Critical L-values from sunrise diagrams with up to 18 loops

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I report my recent progress, in Australia, on the empirical evaluation of special values of **L-functions** by minors of **period matrices** of moments of **Bessel** functions that include those from on-shell equal-mass **sunrise** diagrams, in two space-time dimensions, with up to 18 loops. Previously such relations were known for sunrise diagrams with up to 6 loops.

- Definitions
- To 5 loops, with Jon **Borwein**, Francis **Brown** and Anton **Mellit**
- At 6 loops, with a twist, from Spencer **Bloch**
- To 10 loops, with more twists, from David **Roberts**
- To 18 loops, with a computational challenge at 17 loops

In Memoriam, Jonathan Michael Borwein (20 May 1951 - 2 Aug 2016)

The lists of automorphic forms run dry,
Mere algebraic geometers may tire,
Yet Feynman's diagrams diversify,
As do the mathematics they inspire.

The Bessel moments

$$S_{n,s} \equiv 2^s \int_0^\infty [I_0(t)]^{n-s-1}[K_0(t)]^{s+1} t\mathrm{d}t$$

converge if $s < n \leq 2s+2$, with $s > 1$ in the case $n = 2s+2$. The s-loop the sunrise diagram with $n = s+2$ Bessel functions gives the integral

$$S_{s+2,s} \equiv 2^s \int_0^\infty I_0(t)[K_0(t)]^{s+1} t\mathrm{d}t$$

which engages algebraic geometers via its representation as

$$S_{s+2,s} = \int_0^\infty \ldots \int_0^\infty \frac{\Pi_{k=1}^s \mathrm{d}x_k/x_k}{(1+\Sigma_{i=1}^s x_i)(1+\Sigma_{j=1}^s 1/x^j) - 1}.$$

For $n \leq 8$ Bessel functions, I succeeded in relating such integrals to L-series whose local factors, at the primes, come from moments of Kloosterman sums, defined in finite fields, as follows.

For $a \in \mathbf{F}_q$, with $q = p^k$, let

$$K(a) \equiv \sum_{x \in \mathbf{F}_q^*} \exp\left(\frac{2\pi \mathrm{i}}{p} \mathrm{Trace}\left(x + \frac{a}{x}\right)\right),$$

with a trace of Frobenius in $\mathbf{F}_q$ over $\mathbf{F}_p$. Then we obtain integers

$$c_n(q) \equiv -\frac{1+S_n(q)}{q^2}$$

$$S_n(q) \equiv \sum_{a\in\mathbf{F}_q^*} \sum_{k=0}^{n} [g(a)]^k [h(a)]^{n-k}$$

with $K(a) = -g(a) - h(a)$ and $g(a)h(a) = q$. Then

$$Z_n(p,T) = \exp\left(-\sum_{k>0} \frac{c_n(p^k)}{k} T^k\right)$$

is a polynomial in T. For $n < 8$, the appropriate L-function is

$$L_n(s) = \prod_p \frac{1}{Z_n(p,p^{-s})}$$

with a modification at $n = 8$:

$$L_8(s) = \prod_p \frac{Z_4(p,p^{2-s})}{Z_8(p,p^{-s})}.$$

For $n > 8$, there was no result, until last month. Thanks to advice from **David Roberts**, I have progressed to $n = 18$.

There is a computational challenge for odd n, which becomes severe at $n = 19$. The case $n = 20$ is completed.

1 One Bessel function

Since $c_1(q) = 0$, we have $L_1(s) = 1$ and

$$S_{1,0} = \int_0^\infty K_0(t) t \mathrm{d}t = 1 = L_1(0).$$

2 Two Bessel functions

Since $c_2(q) = 0$, we have $L_2(s) = 1$ and

$$S_{2,1} = 2\int_0^\infty K_0^2(t) t \mathrm{d}t = 1 = L_2(1).$$

3 Three Bessel functions

The local factor involves a Kronecker symbol

$$Z_3(p,T) = 1 - \left(\frac{p}{3}\right) T$$

so $L_3(s)$ is a Dirichlet L-series with conductor $N_3 = 3$. We have the evaluations

$$S_{3,1} = 2L_3(1) = \frac{2\pi}{\sqrt{3^3}}, \quad S_{3,2} = 3L_3(2) = 3\sum_{k\geq 0}\left(\frac{1}{(3k+1)^2} - \frac{1}{(3k+2)^2}\right).$$

4 Four Bessel functions

$$L_4(s) = \prod_{p>2} \frac{1}{1-p^{-s}} = (1-2^{-s})\zeta(s),$$

$$S_{4,2} = 2L_4(2) = \frac{\pi^2}{4}, \quad S_{4,3} = 8L_4(3) = 7\zeta(3).$$

Remark: From now on, I omit cases $S_{n,n-1}$ with no I_0 in the integrand. Those so-called vacuum integrals have an interesting story of their own. Here, the sunrise integrals $S_{n,n-2}$ are the main focus.

5 Five Bessel functions

The conductor is $N_5 = 5N_3 = 15$ and the L-function is that of a modular form with modular weight 3 and level 15:

$$\begin{aligned}
\eta_n &= q^{n/24} \prod_{k>0} (1-q^{nk}) \\
f_{3,15} &= (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3 = \sum_{n>0} A_5(n) q^n \\
L_5(s) &= \sum_{n>0} \frac{A_5(n)}{n^s}
\end{aligned}$$

$$
\begin{aligned}
L_5(2) &= \frac{\sqrt{3}}{360\pi}\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)\\
S_{5,2} &= 3L_5(2)\\
S_{5,3} &= \frac{2}{5}(2\pi)^2 L_5(1).
\end{aligned}
$$

This connection was discovered by studying the denominator of

$$S_{5,3} = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{(abc+ab+bc+ca)(a+b+c)+ab+bc+ca}$$

in finite fields.

Remark: The Hodge vector is $H_5 = (1,0,1)$.

6 Six Bessel functions: the Laporta frontier

Here the modular form is

$$f_{4,6} = (\eta_1\eta_2\eta_3\eta_6)^2$$

with weight 4 and level 6. I discovered and checked to 1000 digits that

$$S_{6,2} = 6L_6(2), \quad S_{6,3} = 12L_6(3), \quad S_{6,4} = 2(2\pi)^2 L_6(2).$$

Remark: The Hodge vector is $H_6 = (1,0,0,1)$.

7 Seven Bessel functions

The conductor is $N_7 = 7N_5 = 105$. For p coprime to 105,

$$Z_7(p,T) = \left(1 - \left(\frac{p}{105}\right) p^2 T\right)\left(1 + \left(\frac{p}{105}\right)(2p^2 - |\lambda_p|^2)T + p^4T^2\right)$$

where $\lambda_p \in \mathbf{Q}(\sqrt{-1}, \sqrt{6}, \sqrt{14})$ is the Hecke eigenvalue of a weight-3 newform on $\Gamma_0(525)$, with modular weight 3 and quartic nebentypus. At the bad primes, I obtained quadratic polynomials

$$Z_7(3,T) = 1 - 10T + 3^4T^2,\ Z_7(5,T) = 1 - 5^4T^2,\ Z_7(7,T) = 1 + 70T + 7^4T^2$$

that required no modification and give the functional equation

$$\Lambda_7(s) \equiv \left(\frac{105}{\pi^3}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) L_7(s) = \Lambda_7(5-s)$$

suggested by Anton Mellit. Then Tim Dokchitser's `computel` gave

$$S_{7,4} \equiv 2^4 \int_0^\infty [I_0(t)]^2[K_0(t)]^5 t\mathrm{d}t = 20\zeta(2)L_7(2).$$

Remark: The Hodge vector is $H_7 = (1,0,1,0,1)$.

8 Eight Bessel functions: removing factors

The conductor is $N_8 = N_6 = 6$, but there is a subtlety. At the odd primes one must **remove** from $Z_8(p,T)$ a factor $(1-p^2T)$ to obtain the L-function

$$L_8(s) = \prod_p \frac{Z_4(p,p^{2-s})}{Z_8(p,p^{-s})}$$

of the modular form

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

with weight 6 and level 6. I discovered and checked to 1000 digits that

$$S_{8,3} = 8L_8(3), \quad S_{8,4} = 36L_8(4), \quad S_{8,5} = 216L_8(5), \quad S_{8,6} = (2\pi)^2 S_{8,4}.$$

Remark: The Hodge vector is $H_8 = (1,0,0,0,0,1)$.

Signpost: From now on, the Hodge vector, H_n, for n Bessels is obtained from that for $n-4$ Bessels by prepending $(1,0)$ and appending $(0,1)$. In particular,

$$\begin{pmatrix} H_9 \\ H_{10} \\ H_{11} \\ H_{12} \end{pmatrix} = \begin{pmatrix} (1,0,1,0,1,0,1) \\ (1,0,1,0,0,1,0,1) \\ (1,0,1,0,1,0,1,0,1) \\ (1,0,1,0,0,0,0,1,0,1) \end{pmatrix}.$$

9 Nine Bessel functions

9.1 Adding a factor at $p = 3$

The Hodge vector is $H_9 = (1, 0, 1, 0, 1, 0, 1)$ with "motivic" weight 6 and degree 4. We expect a conductor $N_9 \geq N_7 = 105$. The local factors at $p = 5$ and $p = 7$ are cubic, but at $p = 3$ the factor $Z_9(p, T)$ is merely quadratic. So there are two possibilities: leave it alone and try conductor 315, or **add** an extra factor $(1 - 3^3 T)$ by hand and hope for conductor 105. The latter works.

$$L_9(s) = \frac{1}{1 - 3^{3-s}} \prod_p \frac{1}{Z_9(p, p^{-s})}$$

has conductor $N_9 = 105$ and a functional equation

$$\Lambda_9(s) \equiv \left(\frac{105}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s-2}{2}\right) \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L_9(s) = \Lambda_9(7 - s)$$

enabling one to evaluate $L_9(4)$, or equivalently $L_9(3)$, for which we expect to have **two** evaluations in terms of integrals of Bessel functions, since $L_5(2)$, or equivalently $L_5(1)$, had two evaluations for the $n = 5$ problem.

9.2 Choosing the determinants

Since the degree is now 4, we search for a pair of 2×2 matrices whose determinants evaluate $L_9(4)$. To specify the choice, let

$$M_{n,s,k} \equiv 2^s \int_0^\infty [I_0(t)]^{n-s-1}[K_0(t)]^{s+1} t^{2k-1} \mathrm{d}t$$

so that $S_{n,s} = M_{n,s,1}$. Now consider the determinants

$$D_{9,7} = \begin{vmatrix} M_{9,7,1} & M_{9,7,2} \\ M_{9,5,1} & M_{9,5,2} \end{vmatrix}, \qquad D_{9,6} = \begin{vmatrix} M_{9,6,1} & M_{9,6,2} \\ M_{9,4,1} & M_{9,4,2} \end{vmatrix}$$

in which the power of t increases by two in the second column and the power of I_0 increases by two in the second row. Then I find that

$$240\pi^2 L_9(4) = D_{9,6}, \qquad \sqrt{105} D_{9,7} = (4\pi)^2 D_{9,6}.$$

Notation: Let $D_{n,s}$ stand for the $k \times k$ determinant with $S_{n,s}$ at the top left, size $k = \lceil n/4 \rceil - 1$, powers of t increasing by two as we move right and powers of I_0 increasing by two as we move down.

10 Ten Bessel functions

The Hodge vector is $H_{10} = (1, 0, 1, 0, 0, 1, 0, 1)$, with degree 4. **David Roberts** suggested that conductor $N_{10} = 5N_8 = 30$ could be achieved by adding a factor $(1-2^3T)$ to $Z_{10}(2,T)$. Then we obtain the evaluations

$$18(2\pi)^6L_{10}(4)=D_{10,8},\quad 60(8\pi)^2L_{10}(5)=D_{10,7}.$$

11 Eleven Bessel functions

The Hodge vector is $H_{11}=(1,0,1,0,1,0,1,0,1)$, with degree 5. The conductor is $N_{11}=11N_9=1155$. I followed the practice at $n=9$ and added a factor $(1-3^4T)$ to $Z_{11}(3,T)$.

From experience at $n=7$, I expected only one evaluation and found

$$21(2\pi)^6L_{11}(4)=5D_{11,8}.$$

12 Twelve Bessel functions: a sign change

The Hodge vector is $H_{12}=(1,0,1,0,0,0,0,1,0,1)$, with degree 4. The conductor is $N_{12}=6N_{10}=180$. We follow the practice at $n=8$, removing $(1-p^4T)$ at all odd primes. Then comes a new feature, observed by **David Roberts**: the functional equation has a odd sign, with $\Lambda_{12}(s)=-\Lambda_{12}(10-s)$.

As in the case of $n = 8$, we obtain 4 results. The sign makes one of them trivial: $L_{12}(5) = 0$. Another relates determinants: $D_{12,10} = (2\pi)^4 D_{12,8}$. Then we have the interesting evaluations

$$3^4 5 (4\pi)^6 L_{12}(6) = D_{12,10}, \qquad 2(2^5 3^3 5\pi)^2 L_{12}(7) = 11 D_{12,9}$$

with 11 in the numerator of a ratio of the L-value over the determinant.

13 Thirteen Bessel functions and the prime 149

Adding a factor $(1 - 3^5 T)$ to $Z_{13}(3, T)$, I obtained a conductor $N_{13} = 13 N_{11} = 15015$. Taking 3×3 determinants, I found that

$$2^5 3^3 35 (4\pi)^6 L_{13}(6) = 149 D_{13,10}, \qquad \sqrt{15015} D_{13,11} = 5(4\pi)^3 D_{13,10}.$$

The following output shows the accuracy achieved with 10000 to 40000 terms in the L-series:

```
10000:  149.0000000000000000000000085...
20000:  148.99999999999999999999999999999985...
30000:  149.00000000000000000000000000000000000020...
40000:  148.9999999999999999999999999999999999999999999945...
```

14 Fourteen Bessel functions

Adding a factor $(1 - 2^5T)$ to $Z_{14}(2, T)$, I obtained a conductor $N_{14} = 7N_{12} = 1260$. The sign is odd and the evaluations are

$$L_{14}(6) = 0, \quad 2^5 5^2 7 (12\pi)^6 L_{14}(7) = 143 D_{14,11}$$

with a composite numerator $143 = 11 \times 13$.

15 Fifteen Bessel functions

Adding a factor $(1 - 3^6T)$ to $Z_{15}(3, T)$, I obtained conductor $N_{15} = 15N_{13} = 225225$ and the evaluation

$$2^{10} 3^2 5 (2\pi)^{12} L_{15}(6) = 371 D_{15,12}$$

with a composite numerator $371 = 7 \times 53$, supported by this output

```
10000:  371.000000000024...
20000:  370.99999999999999970...
30000:  370.9999999999999999908...
40000:  370.99999999999999999983...
```

16 Sixteen Bessel functions: add and remove factors

Here it is necessary to **add** a factor $(1-2^6T)$ to $Z_{16}(2,T)$ and to **remove** $(1-p^6T)$ from $Z_{16}(p,T)$ for odd prime p. This gives a conductor $N_{16}=N_{14}=1260$ and an odd sign. As expected, I obtained 4 relations: $L_{16}(7)=0$, from the sign, the determinant relation $D_{16,14}=(2\pi)^6D_{16,12}$ and the evaluations

$$2^{10}3^75^27(2\pi)^{12}L_{16}(8)=D_{16,14},\qquad 2(3^35^37^2)(24\pi)^6L_{16}(9)=13D_{16,13}.$$

Signpost: It looked as if I could tackle the next 4 cases, up to $n=20$, with some doubts about the case $n=19$, for which I expected a large conductor $N_{19}\geq 17\times 19\times N_{15}=72747675$. Accordingly I ran process on 20 remote nodes, during the conference at Creswick, Victoria, Australia, in January 2017, to accumulate Kloosterman data on $c_n(q)$ with $n\in[17,20]$ and $q<50000$. This has been extended to $q<70000$, in recent days.

17 Seventeen Bessel functions

Adding a factor $(1 - 3^7T)$ to $Z_{17}(3,T)$, I obtained a conductor $N_{17} = 17N_{15} = 3828825$. Taking 4×4 determinants, I found that

$$6^5 5^2 7(4\pi)^{12} L_{17}(8) = 29 D_{17,14}, \qquad \sqrt{N_{17}} D_{17,15} = 35(4\pi)^4 D_{17,14}.$$

The following output shows the accuracy achieved:

```
10000:  28.999999962...
20000:  28.9999999912...
30000:  28.9999999985...
40000:  29.0000000000048...
50000:  28.9999999999968...
```

18 Eighteen Bessel functions: an even sign

Adding a factor $(1 - 2^7T)$ to $Z_{18}(2,T)$, I obtained a conductor $N_{18} = 6N_{16} = 7560$. The sign is **even** and the evaluations are

$$2^{11} 3^5 5^4 7(2\pi)^{20} L_{18}(8) = 323 D_{18,16}, \quad 3^8 5^3 7^2 (8\pi)^{12} L_{18}(9) = 9061 D_{18,15}$$

with composite numerators $323 = 17 \times 19$ and $9061 = 13 \times 17 \times 41$.

19 Nineteen Bessel functions: a challenge

Here I expected a single relation, as for the cases $n = 7$, 11 and 15. I added a factor $(1 - 3^8T)$ to $Z_{19}(3, T)$, guessed a conductor $N_{19} = 19N_{17} = 72747675$ and sought to compute

$$C_{19} \equiv 2^{14}3^35^3(2\pi)^{20}\frac{L_{19}(8)}{D_{19,16}}$$

which I supposed to be rational, on the basis of previous cases. The convergence was painfully slow:

```
40000:  778.99861...
50000:  778.99879...
60000:  778.99896...
70000:  778.99905...
```

leading me to suspect the choice of conductor.

The Hodge vector $H_{19} = (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1)$ gives degree 9 for the local factors at primes $p > 19$. I verified that at the bad primes $p = 7$, 11, 13 and 17, the local factor is an irreducible polynomial of degree 8, consistent with the conductor N_{19} containing no square of those primes. The data at $p = 3$ and 5 are consistent with those primes occurring to the second power. At $p = 19$, I found that $Z_{19}(p, T)$ factorizes into a simple quadratic and irreducible sextic:

$$\frac{Z_{19}(19,T)}{1-19^{16}T^2} = 1 + (19T + 19^{33}T^5)A + (19^4T^2 + 19^{20}T^4)B + 19^9T^3C + 19^{48}T^6$$

with $A = 591875130, B = 2118367677674895, C = 143463945549827119 48$. That does not seem to require the conductor to contain the square of 19. Nor could I get better convergence by adding factors to my best bet

$$N_{19} \stackrel{?}{=} 3 \times 5 \times 7 \times 11 \times 13 \times 15 \times 17 \times 19 = \frac{19!!}{9} = 72747675.$$

20 Twenty Bessel functions and the prime 131

I **add** a factor $(1-2^8T)$ to $Z_{20}(2,T)$ and **remove** $(1-p^8T)$ from $Z_{20}(p,T)$ for odd prime p. This gives a conductor $N_{20} = 5N_{18} = 37800$ and an odd sign. As expected, I obtain 4 relations: $L_{20}(9) = 0$, from the sign, the determinant relation $D_{20,18} = (2\pi)^8 D_{20,16}$ and the evaluations

$$2^{18}3^{11}5^6 7^3 (2\pi)^{20} L_{20}(10) = 1441 D_{20,18}, \quad 6(5^7 7^3)(24\pi)^{12} L_{20}(11) = 7429 D_{20,17}$$

with composite numerators $1441 = 11 \times 131$ and $7429 = 17 \times 19 \times 23$, supported by this output:

```
10000:  1441.000000000015...
20000:  1440.99999999999999907...
30000:  1440.9999999999999999988...
40000:  1440.999999999999999999989...
50000:  1440.999999999999999999999940...
60000:  1441.00000000000000000000000099...
70000:  1440.9999999999999999999999988...
-------------------------------------------
10000:  7429.00000000000059...
20000:  7428.999999999999999964...
30000:  7428.9999999999999999999957...
40000:  7428.999999999999999999999961...
50000:  7428.99999999999999999999999977...
60000:  7429.000000000000000000000000037...
70000:  7428.999999999999999999999999999957...
```

Finis