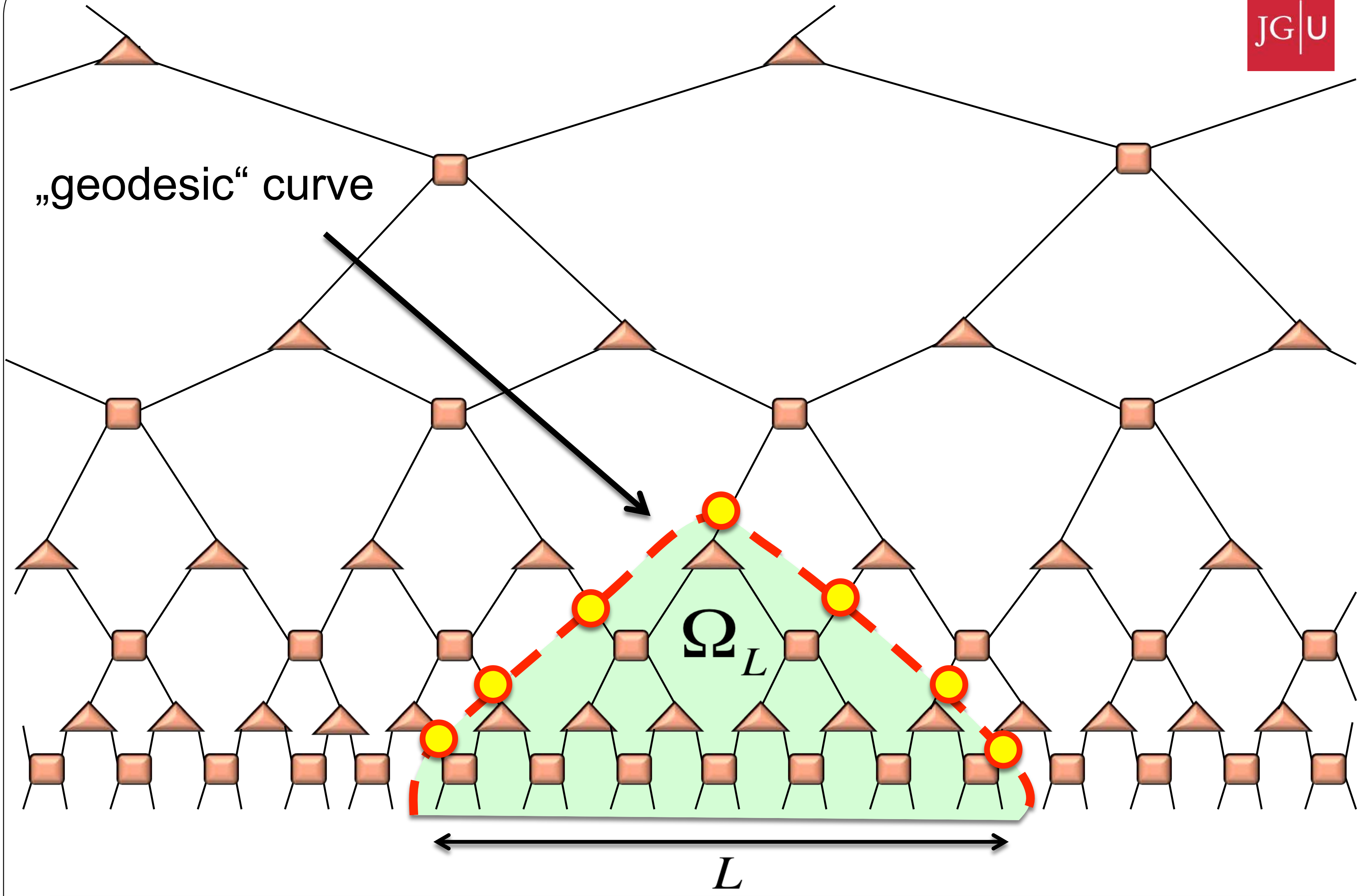
entangle locally

coarse-grain

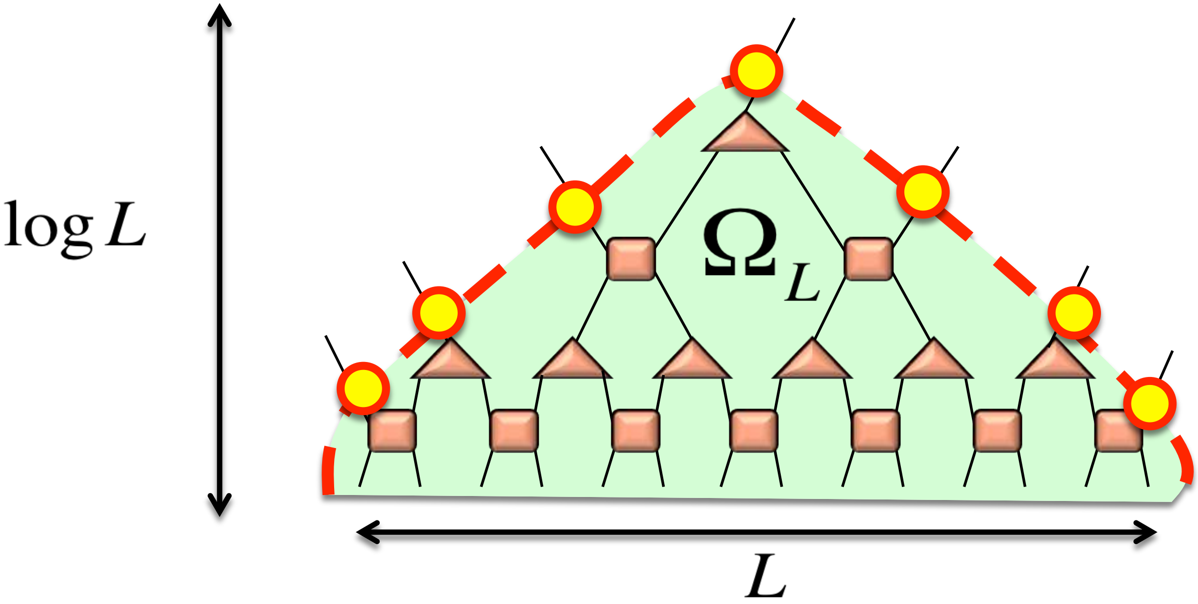
entangle locally

Extra dimension defines an RG flow: Entanglement Renormalization

Entropy of 1d MERA



Entanglement as boundary in holographic geometry: $S(L) \leq \log(\chi)\left|\partial\Omega_L\right|$



Entanglement as boundary in holographic geometry: $S(L) \leq \log(\chi)\left|\partial\Omega_L\right|$

Constant contribution at every layer

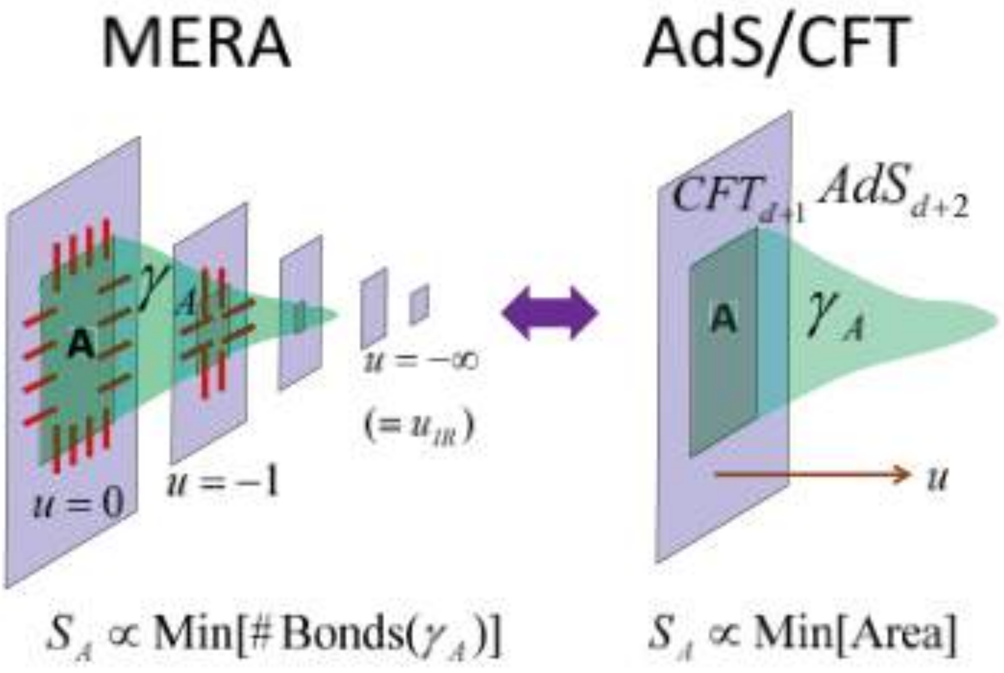
1d MERA can produce logarithmic violations to the area-law: $S(L) \approx \log L$

(like 1d critical systems!)

MERA & AdS/CFT

e.g. B. Swingle, PRD 86, 065007 (2012), G. Evenbly, G. Vidal, JSTAT 145:891-918 (2011)

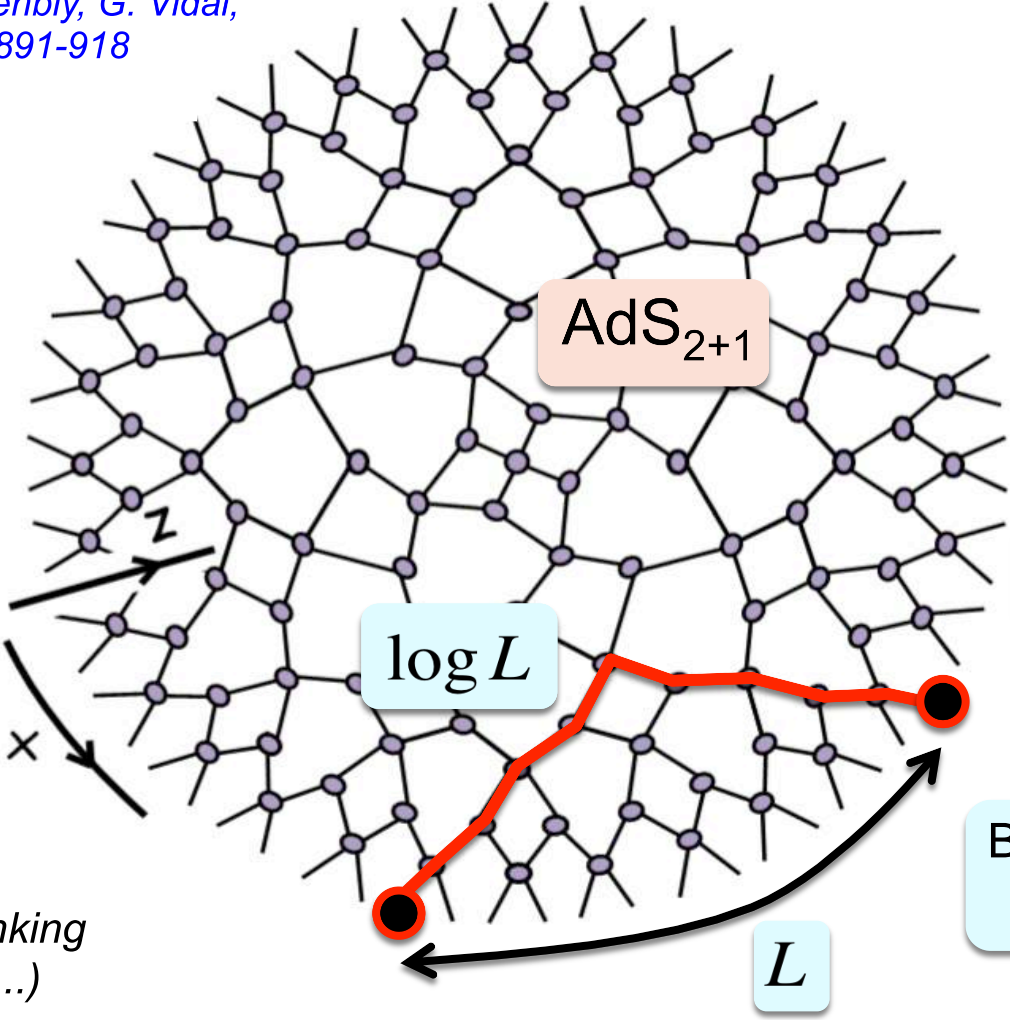
Emergent space-time



Picture from M. Nozaki, S. Ryu, T. Takayanagi, JHEP10(2012)193

MERA entropy ~ Ryu-Takayanagi prescription

Picture from G. Evenbly, G. Vidal,
(2011) JSTAT 145:891-918



AdS$_{2+1}$

$\log L$

L

CFT$_{1+1}$

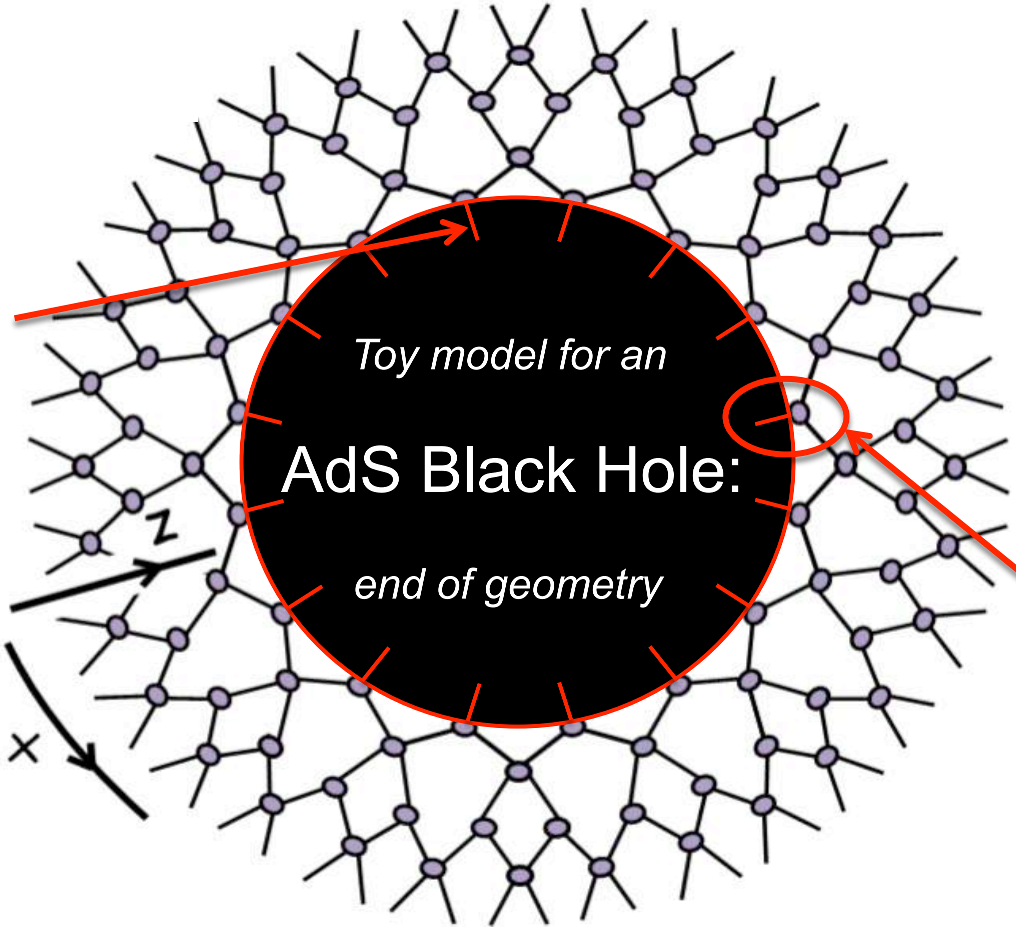
(time slice)

Bulk is a discretized AdS space

(and we were not thinking about gravity at all...)

For a scale-invariant MERA, the tensors of a critical model with a CFT limit correspond to a „gravitational“ description in a discretized AdS space:
„lattice“ realization of AdS/CFT correspondence

Let‘s now play some jazz…



Product state = trivial fixed point

Toy model for an

AdS Black Hole:

end of geometry

(time slice)

If arbitrary, then we can have non-trivial thermal states.

If isommetry, then all information is encoded in the network of correlations and

$$\rho_{in} = I$$

Finite correlation length (gapped systems) = finite number of layers

$$\left.\begin{array}{l} \rho_{in} = tr_{out}\left(|\Psi\rangle\langle\Psi|\right) \\ \rho_{out} = tr_{in}\left(|\Psi\rangle\langle\Psi|\right) \end{array}\right\}$$

*Same **thermal** spectrum (entanglement Hamiltonian)*
finite temperature, scale invariance broken

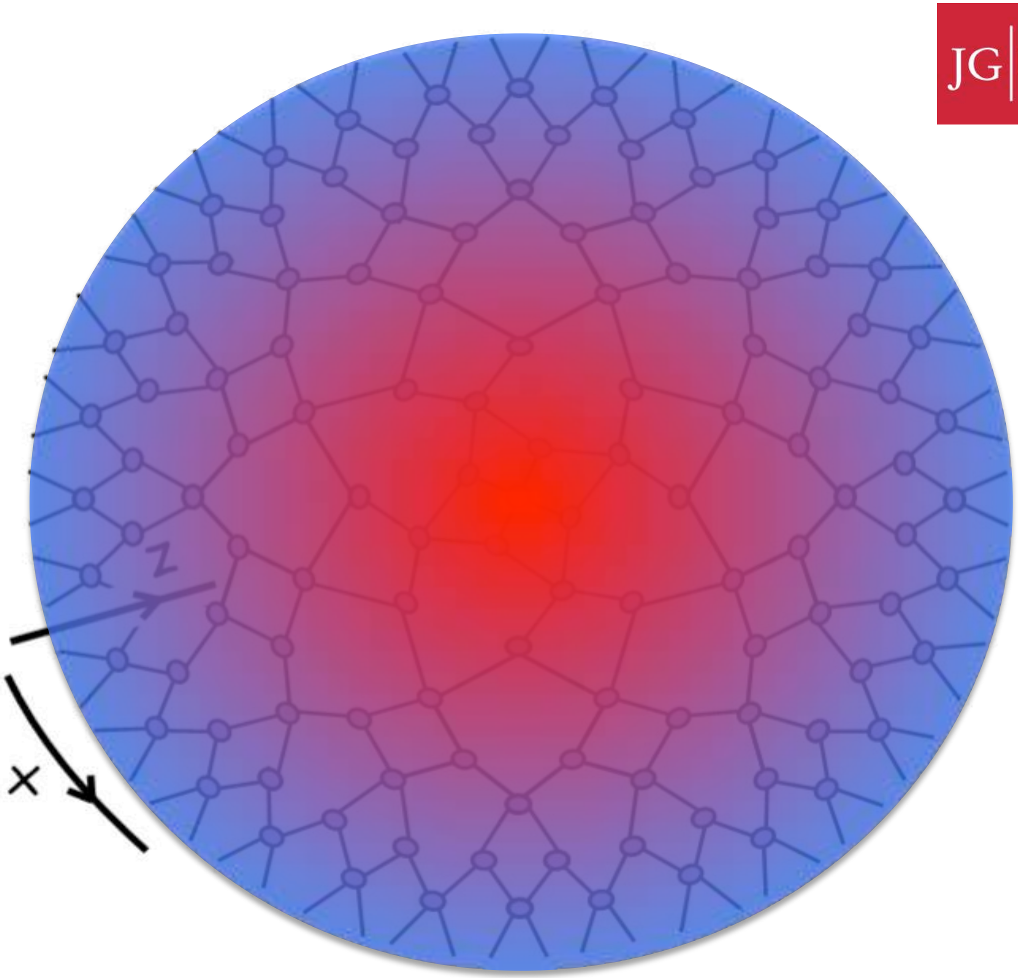
cMERA

(continuum)

$$|\psi\rangle = Pe^{-i\int_{u2}^{u1}(K(u)+L)du}|\Omega\rangle$$

J. Haegeman et al,
Phys. Rev. Lett. 110, 100402 (2013)

$K(u)$	Disentangler generator
L	Isommetry generator



$$g_{uu}(u)du^2 = \mathcal{N}^{-1}\left(1 - \left|\langle\Psi(u)|e^{iL\cdot du}|\Psi(u+du)\rangle\right|^2\right)$$

Measures the density of strength of disentanglers.
Compatible with AdS metric

M. Nozaki, S. Ryu, T. Takayanagi, JHEP10(2012)193

curvature ~ change of entanglement at every length scale

Thank you!