From de Jonquières' Counts to Cohomological Field Theories

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Women at the Intersection of Mathematics and High Energy Physics

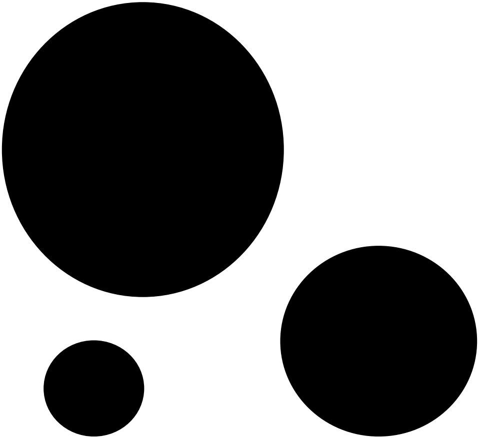
9 March 2017

What is Enumerative Geometry?

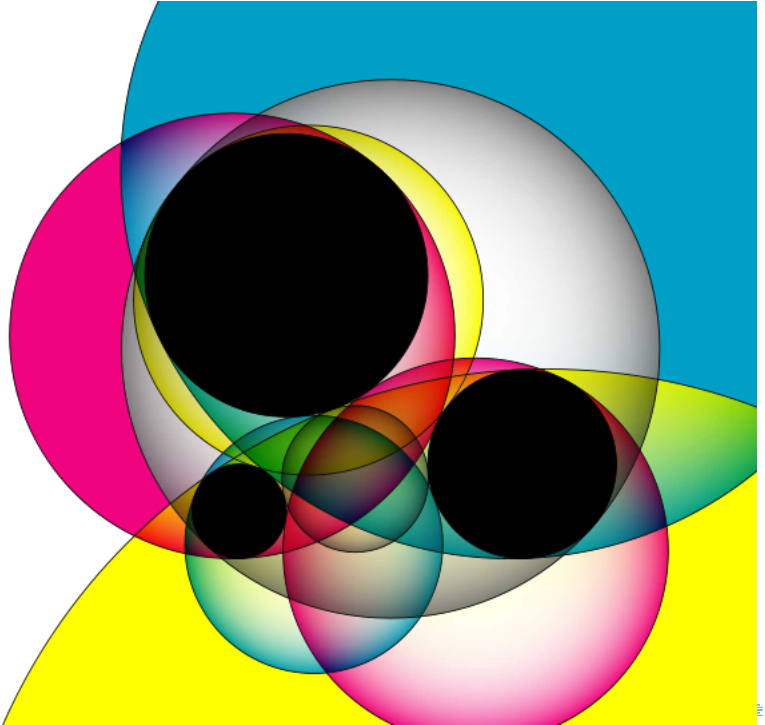
How many geometric structures of a given type satisfy a given collection of geometric conditions?

What is Enumerative Geometry?

Appolonius' problem (approx. 200 BC)



8 circles tangent to 3 other circles



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- **317.206.375** cubics on a quintic threefold (1991 Ellingsrud-Strømme)

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Number of rational curves of any degree on quintic threefold?

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Mirror symmetry!

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- 1995 Kontsevich
- 1996 Givental
- Clemens conjecture?

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How many points in the plane lie at the intersection of two given lines?

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How many points in the plane lie at the intersection of two given lines?

Answer

It depends!

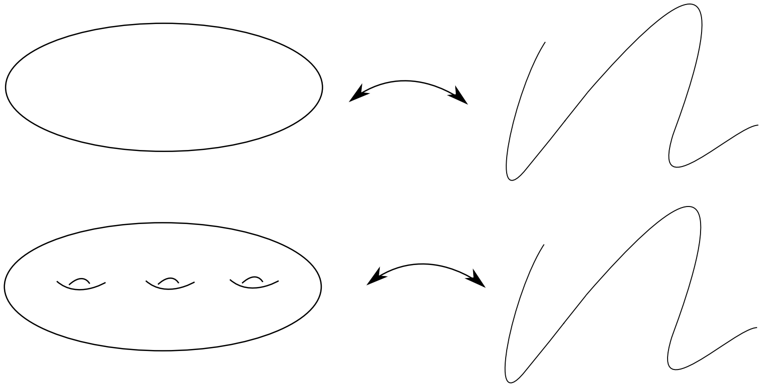
- Lines in general position $\Rightarrow$ exactly one
- Parallel lines $\Rightarrow$ none
- Lines coincide $\Rightarrow$ an infinite number of points

What is Enumerative Geometry?

- Parameter (moduli) space
- Compactify!
- Do excess intersection theory

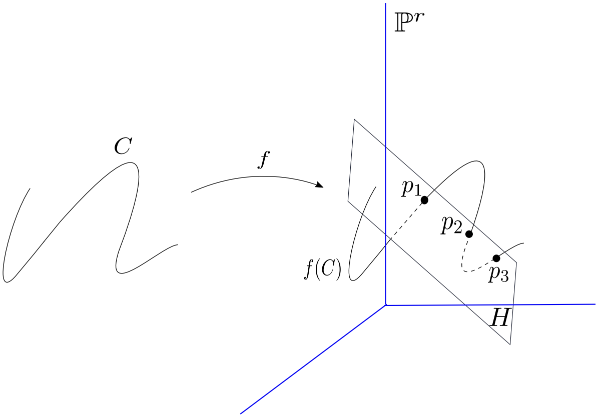
Ernest de Jonquières

Disclaimer



de Jonquières' multitangency conditions

C smooth, genus g
$f : C \to \mathbb{P}^r$ non-degenerate
Degree of $f = \#\{f(C) \cap H\} =: d$



$$d = 3$$

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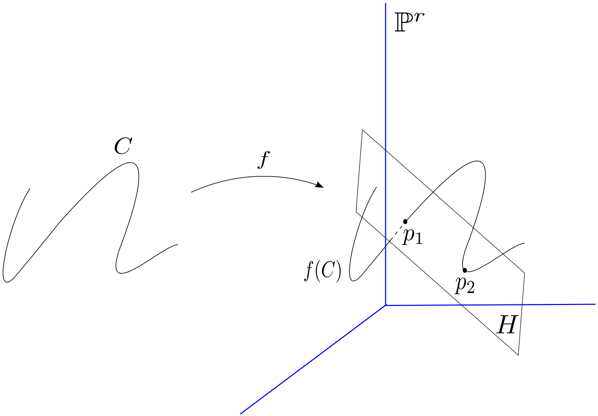

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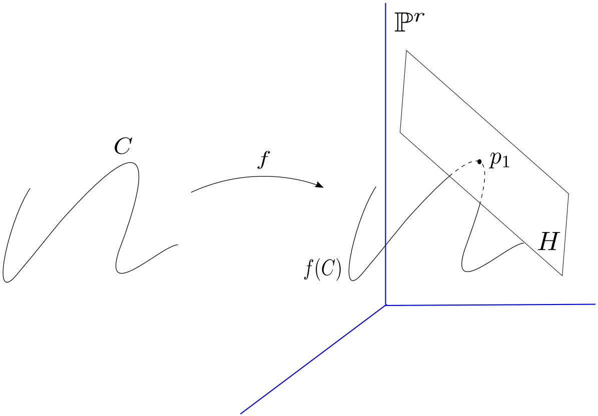

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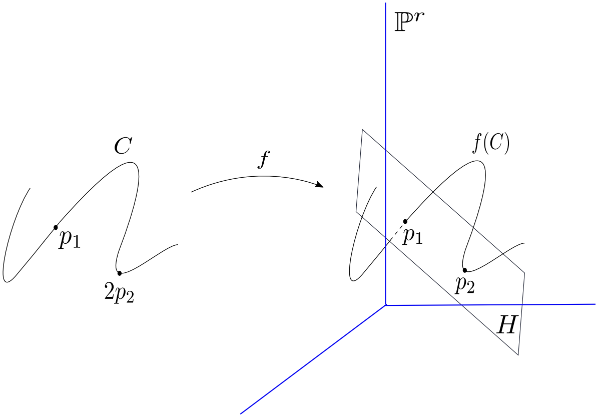

$$f^{-1}\{f(C) \cap H\} = p_1 + 2p_2$$

de Jonquières' multitangency conditions

de Jonquières counts the number of pairs (p_1, p_2) such that there exists a hyperplane $H \subset \mathbb{P}^r$ with

$$f^{-1}\{f(C) \cap H\} = p_1 + 2p_2$$

de Jonquières' multitangency conditions

de Jonquières (and Mattuck, Macdonald) count the n-tuples

$$(p_1, \ldots, p_n)$$

such that there exists a hyperplane $H \subset \mathbb{P}^r$ with

$$f^{-1}\{f(C) \cap H\} = a_1 p_1 + \ldots + a_n p_n$$

where

$$a_1 + \ldots + a_n = d$$

de Jonquières' multitangency conditions

The (virtual) de Jonquières numbers are the coefficients of

$$t_1 \cdot \ldots \cdot t_n$$

in

$$(1 + a_1^2 t_1 + \ldots + a_n^2 t_n)^g (1 + a_1 t_1 + \ldots + a_n t_n)^{d-r-g}$$

Constructing the moduli space

An embedding $f : C \to \mathbb{P}^r$ of degree d is given by

A pair (L, V)

- a line bundle L of degree d on C
- an $(r + 1)$-dimensional vector space V of sections of L

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$$\text{Choose } (\sigma_0, \ldots, \sigma_r) \text{ basis of } V$$
$$\Downarrow$$
$$f : C \to \mathbb{P}^r$$
$$p \mapsto [\sigma_0(p) : \ldots : \sigma_r(p)]$$

Constructing the moduli space

Space of all divisors of degree d on C

$$C_d = \underbrace{C \times \ldots \times C}_{d \text{ times}} / S_d$$

For example

$$p_1 + 2p_2 \in C_3$$

Constructing the moduli space

Space of all divisors of degree d on C

$$C_d = \underbrace{C \times \ldots \times C}_{d \text{ times}} / S_d$$

For example

$$p_1 + 2p_2 \in C_3$$

We define de Jonquières divisors

$$p_1 + \ldots + p_n \in C_n$$

such that

$$f^{-1}\{f(C) \cap H\} = a_1 p_1 + \ldots + a_n p_n$$

Constructing the moduli space

$D = p_1 + \ldots + p_n$ is de Jonquières divisor

$$\Updownarrow$$

there exists a section σ whose zeros are

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the map

$$\begin{aligned} \beta_D : V &\to V|_{a_1 p_1 + \ldots + a_n p_n} \\ \sigma &\mapsto \sigma|_{a_1 p_1 + \ldots + a_n p_n} \end{aligned}$$

has nonzero kernel

Constructing the moduli space

the map

$$\beta_D : V \to V|_{a_1p_1+\ldots+a_np_n}$$
$$\sigma \mapsto \sigma|_{a_1p_1+\ldots+a_np_n}$$

has nonzero kernel

$$\Updownarrow$$

$$rank(\beta_D) \leq \dim V - 1 = r$$

Constructing the moduli space

$$
\begin{tikzcd}
V \arrow[rr, "\beta_D"] \arrow[dr] & & V|_{a_1p_1+\ldots+a_np_n} \arrow[dl] \\
& D = p_1 + \ldots + p_n \in C_n &
\end{tikzcd}
$$

Constructing the moduli space

$$\begin{tikzcd} V \otimes \mathcal{O}_{C_n} \arrow[rr, "\beta"] \arrow[dr] & & \mathcal{F} \arrow[dl] \\ & C_n & \end{tikzcd}$$

De Jonquières divisors:

$$DJ_n = \{D \in C_n \mid rank(\beta_D) \leq r\}$$

determinantal variety

To summarise...

- Fix curve C of genus g
- Fix embedding given by (L, V)
- $C_n :=$ space of divisors of degree n
- Defined de Jonquières divisors via multitangency conditions
- Described space DJ_n of de Jonquières divisors as determinantal variety over C_n

Analysing the moduli space

$$\dim DJ_n \geq n - d + r$$

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Relevant questions

- $n - d + r < 0 \Rightarrow$ non-existence of de Jonquières divisors
- $n - d + r \geq 0 \Rightarrow$ existence of de Jonquières divisors
- $n - d + r = 0 \Rightarrow$ finite number of de Jonquières divisors
- $\dim DJ_n = n - d + r$

Taking a variational perspective

- Allow C to vary in $\mathcal{M}_{g,n}$
- Vary the de Jonquières structure with it

Taking a variational perspective

$\mathcal{M}_{g,n}$ = moduli space of smooth curves of genus g with n marked points

$$(C; p_1, \ldots, p_n) \in \mathcal{M}_{g,n}$$

- $\dim \mathcal{M}_{g,n} = 3g - 3 + n$
- compactification $\overline{\mathcal{M}}_{g,n}$

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Question

What is the cohomology of $\overline{\mathcal{M}}_{g,n}$?

Taking a variational perspective

$L = K_C =$ bundle of differential forms on C

$(L, V) = (K_C, \Gamma(C, K_C))$

Now $d = 2g - 2$ and $r = g - 1$

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Now $d = 2g - 2$ and $r = g - 1$

Fix partition $\mu = (a_1, \ldots, a_n)$ of $2g - 2$

$\mathcal{H}_g(\mu) = \{(C; p_1, \ldots, p_n)$ such that
K_C admits the de Jonquières divisor $a_1 p_1 + \ldots + a_n p_n\}$

Taking a variational perspective

$\mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n}$ determinantal subvariety

- flat surfaces, dynamical systems, Teichmüller theory: Masur, Eskin, Zorich, Kontsevich,... Bainbridge-Chen-Gendron-Grushevsky-Möller ('16), ...
- algebraic geometry: Diaz ('84), Polishchuk ('03), Farkas-Pandharipande ('15)

Taking a variational perspective

$\mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n}$
Take closure: $\overline{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$

Question

What is the fundamental class $[\overline{\mathcal{H}}_g(\mu)]$?

Answer

(potentially) Cohomological field theory!

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

CFT

- 2-dimensional QFT invariant under conformal transformations
- defined over compact Riemann surfaces

Stick to holomorphic side

CFT = 2-dimensional QFT covariant w.r.t. holomorphic coordinate changes

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

Infinitesimal change of holomorphic coordinate

$$z \mapsto z + \epsilon f(z)$$

Local holomorphic vector field

$$f(z)\frac{d}{dz}$$

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$$\Downarrow$$

Virasoro algebra:

$$L_n = -z^{n+1}\frac{d}{dz} \Rightarrow [L_n, L_m] = (m-n)L_{m+n}, n \in \mathbb{Z}$$

etc...

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

Local meromorphic vector field

$$f(z)\frac{d}{dz}$$

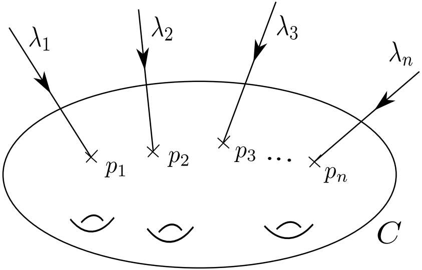
$$\Updownarrow$$

Infinitesimal deformation of complex structure

$$\Updownarrow$$

Infinitesimal deformation of an algebraic curve

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

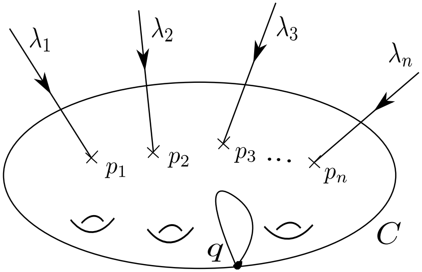


$(C; p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$

$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ representation labels

$V_{\vec{\lambda}}(C; p_1, \ldots, p_n)$ space of conformal blocks

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT



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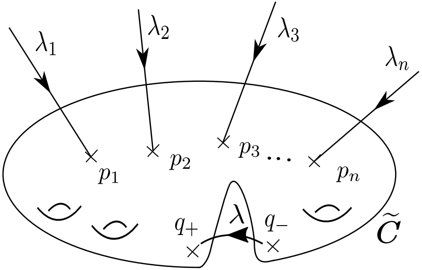

$(\widetilde{C}; p_1, \ldots, p_n, q_+, q_-) \in \overline{\mathcal{M}}_{g,n+2}$

$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n, \lambda, \lambda^\dagger)$ representation labels

$V_{\vec{\lambda}}(\widetilde{C}; p_1, \ldots, p_n, q_+, q_-)$ space of conformal blocks

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

Verlinde bundle

$$\mathcal{V}_{\vec{\lambda}} \to \overline{\mathcal{M}}_{g,n}$$

Each fibre is given by space of conformal blocks

$$V_{\vec{\lambda}}(C; p_1, \ldots, p_n) \to (C; p_1, \ldots, p_n)$$

$\overline{\mathcal{M}}_{g,n}$ and CohFT

The characters $ch(\mathcal{V}_{\vec{\lambda}})$ define a CohFT on $\overline{\mathcal{M}}_{g,n}$!

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A CohFT

- a vector space of fields U
- a non-degenerate pairing η
- a distinguished vector $\mathbf{1} \in U$
- a family of correlators

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (U^*)^{\otimes n}$$

satisfying gluing...

$\overline{\mathcal{M}}_{g,n}$ and CohFT

Quantum multiplication $*$ on U

$$\eta(v_1 * v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3) \in \mathbb{Q}$$

$(U, *)$ Frobenius algebra of the CohFT

Teleman: classification of all CohFT with semisimple Frobenius algebra

$\overline{\mathcal{M}}_{g,n}$ and CohFT

$\mathcal{H}_g(\mu) = \{(C; p_1, \ldots, p_n)$ such that
K_C admits the de Jonquières divisor $a_1p_1 + \ldots + a_np_n\}$

$[\overline{\mathcal{H}}_g(\mu)] =?$

Maybe $[\overline{\mathcal{H}}_g(\mu)]$ is one of the $\Omega_{g,n}$

$\overline{\mathcal{H}}_g(\mu)$ and CohFT

$[\overline{\mathcal{H}}_g(\mu)]$ is not a CohFT class!

Conjecture (Pandharipande, Pixton, Zvonkine): it is related to one

Witten R-spin class

$$W^R_{g,\mu} \in H^{2g-2}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

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What if $L \neq K_C$?