

The duality of Wilson loop form factors

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Based on work arXiv:1612.05197 in collaboration with
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Outline

- $\mathcal{N} = 4$ SYM in the planar limit
- Duality between Correlation Functions, Wilson Loops, and Scattering Amplitudes
- Duality of the Wilson loop form factors (two polygonal light-like contours)
- MHV example of the duality
- Supersymmetric extension of the duality
- LHC superspace formulation of the Wilson loop
- Diagrammatic proof of the duality

$N = 4$ SYM theory and its symmetries

- Gauge group $SU(N_c)$
- global R-symmetry group $SU(4)$
- Field content

gauge field $\mathcal{A}_\mu \sigma_{\alpha\dot{\alpha}}^\mu = \mathcal{A}_{\alpha\dot{\alpha}}$

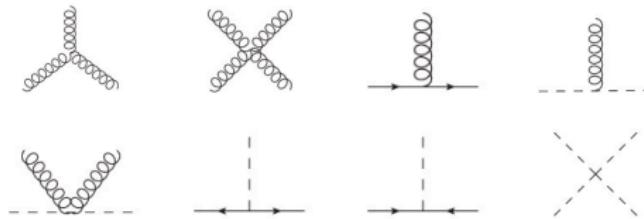
6 real scalars ϕ_{AB} ,

4 gauginos $\psi_{\alpha A}$

4 antigauginos $\bar{\psi}_{\dot{\alpha}}^A$

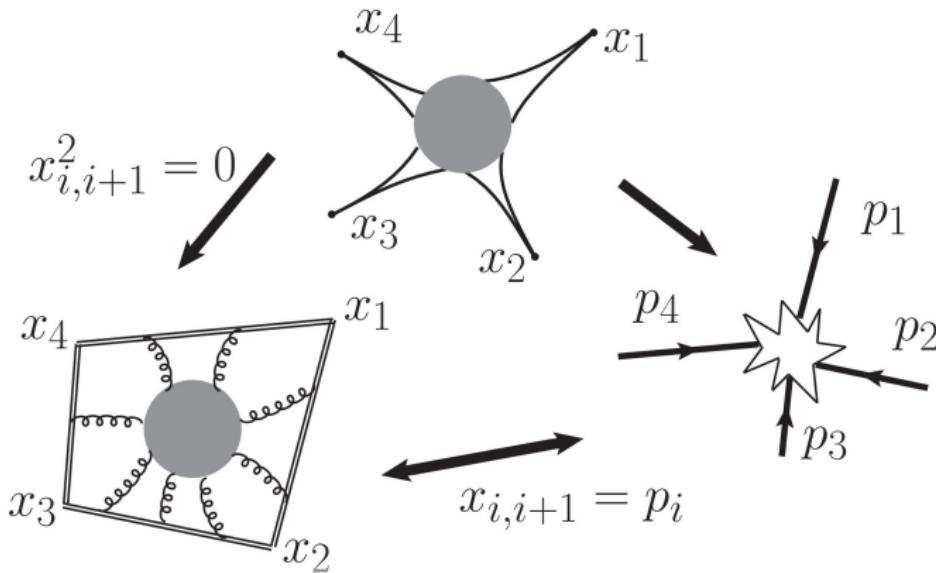
$A, B = 1, \dots, 4$

- Interactions



- 16 supercharges $Q_A^\alpha, \bar{Q}_{\dot{\alpha}}^A$ (on shell supersymmetry)
- Conformal symmetry
- Superconformal symmetry $PSU(2, 2|4)$ @ quantum level

Duality between $N = 4$ SYM observables

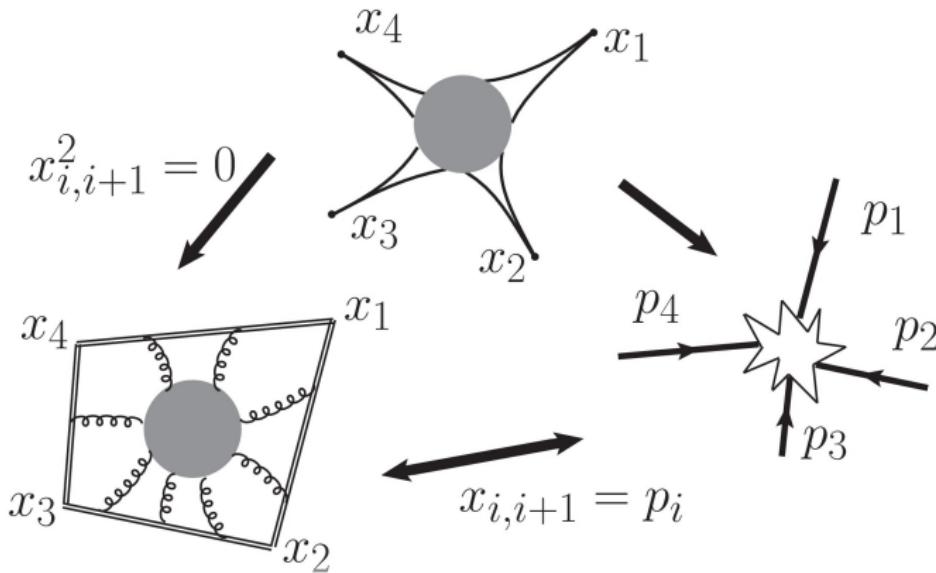


Alday, Maldacena '07;
Drummond, Henn, Korchemsky,
Sokatchev '07 '08;
Alday, Eden, Korchemsky,
Maldacena, Sokatchev '07;
Mason, Skinner '10;
Caron-Huot '10;
Adamo, Bullimore, Mason, Skinner
'11;
Brandhuber, Heslop, Travaglini '07

- Correlators of half-BPS operators $\mathcal{O} = \text{tr } \phi^2$. **UV & IR finite.** Conformal symmetry!!!

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle$$

Duality between $N = 4$ SYM observables



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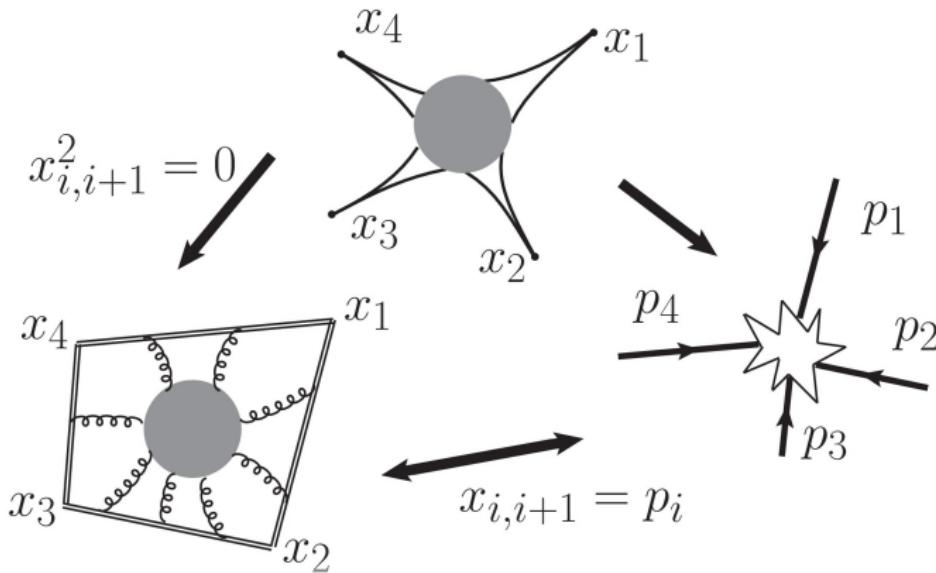
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Brandhuber, Heslop, Travaglini '07

- Polygonal light-like Wilson loops on the contour $C_n = p_1 \cup \dots \cup p_n$. **UV div**

$$W[C_n] = \left\langle \text{tr Pexp} \left(ig \oint_{C_n} dx \cdot \mathcal{A} \right) \right\rangle$$

Duality between $N = 4$ SYM observables

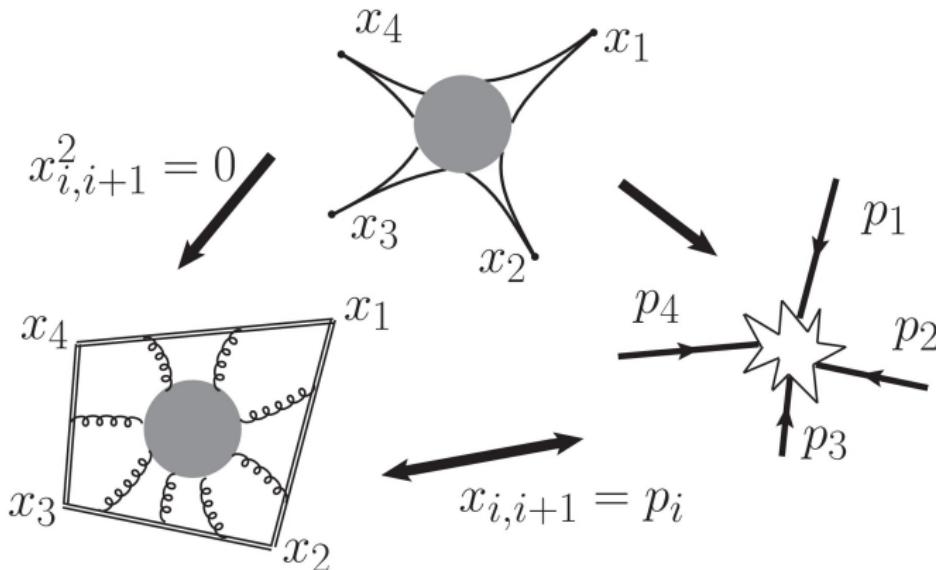


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- Loop corrections of the MHV amplitudes. IR div

$$\mathcal{A}_n^{\text{MHV}}(p_1^+, \dots, p_i^-, \dots, p_j^-, \dots, p_n^+)$$

Duality between $N = 4$ SYM observables

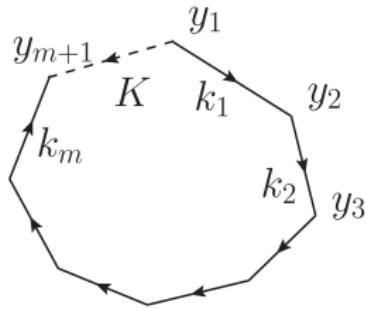
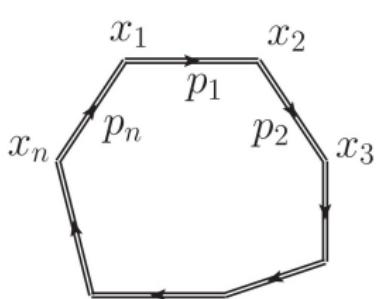


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- Conformal and dual conformal symmetries are anomalous.
- Correlator \leftrightarrow Amplitude works at the level of INTEGRANDS
- Dualities work for supersymmetric observables

"MHV" Wilson loop form factors and their duality

Polygonal light-like contours

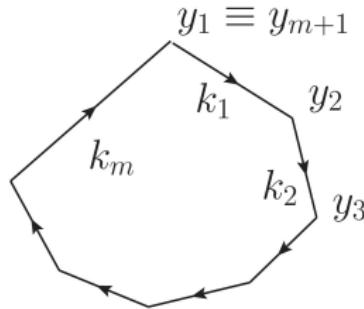
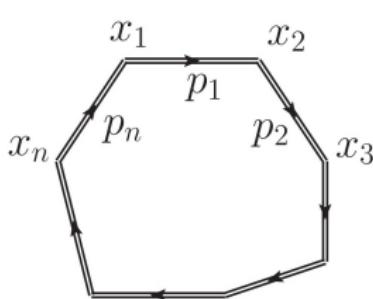


$$p_i^2 = 0 , \quad \sum_i p_i = 0 \qquad\qquad k_i^2 = 0 , \quad \sum_i k_i \equiv K = y_1 - y_{m+1} \neq 0$$

$$\left\langle \text{tr Pexp} \left(ig \oint_{p_1 \cup \dots \cup p_n} dx \cdot \mathcal{A} \right) \middle| k_1^+ k_2^+ \dots k_m^+ \right\rangle$$

"MHV" Wilson loop form factors and their duality

Polygonal light-like contours



$$p_i^2 = 0 , \quad \sum_i p_i = 0$$

$$k_i^2 = 0 , \quad \sum_i k_i = 0$$

$$\left\langle \text{tr Pexp} \left(ig \oint_{p_1 \cup \dots \cup p_n} dx \cdot \mathcal{A} \right) \middle| k_1^+ k_2^+ \dots k_m^+ \right\rangle \sim \left\langle \text{tr Pexp} \left(ig \oint_{k_1 \cup \dots \cup k_m} dx \cdot \mathcal{A} \right) \middle| p_1^+ p_2^+ \dots p_n^+ \right\rangle$$

in the planar limit

Form factors of $SL(2)$ light-ray operators

[Derkachov, Korchemsky, Manashov '13]

$$\mathbb{O}(z_1, \dots, z_m) = \frac{\text{---} \atop \text{---}}{n z_1 \quad n z_2 \quad \cdots \quad n z_m} \quad , \quad \bar{n}^2 = 0$$

$$f(z_1, \dots, z_m; y_1, \dots, y_m) \equiv \langle \mathbb{O}(z_1, \dots, z_m) | p_1 \bar{n}, \dots, p_m \bar{n} \rangle$$

where $\bar{n}^2 = 0$, $n \cdot \bar{n} \neq 0$, $\sum_{i=1}^m p_i = 0$, $p_i \equiv y_i - y_{i+1}$

One-loop integrability of the dilatation operator implies

$$f(z_1, \dots, z_m; y_1, \dots, y_m) = f(y_1, \dots, y_m; z_1, \dots, z_m)$$

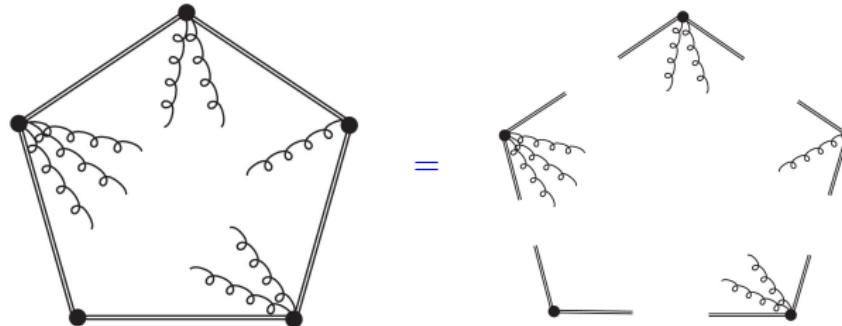
Now we unfold contour and consider form factors of polygonal Wilson loop

Tree level "MHV" Wilson loop form factors

Color ordered MHV form factor of the "wedge"

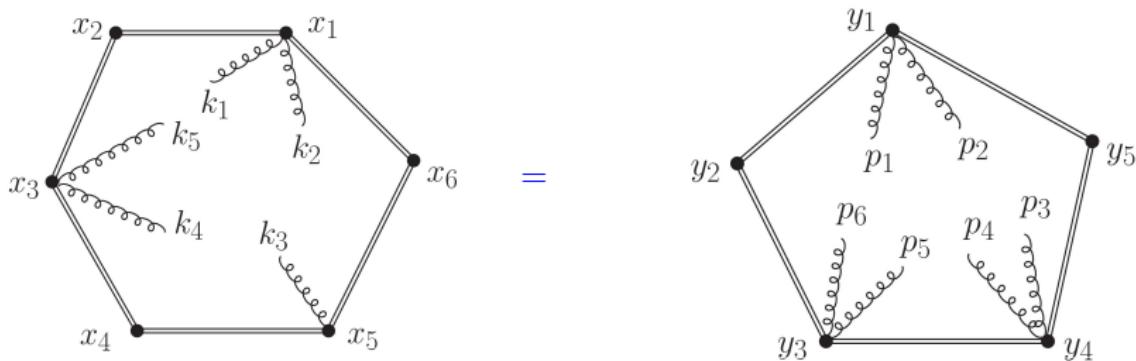
$$= g^3 \frac{\langle p_1 p_2 \rangle}{\langle p_1 k_1 \rangle \langle k_1 k_2 \rangle \langle k_2 k_3 \rangle \langle k_3 p_2 \rangle} \exp(ix(k_1 + k_2 + k_3))$$

Factorization of the Wilson loop form factor



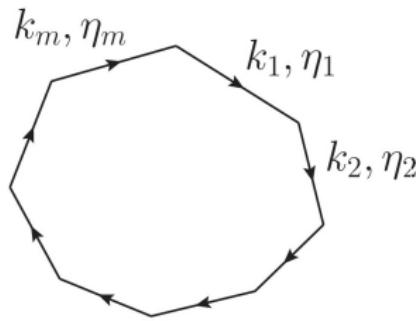
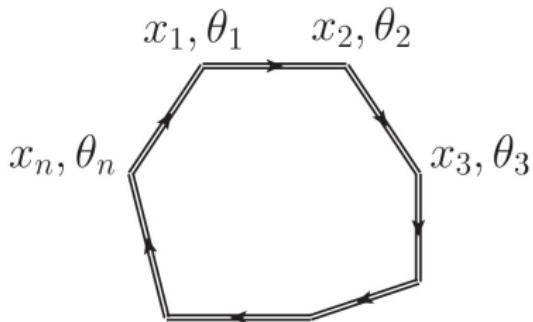
Tree level "MHV" Wilson loop form factors. Example

[S. Caron-Huot]



$$\exp\left(ix_1\underbrace{(k_1+k_2)}_{y_{13}}+ix_3\underbrace{(k_4+k_5)}_{y_{41}}+ix_5\underbrace{(k_3)}_{y_{34}}\right) = \exp\left(iy_1\underbrace{(p_1+p_2)}_{x_{13}}+iy_3\underbrace{(p_5+p_6)}_{x_{51}}+iy_4\underbrace{(p_3+p_4)}_{x_{35}}\right)$$

Supersymmetric Wilson loop form factors



Light-like chiral contour. θ^A , $A = 1, \dots, 4$

Superconnections $\mathcal{A}_{\alpha\dot{\alpha}}(x, \theta)$, $\mathcal{A}_{\alpha A}(x, \theta)$

$$x_i - x_{i+1} \equiv p_i \quad , \quad \theta_i^A - \theta_{i+1}^A \equiv |p_i\rangle \omega_i^A$$

$$\sum_i p_i = 0 \quad , \quad \sum_i |p_i\rangle \omega_i = 0$$

Superstate (vector multiplet: G_\pm, Γ_\pm, S)

$$|k, \eta\rangle = G_+ + \eta_A \Gamma_+^A + \dots + (\eta)^4 G_-$$

$$y_i - y_{i+1} \equiv k_i \quad , \quad \psi_{i,A} - \psi_{i+1,A} \equiv |k_i\rangle \eta_{i,A}$$

$$\sum_i k_i = 0 \quad , \quad \sum_i |k_i\rangle \eta_i = 0 \quad \text{Constraint!}$$

$$W_{n,m}(\{x, \theta\}\{y, \psi\}) \equiv \prod_{i=1}^m \langle k_i k_{i+1} \rangle \cdot \left\langle \text{tr Pexp } ig \oint_{p_1 \cup \dots \cup p_n} (dx^{\alpha\dot{\alpha}} \mathcal{A}_{\alpha\dot{\alpha}} + d\theta^{\alpha A} \mathcal{A}_{\alpha A}) \right| k_1, \eta_1; \dots; k_m, \eta_m \right\rangle$$

Duality:

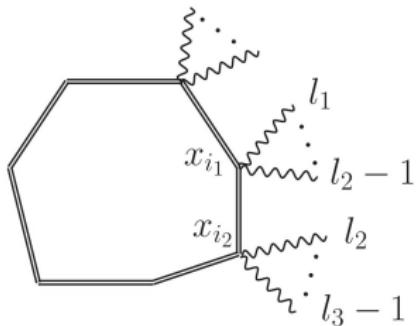
$$W_{\textcolor{red}{n}, \textcolor{blue}{m}}(\{x, \theta\}\{y, \psi\}) = W_{\textcolor{blue}{m}, \textcolor{red}{n}}(\{y, \psi\}\{x, \theta\})$$

Grassmann structure of the Wilson loop form factors in Born approximation

Recall: Vacuum expectation value of the Wilson loop

$$W_n = \sum_{\kappa=0}^{n-4} W_n^{(\kappa)}, \quad \text{where} \quad W_n^{(\kappa)} \sim (\theta)^{4\kappa}$$

Form factors: Grassmann variables $\theta^A, \psi_A \Rightarrow SU(4)$ invariants: $(\theta)^4, (\psi)^4, (\theta \cdot \psi)$



$$W_{n,m} = \sum_{\text{clusters}} e^{ix_{i_1}y_{1l_2} + \dots} e^{i\langle \theta_{i_1} \psi_{1l_2} \rangle + \dots} \widehat{W}_{n,m}$$

Grassmann expansion $\widehat{W}_{n,m} = \sum_{\kappa, \sigma \geq 0} W_{n,m}^{(\kappa, \sigma)}$

$$W_{n,m}^{(\kappa, \sigma)} \sim (\theta)^{4\kappa} (\eta)^{4\sigma} \quad \text{---} \quad N^\kappa \text{MHV} \times N^\sigma \text{MHV}$$

Duality: $N^\kappa \text{MHV} \times N^\sigma \text{MHV} \leftrightarrow N^\sigma \text{MHV} \times N^\kappa \text{MHV}$

How to describe supersymmetric Wilson loop?

Superconnections (at $\bar{\theta} \rightarrow 0$)

$$\nabla_{\alpha\dot{\alpha}} = \partial/\partial x^{\alpha\dot{\alpha}} + \mathcal{A}_{\alpha\dot{\alpha}}(x, \theta) \quad , \quad \nabla_A^\alpha = \partial/\partial \theta_\alpha^A + \mathcal{A}_A^\alpha(x, \theta)$$

Defining constraints of $\mathcal{N} = 4$ SYM

$$\{\nabla_A^\alpha, \nabla_B^\beta\} = \epsilon^{\alpha\beta} W_{AB} + \cancel{W_{AB}^{(\alpha\beta)}} \quad , \quad \dots$$

put all fields **on shell**

- Complicated expressions for $\mathcal{A}_{\alpha\dot{\alpha}}(x, \theta)$ and $\mathcal{A}_A^\alpha(x, \theta)$ in terms of standard fields

$$\mathcal{A}_{\alpha\dot{\alpha}}(x, \theta) = \mathcal{A}_{\alpha\dot{\alpha}}(x) + \theta_\alpha \cdot \bar{\psi}_{\dot{\alpha}} + \dots \quad , \quad \mathcal{A}_{\alpha A}(x, \theta) = \phi_{AB}(x) \theta_\alpha^B + \dots$$

- On-shell superfields. Chiral anomaly of the Wilson loop [Belitsky, Korchemsky, Sokatchev '11]
- We need chiral SUSY **off shell**
- Twistor [Witten '04; Mason '05; Mason & Skinner '10; Adamo, Boels, Bullimore, Mason, Skinner] vs Lorentz Harmonic [Sokatchev '95; D.C. & Sokatchev '16]
approach to **off-shell** formulation of $\mathcal{N} = 4$ SYM
- But the twistor Wilson loop of Mason & Skinner is NOT gauge invariant

$\mathcal{N} = 4$ SYM in the Lorentz Harmonic Chiral superspace

Harmonic variables u^{\pm}_{α} on $SU(2)/U(1) \sim S^2$, a pair of spinors

Rosly '83; Galperin, Ivanov, Kalitsyn, Ogievetsky, Sokatchev '84; Sokatchev '87

$$(u^+_{\alpha}, u^-_{\alpha}) \in SU(2) \quad , \quad u^{+\alpha} u^-_{\alpha} = 1$$

A harmonic field of given $U(1)$ charge contains an infinite number of irrep., e.g.

$$f^{++}(x, u) = f^{(\alpha\beta)}(x) u^+_{\alpha} u^+_{\beta} + f^{(\alpha\beta\gamma)}(x) u^+_{\alpha} u^+_{\beta} u^-_{\gamma} + f^{(\alpha\beta\gamma\delta)}(x) u^+_{\alpha} u^+_{\beta} u^-_{\gamma} u^-_{\delta} + \dots$$

	τ -frame	Analytic frame
Superfields	$A(x, \theta)$	$A(x, \theta, u)$
Gauge transform	$\nabla \rightarrow (e^{\tau(x, \theta)})_{\text{Adj}} \nabla$	$\nabla \rightarrow (e^{\Lambda(x, \theta^+, u)})_{\text{Adj}} \nabla$
Connections	$\nabla^{++} = \partial^{++} \equiv u^+ \frac{\partial}{\partial u^-}$ $\nabla_A^+ = u_{\alpha}^+ \nabla_A^{\alpha}, \quad \{\nabla_A^+, \nabla_B^+\} = 0$	$\xrightarrow{\substack{\text{generalized} \\ \text{gauge} \\ \text{transform} \\ h(x, \theta, u)}}$ $\nabla^{++} = \partial^{++} + A^{++}$ $\nabla_A^+ = u_{\alpha}^+ \frac{\partial}{\partial \theta_{\alpha}^A}$

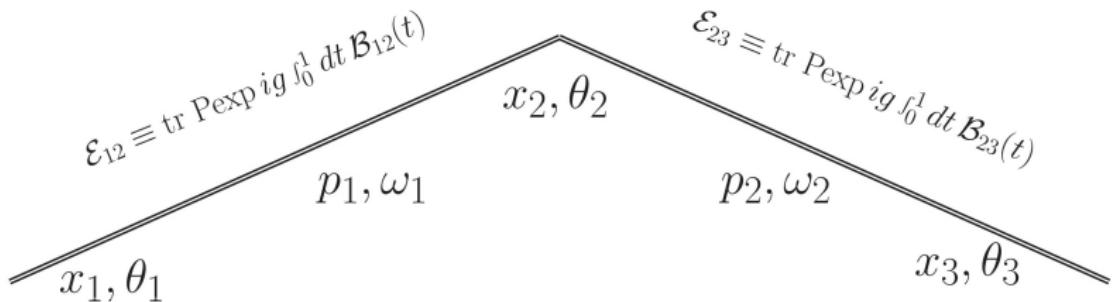
Off-shell fields of the theory: gauge connections covariantizing ∂^{++} and $\partial_{\dot{\alpha}}^+ \equiv u^{+\alpha} \partial_{\alpha\dot{\alpha}}$

$$A^{++}(x, \theta^+, u) \quad , \quad A_{\dot{\alpha}}^+(x, \theta^+, u)$$

Wilson Loop in the Lorentz Harmonic Chiral superspace

Classical Wilson loop

$$\mathcal{W}_n = \text{tr } \mathcal{E}_{12} \mathcal{E}_{23} \dots \mathcal{E}_{n1}$$



$$\begin{aligned}\mathcal{B}_{i,i+1}(t) &\equiv p_i^{\alpha\dot{\alpha}} \mathcal{A}_{\alpha\dot{\alpha}}(x(t), \theta(t)) \\ &+ \omega_i^A p_i^\alpha \mathcal{A}_{\alpha A}(x(t), \theta(t))\end{aligned}$$

bridge transform

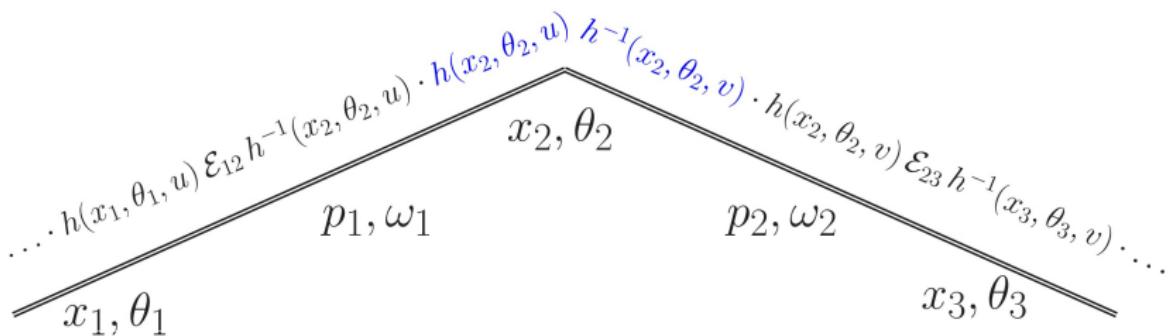
$$\mathcal{B}_{i,i+1}(t) \equiv p_i^{\dot{\alpha}} A_{\dot{\alpha}}^+(x(t), \langle p_i \theta_i \rangle, p_i^\alpha)$$

where $x(t) = x_i - t p_i, \dots$

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bridge
transform

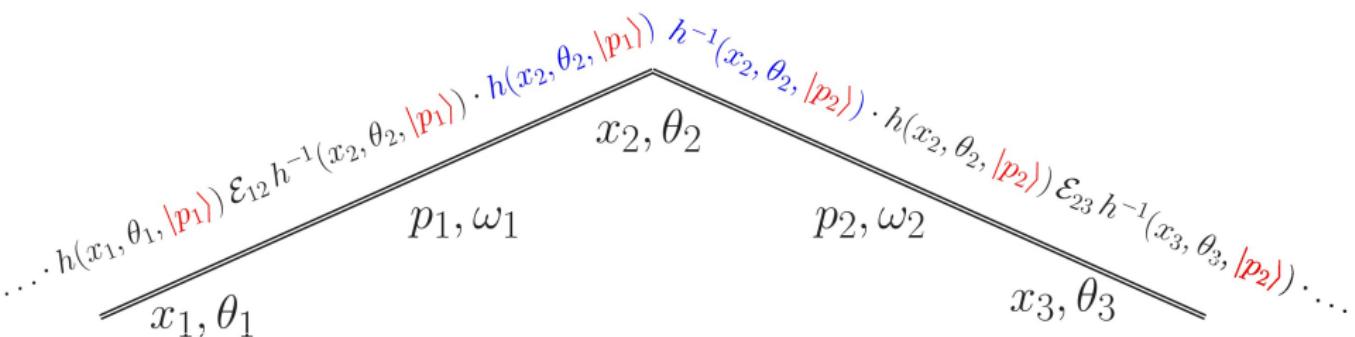
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bridge transform

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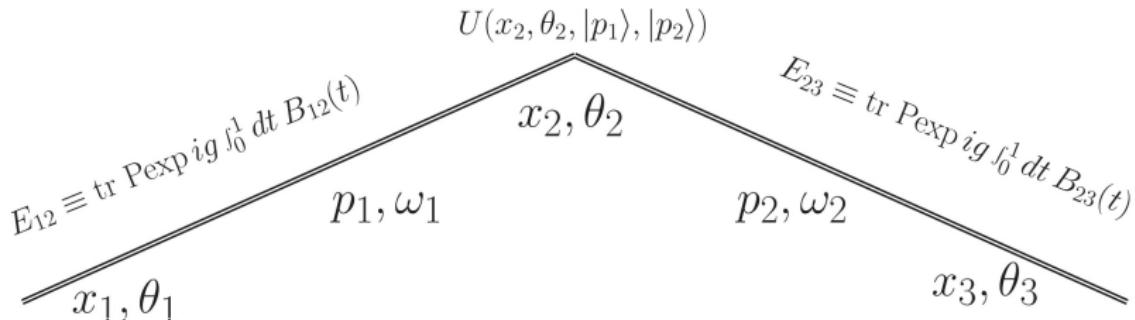
Wilson Loop in the Lorentz Harmonic Chiral superspace

Classical Wilson loop

$$\mathcal{W}_n = \text{tr } \mathcal{E}_{12} \mathcal{E}_{23} \dots \mathcal{E}_{n1}$$

$$= \text{tr } E_{12} U_2 E_{23} U_3 \dots E_{n1} U_1$$

gauge invariant & completes the twistor formula [Mason & Skinner '10]



$$\begin{aligned} \mathcal{B}_{i,i+1}(t) &\equiv p_i^{\alpha\dot{\alpha}} \mathcal{A}_{\alpha\dot{\alpha}}(x(t), \theta(t)) \\ &+ \omega_i^A p_i^\alpha \mathcal{A}_{\alpha A}(x(t), \theta(t)) \end{aligned}$$

bridge
transform

$$\mathcal{B}_{i,i+1}(t) \equiv p_i^{\dot{\alpha}} A_{\dot{\alpha}}^+(x(t), \langle p_i \theta_i \rangle, p_i^\alpha)$$

where $x(t) = x_i - t p_i, \dots$

Classical Wilson loop

$$\mathcal{W}_n = \text{tr } E_{12} \ U_2 \ E_{23} \ U_3 \ \dots \ E_{n1} \ U_1$$

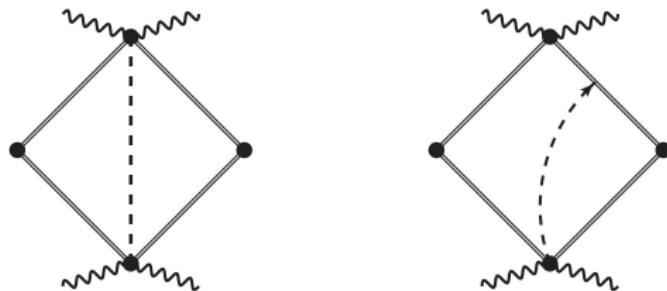
$E_{i,i+1} = E_{i,i+1}[A_{\dot{\alpha}}^+]$	sits on edge 	Wilson line	Nonlocal in x-space
$U_i = U_i[A^{++}]$	sits at vertex 	Transform between two analytic frames $u^+ = p_{i-1}\rangle \rightarrow u^+ = p_i\rangle$	Nonlocal in harmonic space

Quantization of the theory in the light-cone gauge

$$\langle A^{++} A^{++} \rangle \quad , \quad \langle A^{++} A_{\dot{\alpha}}^+ \rangle \quad , \quad \langle 0 | A^{++} | k, \eta \rangle$$

The main role is played by **vertex graphs**

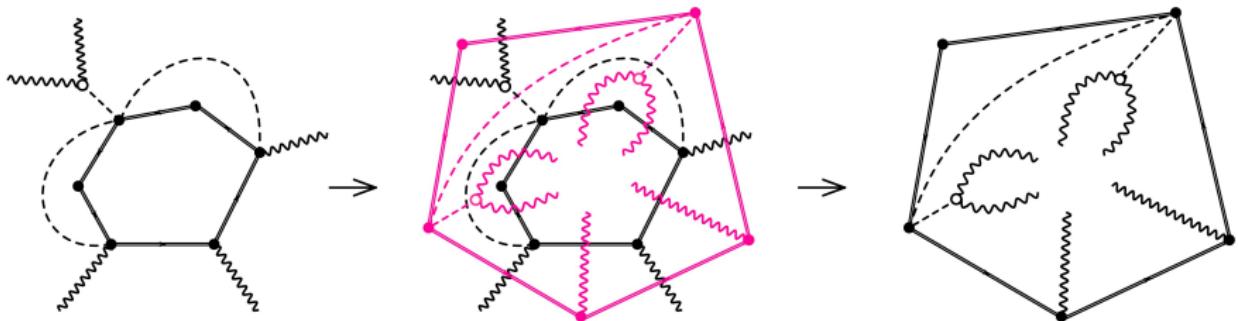
Edge graphs restore gauge invariance



$N^2\text{MHV} \times \text{NMHV}$ is dual to $\text{NMHV} \times N^2\text{MHV}$

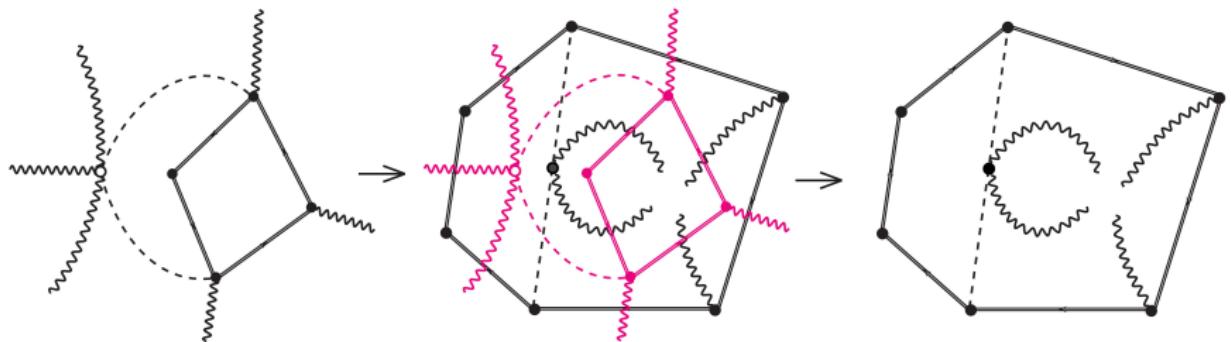
Drop E -pieces of the Wilson loop. Then the duality works graph by graph.

Duality of Wilson loop form factors \sim Duality of planar graphs



$$\widehat{W}_{6,5}^{(2,1)}(\{x,\theta\}\{y,\psi\}) = \widehat{W}_{5,6}^{(1,2)}(\{y,\psi\}\{x,\theta\})$$

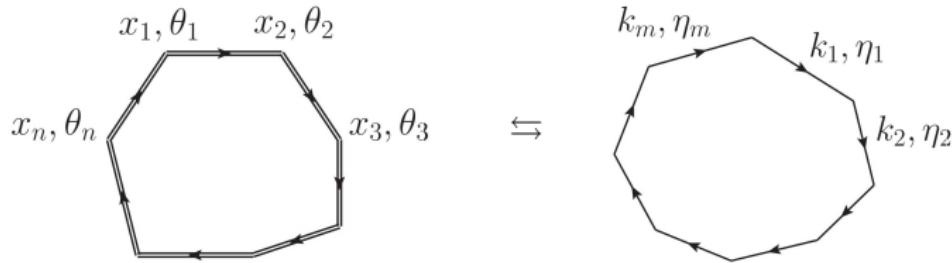
One loop MHV form factor of the bosonic Wilson loop



UV and IR divergences, but the duality works at the level of the INTEGRAND !

Summary

- Supersymmetric Wilson loop form factors
- New duality which exchanges two polygonal light-like contours



- Lorentz Harmonic superspace formulation of the Wilson loop
- Deep reason for the duality?