Numbers and Patterns in Feynman graphs

Karen Yeats

Mainz Institute for Theoretical Physics

① Set up

- Imagine you are reading a QFT ~~text book~~
  Say we have a scalar ϕ^4 field theory
  $$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$
  and you want to understand the theory perturbatively
  naively you say sum over all possibilities for what can happen — Feynman diagrams
  or more physically use the path integral $Z = \int d\phi\, e^{-\int d^4y\,\mathcal{L} + J\phi}$
  expand in λ and J and it builds graphs
  by joining half edges into corollas
  and pairing half edges into edges to get Feynman diagrams

  what's a mathematician to do?
  let's just take as our starting point to be the Feynman diagrams and their Feynman integrals

- For ϕ^4 theory graphs are 
  Feynman rules $\xrightarrow{p} \;\to\; \frac{1}{p^2+m^2}$ (you you $i\varepsilon$)
  Let G be such a graph
  eg $G =$  in momentum space $\rightsquigarrow$ divergence we'll talk about later

- I want to be in parametric space and let's say massless
  $$\int_0^\infty e^{-ap^2}\,da = \frac{1}{p^2}$$ this associates an edge to each variable
  so from $$\int \frac{\prod d^4k_i}{\prod p_e^2} \;\to\; \int \prod_e d^4k_e \int_{a_e\geq 0} \prod_e da_e\; e^{\sum -a_e p_e^2}$$
  Gaussian so easy (or look it up) to do k_e integrals
  get $$\int_{a_e\geq 0} \prod_e da_e \frac{1}{\sqrt{\det M}^4}$$ where M is matrix of the quadratic form
  $$= \int_{a_e\geq 0} \prod_e da_e \frac{1}{(\det M)^2}$$

  but what is M?
  ijth entry is $-a_e$ if k_i and k_j appear in momentum of e
  $\rightsquigarrow$ dual graph Laplacian matrix (with variables)

  by the matrix-tree theorem
  $$\det M = \Psi_G = \sum_T \prod_{e\notin T} a_e$$

Mainz Institute for Theoretical Physics

- what about divergence etc?
  lets restrict to log divergent with no subdivergences
  then let $P_G = \int_{a_e \geq 0} \frac{da_e}{\Psi_G^2} \delta(a_1 + \dots + = 1)$

- Points — P_G is a sensible object — physically, alg. geom'ly, comb'ly

  — it is special but not so special
  ~ more complicated theories give numerator structure
  but Ψ_G controls singularities
  theories with overall symmetries may cause cancellations
  but for realistic theories problems are postponed
  not eliminated)
  ~ get the kinds of numbers we see in scattering amplitudes
  (already saw ζ_3 in Valentina's talk)

  ~ its mathematically clean — sometimes difficult but
  we can see what's going on

  ~ can extend to include subdivs

② What is known about P_G

MZV $\zeta(s_1, s_2, \dots, s_k) = \sum_{n_1 > n_2 > \dots > n_k > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_k^{s_k}}$

(interesting number theoretically and motivically also nice algebraic/comb properties
extension of special values of Riemann ζ
special values of multiple polylogs)

Calculations by Broadhurst Kreimer (95)
all up to 7 loops except 3 families (1 now solved))
eg
zigzags, $K_{3,4}$

Calculated by Schnetz (2010)
many more at 8 and 9 loops

Proof by Schnetz + Brown of zigzag family (2015)

Non MZV egs by Doryn (2011) + Brown-Schnetz (2012)

Tell the motivic story — thought all MZVs
(first half) — then should all come from MTMs
(will lead to (2)) but false

Mainz Institute for Theoretical Physics

③ Symmetries of Feynman periods

- planar duality
- completion (Broadhurst Kreimer)
- Schnetz twist (Schnetz)
- 3-vertex (completed) product
- ~~note subdiv ↔ 4 edge cuts in completion~~

④ Other graph invariants with these symmetries

in chronological order (for all conjecture if $P_{G_1} = P_{G_2}$ then $\mathrm{inv}(G_1) = \mathrm{inv}(G_2)$)

	• c_2 (Schnetz, Brown-Schnetz)	• Graph perm (Crump, DeVos, –)	• Hepp (Panzer)
Def	$-\ c_2^{(p)}(G) = \dfrac{[\Psi_G]_p}{p^2} \bmod p$ – why – root of motivic theory (Kontsevich conj) Belkale Brosnan	$-\ \mathrm{perm}\begin{bmatrix} E \\ E \end{bmatrix} \bmod 3$ + blocks	$-\ \mathcal{F}_G = \{p = \gamma_0 \subsetneq \gamma_1 \subsetneq \gamma_2 \dots \subsetneq G\}$ bridgeless, max length $w(\gamma) = \lvert E(\gamma)\rvert - 2\,\ell(\gamma)$ $\mathcal{H}(G) = \sum\limits_{\gamma_0,\gamma_1,\dots \in \mathcal{F}_G} \dfrac{1}{w(\gamma_1)\,w(\gamma_2)\dots} \prod \lvert E(\gamma_k)\rvert - \lvert E($
Key results	– $c_2^{(p)}(G) = 0$ wt drop – $c_2^{(p)}(G)$ const MZV – Fourier coeff seqs of modular forms	– all symmetries	
Advantages	Number theory connects clear conn to P_G	all symmetry known connects to flow theory in graph	appropriately scaled predicts value of P_G well conjecture strengthened to iff
Disadvantages	props conjectural	not clear what about P_G it predicts	very new & not well understood

Point: try to understand P_G with these graph invariants

also nice mathematical objects

If extra time perm proofs