Recent developments in the inflationary scenario

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Inflation

The inflationary scenario is based on the two cornerstone independent ideas (hypothesis):

1. Existence of inflation (or, quasi-de Sitter stage) – a stage of accelerated, close to exponential expansion of our Universe in the past preceding the hot Big Bang with decelerated, power-law expansion.

2. The origin of all inhomogeneities in the present Universe is the effect of gravitational creation of particles and field fluctuations during inflation from the adiabatic vacuum (no-particle) state for Fourier modes covering all observable range of scales (and possibly somewhat beyond).

NB The latter effect requires breaking of the weak and null energy conditions for matter inhomogeneities.

Outcome of inflation

In the super-Hubble regime ($k \ll aH$) in the coordinate representation:

$$ds^2 = dt^2 - a^2(t)(\delta_{lm} + h_{lm})dx^l dx^m, \quad l, m = 1, 2, 3$$

$$h_{lm} = 2\zeta(\mathbf{r})\delta_{lm} + \sum_{a=1}^{2} g^{(a)}(\mathbf{r})\, e_{lm}^{(a)}$$

$$e_l^{l(a)} = 0, \;\; g^{(a)}_{,l}\, e_m^{l(a)} = 0, \;\; e_{lm}^{(a)}\, e^{lm(a)} = 1$$

ζ describes primordial scalar perturbations, g – primordial tensor perturbations (primordial gravitational waves (GW)).

The most important quantities:

$$n_s(k) - 1 \equiv \frac{d \ln P_\zeta(k)}{d \ln k}, \quad r(k) \equiv \frac{P_g}{P_\zeta}$$

In fact, metric perturbations h_{lm} are quantum (operators in the Heisenberg representation) and remain quantum up to the present time. But, after omitting of a very small part, decaying with time, they become commuting and, thus, equivalent to classical (c-number) stochastic quantities with the Gaussian statistics (up to small terms quadratic in $\zeta,\ g$).

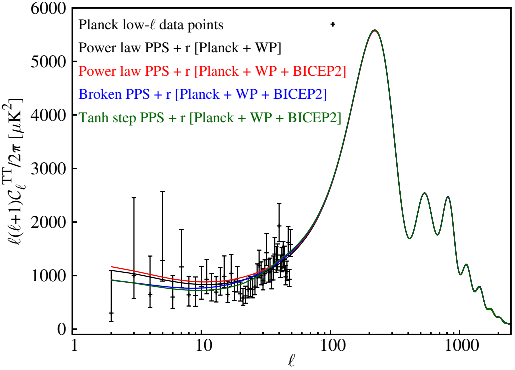
In particular:

$$\hat{\zeta}_k = \zeta_k\, i(\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}}) + \mathcal{O}\left((\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}})^2\right) + ... + \mathcal{O}(10^{-100})(\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}) + ,,,$$

The last term is time dependent, it is affected by physical decoherence and may become larger, but not as large as the second term.

Remaining quantum coherence: deterministic correlation between $\mathbf{k}$ and $-\mathbf{k}$ modes - shows itself in the appearance of acoustic oscillations (primordial oscillations in case of GW).

CMB temperature anisotropy multipoles



Present status of inflation

Now we have numbers: P. A. R. Ade et al., arXiv:1502.01589

The primordial spectrum of scalar perturbations has been measured and its deviation from the flat spectrum $n_s = 1$ in the first order in $|n_s - 1| \sim N^{-1}$ has been discovered (using the multipole range $\ell > 40$):

$$< \zeta^2(\mathbf{r}) >= \int \frac{P_\zeta(k)}{k}\, dk, \quad P_\zeta(k) = \left(2.21^{+0.07}_{-0.08}\right) 10^{-9} \left(\frac{k}{k_0}\right)^{n_s - 1}$$

$$k_0 = 0.05 \mathrm{Mpc}^{-1}, \quad n_s - 1 = -0.035 \pm 0.005$$

Two fundamental observational constants of cosmology in addition to the three known ones (baryon-to-photon ratio, baryon-to-matter density and the cosmological constant). Existing inflationary models can predict (and predicted, in fact) one of them, namely $n_s - 1$, relating it finally to $N_H = \ln \frac{k_B T_\gamma}{\hbar H_0} \approx 67.2$ (note that $(1 - n_s)N_H \sim 2$).

From "proving" inflation to using it as a tool

Present status of inflation: transition from "proving" it in general and testing some of its simplest models to applying the inflationary paradigm to investigate particle physics at super-high energies and the actual history of the Universe in the remote past using real observational data on $n_s(k) - 1$ and $r(k)$.

The reconstruction approach – determining curvature and inflaton potential from observational data – a kind of inverse dynamical problem.

The most important quantities:
1) for classical gravity – $H, \dot{H}$
2) for super-high energy particle physics – m^2_{infl}.

Dynamical origin of scalar perturbations

Local duration of inflation in terms of $N_{tot} = \ln\left(\frac{a(t_{fin})}{a(t_{in})}\right)$ is different in different points of space: $N_{tot} = N_{tot}(\mathbf{r})$. Then

$$\zeta(\mathbf{r}) = \delta N_{tot}(\mathbf{r})$$

Correct generalization to the non-linear case: the space-time metric after the end of inflation at super-Hubble scales

$$ds^2 = dt^2 - a^2(t)e^{2N_{tot}(\mathbf{r})}(dx^2 + dy^2 + dz^2)$$

First derived in A. A. Starobinsky, Phys. Lett. B **117**, 175 (1982) in the case of one-field inflation.

CMB temperature anisotropy

$$T_\gamma = (2.72548 \pm 0.00057)\mathrm{K}$$

$$\Delta T(\theta, \phi) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi)$$

$$< a_{\ell m} a_{\ell' m'} >= C_\ell \delta_{\ell \ell'} \delta_{m m'}$$

Theory: averaging over realizations.
Observations: averaging over the sky for a fixed ℓ.

For scalar perturbations, generated mainly at the last scattering surface (the surface or recombination) at $z_{LSS} \approx 1090$ (the Sachs-Wolfe, Silk and Doppler effects), but also after it (the integrated Sachs-Wolfe effect).
For GW – only the ISW works.

Visualizing small differences in the number of e-folds

For $\ell \lesssim 50$, neglecting the Silk and Doppler effects, as well as the ISW effect due the presence of dark energy,

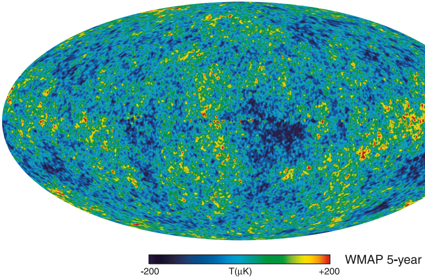
$$\frac{\Delta T(\theta,\phi)}{T_\gamma} = -\frac{1}{5}\zeta(r_{LSS},\theta,\phi) = -\frac{1}{5}\delta N_{tot}(r_{LSS},\theta,\phi)$$

For $n_s = 1$,

$$\ell(\ell+1)C_{\ell,s} = \frac{2\pi}{25}P_\zeta$$

For $\frac{\Delta T}{T} \sim 10^{-5}$, $\delta N \sim 5\times 10^{-5}$, and for $H \sim 10^{14}\,\mathrm{GeV}$, $\delta t \sim 5 t_{Pl}$!

Planck time intervals are seen by the naked eye!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T(μK)
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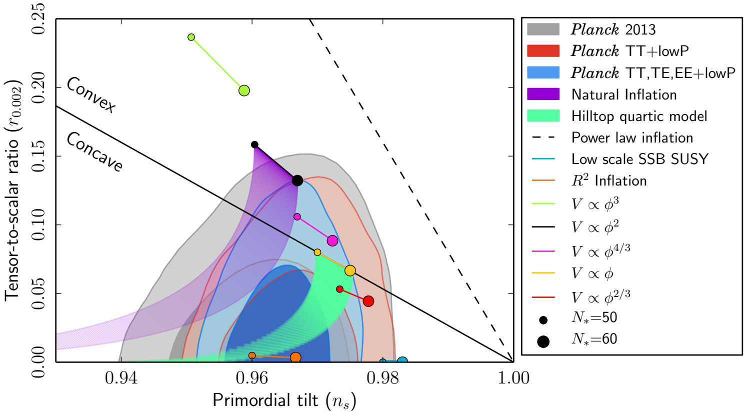
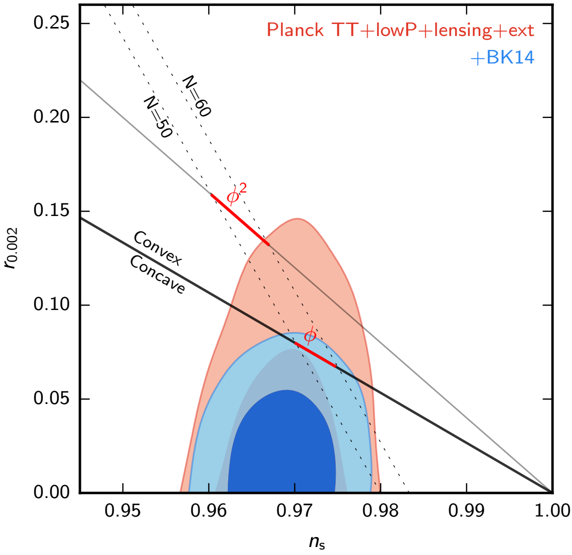
WMAP 5-year

Direct approach: comparison with simple smooth models



Combined results from Planck/BISEP2/Keck Array

P. A. R. Ade et al., Phys. Rev. Lett. 116, 031302 (2016)



The simplest models producing the observed scalar slope

$$f(R) = R + \frac{R^2}{6M^2}$$

$$M = 2.6 \times 10^{-6} \left(\frac{55}{N}\right) M_{Pl} \approx 3.2 \times 10^{13}\,\mathrm{GeV}$$

$$n_s - 1 = -\frac{2}{N} \approx -0.036, \quad r = \frac{12}{N^2} \approx 0.004, \quad N = \ln\frac{k_f}{k}$$

$$H_{dS}(N = 55) = 1.4 \times 10^{14}\,\mathrm{GeV}$$

The same prediction from a scalar field model with $V(\phi) = \frac{\lambda\phi^4}{4}$ at large ϕ and strong non-minimal coupling to gravity $\xi R\phi^2$ with $\xi < 0,\ |\xi| \gg 1$, including the Brout-Englert-Higgs inflationary model.

The Lagrangian density for the simplest 1-parametric model:

$$\mathcal{L} = \frac{R}{16\pi G} + \frac{N^2}{288\pi^2 P_\zeta(k)} R^2 = \frac{R}{16\pi G} + 5 \times 10^8 R^2$$

The quantum effect of creation of particles and field fluctuations works twice in this model:
a) at super-Hubble scales during inflation, to generate space-time metric fluctuations;
b) at small scales after inflation, to provide scalaron decay into pairs of matter particles and antiparticles (AS, 1980, 1981).
The most effective decay channel: into minimally coupled scalars with $m \ll M$. Then the formula

$$\frac{1}{\sqrt{-g}} \frac{d}{dt}(\sqrt{-g}\, n_s) = \frac{R^2}{576\pi}$$

can be used (Ya. B. Zeldovich and A. A. Starobinsky, JETP Lett. 26, 252 (1977)). Scalaron decay into graviton pairs is suppressed (A. A. Starobinsky, JETP Lett. 34, 438 (1981)).

Possible microscopic origins of this model.

1. The specific case of the fourth order gravity in 4D

$$\mathcal{L} = \frac{R}{16\pi G} + AR^2 + BC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + (\text{small rad. corr.})$$

for which $A \gg 1,\ A \gg |B|$. Approximate scale (dilaton) invariance and absence of ghosts in the curvature regime $A^{-2} \ll (RR)/M_P^4 \ll B^{-2}$.

2. Another, completely different way: a non-minimally coupled scalar field with a large negative coupling ξ ($\xi_{conf} = \frac{1}{6}$):

$$L = \frac{R}{16\pi G} - \frac{\xi R\phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi), \quad \xi < 0, \quad |\xi| \gg 1 \ .$$

In this limit, the Higgs-like scalar tree level potential $V(\phi) = \frac{\lambda(\phi^2-\phi_0^2)^2}{4}$ just produces $f(R) = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right)$ with $M^2 = \lambda/24\pi\xi^2 G$ and $\phi^2 = |\xi|R/\lambda$ (plus small corrections $\propto |\xi|^{-1}$).

Inflation in GR

Inflation in GR with a minimally coupled scalar field with some potential.

In the absence of spatial curvature and other matter:

$$H^2 = \frac{\kappa^2}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right)$$

$$\dot{H} = -\frac{\kappa^2}{2}\dot{\phi}^2$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

where $\kappa^2 = 8\pi G$ ($\hbar = c = 1$).

Reduction to the first order equation

It can be reduced to the first order Hamilton-Jacobi-like equation for $H(\phi)$. From the equation for $\dot{H}$, $\frac{dH}{d\phi} = -\frac{\kappa^2}{2}\dot{\phi}$. Inserting this into the equation for H^2, we get

$$\frac{2}{3\kappa^2}\left(\frac{dH}{d\phi}\right)^2 = H^2 - \frac{\kappa^2}{3}V(\phi)$$

Time dependence is determined using the relation

$$t = -\frac{\kappa^2}{2}\int\left(\frac{dH}{d\phi}\right)^{-1} d\phi$$

However, during oscillations of ϕ, $H(\phi)$ acquires non-analytic behaviour of the type $const + \mathcal{O}(|\phi - \phi_1|^{3/2})$ at the points where $\dot{\phi} = 0$, and then the correct matching with another solution is needed.

Inflationary slow-roll dynamics

Slow-roll occurs if: $|\ddot{\phi}| \ll H|\dot{\phi}|, \;\; \dot{\phi}^2 \ll V$, and then $|\dot{H}| \ll H^2$.

Necessary conditions: $|V'| \ll \kappa V, \;\; |V''| \ll \kappa^2 V$. Then

$$H^2 \approx \frac{\kappa^2 V}{3}, \;\; \dot{\phi} \approx -\frac{V'}{3H}, \;\; N \equiv \ln \frac{a_f}{a} \approx \kappa^2 \int_{\phi_f}^{\phi} \frac{V}{V'}\, d\phi$$

First obtained in A. A. Starobinsky, Sov. Astron. Lett. 4, 82 (1978) in the $V = \frac{m^2\phi^2}{2}$ case and for a bouncing model.

Spectral predictions of the one-field inflationary scenario in GR

Scalar (adiabatic) perturbations:

$$P_\zeta(k) = \frac{H_k^4}{4\pi^2\dot{\phi}^2} = \frac{GH_k^4}{\pi|\dot{H}|_k} = \frac{128\pi G^3 V_k^3}{3V_k'^2}$$

where the index k means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$. Through this relation, the number of e-folds from the end of inflation back in time $N(t)$ transforms to $N(k) = \ln \frac{k_f}{k}$ where $k_f = a(t_f)H(t_f)$, t_f denotes the end of inflation.

The spectral slope

$$n_s(k) - 1 \equiv \frac{d\ln P_\zeta(k)}{d\ln k} = \frac{1}{\kappa^2}\left(2\frac{V_k''}{V_k} - 3\left(\frac{V_k'}{V_k}\right)^2\right)$$

is small by modulus – confirmed by observations!

Tensor perturbations (A. A. Starobinsky, JETP Lett. 50, 844 (1979)):

$$P_g(k) = \frac{16GH_k^2}{\pi}; \quad n_g(k) \equiv \frac{d\ln P_g(k)}{d\ln k} = -\frac{1}{\kappa^2}\left(\frac{V'_k}{V_k}\right)^2$$

The consistency relation:

$$r(k) \equiv \frac{P_g}{P_\zeta} = \frac{16|\dot{H}_k|}{H_k^2} = 8|n_g(k)|$$

Tensor perturbations are always suppressed by at least the factor $\sim 8/N(k)$ compared to scalar ones. For the present Hubble scale, $N(k_H) = (50-60)$. Typically, $|n_g| \leq |n_s - 1|$, so $r \leq 8(1-n_s) \sim 0.3$ – confirmed by observations!

Inverse reconstruction of an inflationary model in GR

In the slow-roll approximation:

$$\frac{V^3}{V'^2} = CP_\zeta(k(t(\phi))), \quad C = \frac{12\pi^2}{\kappa^6}$$

Changing variables for ϕ to $N(\phi)$ and integrating, we get:

$$\frac{1}{V(N)} = -\frac{\kappa^4}{12\pi^2}\int \frac{dN}{P_\zeta(N)}$$

$$\kappa\phi = \int dN\sqrt{\frac{d\ln V}{dN}}$$

Here, $N \gg 1$ stands both for $\ln(k_f/k)$ at the present time and the number of e-folds back in time from the end of inflation.
First derived in H. M. Hodges and G. R. Blumenthal, Phys. Rev. D 42, 3329 (1990).

Minimal "scale-free" reconstruction

Minimal inflationary model reconstruction avoiding introduction of any new physical scale both during and after inflation and producing the best fit to the Planck data.

Assumption: the numerical coincidence between $2/N_H \sim 0.04$ and $1 - n_s$ is not accidental but happens for all $1 \ll N \lesssim 60$: $P_\zeta = P_0 N^2$. Then:

$$V = V_0 \frac{N}{N + N_0} = V_0 \tanh^2 \frac{\kappa\phi}{2\sqrt{N_0}}$$

$$r = \frac{8N_0}{N(N + N_0)}$$

$r \sim 0.003$ for $N_0 \sim 1$. From the upper limit on r:

$$N_0 < \frac{0.07N^2}{8 - 0.07N}$$

$N_0 < 57$ for $N = 57$.

Another example: $P_\zeta = P_0 N^{3/2}$.

$$V(\phi) = V_0 \frac{\phi^2 + 2\phi\phi_0}{(\phi+\phi_0)^2}$$

Not bounded from below (of course, in the region where the slow-roll approximation is not valid anymore). Crosses zero linearly.

More generally, the two "aesthetic" assumptions – "no-scale" scalar power spectrum and $V \propto \phi^{2n},\ n = 1, 2...$ at the minimum of the potential – lead to $P_\zeta = P_0 N^{n+1},\ n_s - 1 = -\frac{n+1}{N}$ unambiguously. From this, only $n = 1$ is permitted by observations. Still an additional parameter appears due to tensor power spectrum – no preferred one-parameter model (if the $V(\phi) \propto \phi^2$ model is excluded).

Inflation in $f(R)$ gravity

The simplest model of modified gravity (= geometrical dark energy) considered as a phenomenological macroscopic theory in the fully non-linear regime and non-perturbative regime.

$$S = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\,d^4x + S_m$$

$$f(R) = R + F(R), \quad R \equiv R_\mu^\mu$$

Here $f''(R)$ is not identically zero. Usual matter described by the action S_m is minimally coupled to gravity.

Vacuum one-loop corrections depending on R only (not on its derivatives) are assumed to be included into $f(R)$. The normalization point: at laboratory values of R where the scalaron mass (see below) $m_s \approx \mathrm{const}$.

Metric variation is assumed everywhere. Palatini variation leads to a different theory with a different number of degrees of freedom.

Field equations

$$\frac{1}{8\pi G}\left(R_{\mu}^{\nu}-\frac{1}{2}\,\delta_{\mu}^{\nu}R\right)=-\left(T_{\mu\,(vis)}^{\nu}+T_{\mu\,(DM)}^{\nu}+T_{\mu\,(DE)}^{\nu}\right)\ ,$$

where $G=G_0=const$ is the Newton gravitational constant measured in laboratory and the effective energy-momentum tensor of DE is

$$8\pi G T_{\mu\,(DE)}^{\nu}=F'(R)\,R_{\mu}^{\nu}-\frac{1}{2}\,F(R)\delta_{\mu}^{\nu}+\left(\nabla_{\mu}\nabla^{\nu}-\delta_{\mu}^{\nu}\nabla_{\gamma}\nabla^{\gamma}\right)F'(R)\,.$$

Because of the need to describe DE, de Sitter solutions in the absence of matter are of special interest. They are given by the roots $R=R_{dS}$ of the algebraic equation

$$Rf'(R)=2f(R)\ .$$

The special role of $f(R)\propto R^2$ gravity: admits de Sitter solutions with any curvature.

Degrees of freedom

I. In quantum language: particle content.
1. Graviton – spin 2, massless, transverse traceless.
2. Scalaron – spin 0, massive, mass - R-dependent:
$m_s^2(R) = \frac{1}{3f''(R)}$ in the WKB-regime.

II. Equivalently, in classical language: number of free functions of spatial coordinates at an initial Cauchy hypersurface.
Six, instead of four for GR – two additional functions describe massive scalar waves.

Thus, $f(R)$ gravity is a non-perturbative generalization of GR. It is equivalent to scalar-tensor gravity with $\omega_{BD} = 0$ if $f''(R) \neq 0$.

Transformation to the Einstein frame and back

In the Einstein frame, free particles of usual matter do not follow geodesics and atomic clocks do not measure proper time.

From the Jordan (physical) frame to the Einstein one:

$$g^{E}_{\mu\nu} = fag^{J}_{\mu\nu}, \quad \kappa\phi = \sqrt{\frac{3}{2}}\ln f', \quad V(\phi) = \frac{f'R - f}{2\kappa^2 f'^2}$$

where $\kappa^2 = 8\pi G$.

Inverse transformation:

$$R = \left(\sqrt{6}\kappa\frac{dV(\phi)}{d\phi} + 4\kappa^2 V(\phi)\right)\exp\left(\sqrt{\frac{2}{3}}\kappa\phi\right)$$

$$f(R) = \left(\sqrt{6}\kappa\frac{dV(\phi)}{d\phi} + 2\kappa^2 V(\phi)\right)\exp\left(2\sqrt{\frac{2}{3}}\kappa\phi\right)$$

$V(\phi)$ should be at least C^1.

Reduction to the first order equation

In the absence of spatial curvature and $\rho_m = 0$, it is always possible to reduce these equations to a first order one using either the transformation to the Einstein frame and the Hamilton-Jacobi-like equation for a minimally coupled scalar field in a spatially flat FLRW metric, or by directly transforming the 0-0 equation to the equation for $R(H)$:

$$\frac{dR}{dH} = \frac{(R - 6H^2)f'(R) - f(R)}{H(R - 12H^2)f''(R)}$$

Analogues of large-field (chaotic) inflation: $F(R) \approx R^2 A(R)$ for $R \to \infty$ with $A(R)$ being a slowly varying function of R, namely

$$|A'(R)| \ll \frac{A(R)}{R}\ , \ |A''(R)| \ll \frac{A(R)}{R^2}\ .$$

Analogues of small-field (new) inflation, $R \approx R_1$:

$$F'(R_1) = \frac{2F(R_1)}{R_1}\ , \ F''(R_1) \approx \frac{2F(R_1)}{R_1^2}\ .$$

Thus, all inflationary models in $f(R)$ gravity are close to the simplest one over some range of R.

Perturbation spectra in slow-roll $f(R)$ inflationary models

Let $f(R) = R^2 A(R)$. In the slow-roll approximation $|\ddot{R}| \ll H|\dot{R}|$:

$$P_\zeta(k) = \frac{\kappa^2 A_k}{64\pi^2 A_k'^2 R_k^2}, \quad P_g(k) = \frac{\kappa^2}{12 A_k \pi^2}$$

$$N(k) = -\frac{3}{2}\int_{R_f}^{R_k} dR\, \frac{A}{A' R^2}$$

where the index k means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$.

Smooth reconstruction of inflation in $f(R)$ gravity

$$f(R) = R^2 A(R)$$

$$A = const - \frac{\kappa^2}{96\pi^2} \int \frac{dN}{P_\zeta(N)}$$

$$\ln R = const + \int dN \sqrt{-\frac{2\, d \ln A}{3\, dN}}$$

Here, the additional assumptions that $P_\zeta \propto N^\beta$ and that the resulting $f(R)$ can be analytically continued to the region of small R without introducing a new scale, and it has the linear (Einstein) behaviour there, leads to $\beta = 2$ and the $R + R^2$ inflationary model with $r = \frac{12}{N^2} = 3(n_s - 1)^2$ unambiguously.

For $P_\zeta = P_0 N^2$ ("scale-free reconstruction"):

$$A = \frac{1}{6M^2}\left(1 + \frac{N_0}{N}\right), \quad M^2 \equiv \frac{16\pi^2 N_0 P_\zeta}{\kappa^2}$$

Two cases:

1. $N \gg N_0$ always.

$$A = \frac{1}{6M^2}\left(1 + \left(\frac{R_0}{R}\right)^{\sqrt{3/(2N_0)}}\right), \quad R \gg R_0$$

For $N_0 = 3/2, \;\; R_0 = 6M^2$ we return to the simplest $R + R^2$ inflationary model.

2. $N_0 \gg 1$.

$$A = \frac{1}{6M^2}\left(\frac{1 + \left(\frac{R_0}{R}\right)^{\sqrt{3/(2N_0)}}}{1 - \left(\frac{R_0}{R}\right)^{\sqrt{3/(2N_0)}}}\right)^2, \quad R > R_0$$

Generality of inflation

Some myths regarding the onset of inflation:

1. Inflation begins with $V(\phi) \sim \dot{\phi}^2 \sim M_{Pl}^2$.
2. As a consequence, its formation is strongly suppressed in models with a plateau-type potentials favored by observations.
3. Beginning of inflation in some patch requires causal connection throughout the patch.
4. One of weaknesses of inflation is that it does not solve the singularity problem.

Theorem. In inflationary models in GR and $f(R)$ gravity, there exists an open set of classical solutions with a non-zero measure in the space of initial conditions at curvatures much exceeding those during inflation which have a metastable inflationary stage with a given number of e-folds.

For the GR inflationary model this follows from the generic late-time asymptotic solution for GR with a cosmological constant found in A. A. Starobinsky, JETP Lett. 37, 55 (1983). For the $R + R^2$ model, this was proved in A. A. Starobinsky and H.-J. Schmidt, Class. Quant. Grav. 4, 695 (1987). For the power-law and $f(R) = R^p$ inflation – in V. Müller, H.-J. Schmidt and A. A. Starobinsky, Class. Quant. Grav. 7, 1163 (1990).

Generic late-time asymptote of classical solutions of GR with a cosmological constant Λ both without and with hydrodynamic matter (also called the Fefferman-Graham expansion):

$$ds^2 = dt^2 - \gamma_{ik} dx^i dx^k$$

$$\gamma_{ik} = e^{2H_0 t} a_{ik} + b_{ik} + e^{-H_0 t} c_{ik} + ...$$

where $H_0^2 = \Lambda/3$ and the matrices a_{ik}, b_{ik}, c_{ik} are functions of spatial coordinates. a_{ik} contains two independent physical functions (after 3 spatial rotations and 1 shift in time + spatial dilatation) and can be made unimodular, in particular. b_{ik} is unambiguously defined through the 3-D Ricci tensor constructed from a_{ik}. c_{ik} contains a number of arbitrary physical functions (two - in the vacuum case, or with radiation).

The appearance of an inflating patch does not require that all parts of this patch should be causally connected at the beginning of inflation.

Related academic question: "vacuum" state in quantum GR+Λ

$$H = H_0 = \sqrt{\Lambda/3}$$

Using the result for $P_g(k)$:

$$< h_{ik} h^{ik} > = \frac{16 G H_0^2 N}{\pi} \ .$$

where $N \gg 1$ is now the number of e-folds from the beginning of the expanding de Sitter stage. The assumption of small perturbations breaks down for $N \gtrsim 1/GH_0^2$.

Still ongoing discussion on the final outcome of this effect. My opinion - no screening of the background cosmological constant, instead - stochastic drift through an infinite number of locally de Sitter, but globally non-equivalent vacua.

What was before inflation?

In classical gravity (GR or modified $f(R)$): generic space-like curvature singularity.

Generic initial conditions near a curvature singularity in these models: anisotropic and inhomogeneous (though quasi-homogeneous locally).

1. Modified gravity models (the $R+R^2$ and Higgs ones).

Two types singularities with the same structure at $t \to 0$:

$$ds^2 = dt^2 - \sum_{i=1}^{3} |t|^{2p_i}\, a_l^{(i)} a_m^{(i)}\, dx^l dx^m,\ 0 < s \le 3/2,\ u = s(2-s)$$

where $p_i < 1,\ \ s = \sum_i p_i,\ \ u = \sum_i p_i^2$ and $a_l^{(i)},\ \ p_i$ are functions of $\mathbf{r}$. Here $R^2 \ll R_{\alpha\beta}R^{\alpha\beta}$.

Type A. $1 \le s \le 3/2,\ \ R \propto |t|^{1-s} \to +\infty$

Type B. $0 < s < 1,\ \ R \to R_0 < 0,\ \ f'(R_0) = 0$

2. GR model with a very flat potential.
A similar behaviour but with $s = 1, \; u < 1$ and all $p_i < 1$.
Potential $V(\phi)$ is not important if grows slower than an exponent. No infinite number of BKL oscillations in the limit $t \longrightarrow 0$: they stop when all p_i become positive.

In both cases, spatial gradients may become important for some period before the beginning of inflation.

What is needed for beginning of inflation in classical (modified) gravity, is:
1) the existence of a sufficiently large compact expanding region of space with the Riemann curvature much exceeding that during the end of inflation ($\sim M^2$) – realized near a curvature singularity;
2) the average value $< R >$ over this region positive and much exceeding $\sim M^2$, too, – type A singularity;
3) the average spatial curvature over the region is either negative, or not too positive.

Recent numerical studies confirming this in GR: W. H. East, M. Kleban, A. Linde and L. Senatore, JCAP 1609, 010 (2016); M. Kleban and L. Senatore, JCAP 1610, 022 (2016).

On the other hand, causal connection is certainly needed to have a "graceful exit" from inflation, i.e. to have practically the same amount of the total number of e-folds during inflation N_{tot} in some sub-domain of this inflating patch.

Conclusions

- The typical inflationary predictions that $|n_s - 1|$ is small and of the order of N_H^{-1}, and that r does not exceed $\sim 8(1 - n_s)$ are confirmed. Typical consequences following without assuming additional small parameters: $H_{55} \sim 10^{14}\,\mathrm{GeV},\ m_{infl} \sim 10^{13}\,\mathrm{GeV}$.
- Though the Einstein gravity plus a minimally coupled inflaton remains sufficient for description of inflation with existing observational data, modified (in particular, scalar-tensor or $f(R)$) gravity can do it as well.
- From the scalar power spectrum $P_\zeta(k)$, it is possible to reconstruct an inflationary model both in the Einstein and $f(R)$ gravity up to one arbitrary physical constant of integration.

- In the Einstein gravity, the simplest inflationary models permitted by observational data are two-parametric, no preferred quantitative prediction for r, apart from its parametric dependence on $n_s - 1$, namely, $\sim (n_s - 1)^2$ or larger.
- In the $f(R)$ gravity, the simplest model is one-parametric and has the preferred value $r = \frac{12}{N^2} = 3(n_s - 1)^2$.
- Thus, it has sense to search for primordial GW from inflation at the level $r > 10^{-3}$!

- Inflation is generic in the $R + R^2$ inflationary model and scalar models with sufficiently flat potentials in GR (as generic e.g. as black holes in GR). Thus, its beginning does not require causal connection of all parts of an inflating patch of space-time (similar to spacelike singularities).
- However, graceful exit from inflation requires approximately the same number of e-folds during it for a sufficiently large compact set of geodesics. To achieve this, causal connection inside this set is necessary (though still may appear insufficient).
- The fact that inflation does not "solve" the singularity problem, i.e. it does not remove a curvature singularity preceding it, can be an advantage, not its weakness.