Exact Results in *N*=2 Gauge Theories

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- Non-perturbative effects are essential in field theories to complete the perturbative expansion and lead to results valid at all couplings

- In supersymmetric theories, tremendous progress has been possible thanks to the developement of localization techniques (Nekrasov ‘02, Nekrasov-Okounkov ’03, Pestun ‘07, Nekrasov-Pestun ‘13)

- In maximally supersymmetric theories these methods allowed us to compute exactly several quantities:
    - Sphere partition function and free energy
    - Wilson loops
    - Correlation functions, amplitudes

- We will focus on SYM theories in *4d* with *N*=2 supersymmetry
  - They are less constrained than the *N*=4 theories
  - They are sufficiently constrained to be analyzed exactly

- We will be interested in studying how S-duality on the quantum effective couplings constrains the prepotential and the observables of N=2 theories

(earlier work by Minahan et al. '96, '97)

- We will make use of these constraints to obtain exact expressions valid at all couplings

This talk is mainly based on:

- M.Billò, M.F., F.Fucito, A.Lerda, J.F.Morales, *``S-duality and the prepotential in N=2*theories (I): the ADE algebras,''* JHEP **1511** (2015) 024, arXiv:1507.07709
- M.Billò, M.F., F.Fucito, A.Lerda, J.F.Morales, *``S-duality and the prepotential in N=2*theories (II): the non-simply laced algebras,''* JHEP **1511** (2015) 026, arXiv:1507.08027

and

- S.K.Ashok, M.Billò, E.Dell'Aquila, M.F., A.Lerda, M.Raman, *``Modular anomaly equations and S-duality in N=2 conformal SQCD,"* JHEP 1510 (2015) 091, arXiv:1507.07476
- S.K.Ashok, M.Billò, E.Dell'Aquila, M.F., A.Lerda, M.Moskovic, M.Raman, *``Chiral observables and S-duality in N=2* U(N) gauge theories'',* arXiv:1607.08327
- S.K.Ashok, M.Billò, E.Dell'Aquila, M.F., R.R.John, A.Lerda, *``Modular and duality properties of surface operators in N=2* gauge theories'',* arXiv:1702.02833

but it builds on a very vast literature…

Plan of the talk

1. $\mathcal{N}$=4 SYM
2. $\mathcal{N}$=2* SYM
3. $\mathcal{N}$=2* SYM with surface operators
4. Conclusions

$\mathcal{N}$=4 SYM

N=4 SYM

- Consider *N* =4 SYM in d=4
  - This theory is maximally supersymmetric (16 SUSY charges)
  - The field content is

A		1 vector
λ^a	$(a = 1, \cdots, 4)$	4 Weyl spinors
X^i	$(i = 1, \cdots, 6)$	6 real scalars

  - All fields are in the adjoint repr. of the gauge group G
  - The β–function vanishes to all orders in perturbation theory
  - If $\langle X^i \rangle = 0$, the theory is superconformal (*i.e.* invariant under $\mathrm{SU}(2,2|4)$) also at the quantum level

N=4 SYM

- The dynamics of *N* =4 SYM is described by the (complexified) coupling constant τ

$$\tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2} \qquad \in \mathbb{H}_+$$

that contains the gauge coupling and the θ angle of the gauge theory :

$$\mathcal{L} \propto \mathrm{Im}\,\tau \int_{\mathbb{R}_4} \mathrm{Tr}\,(F \wedge *F) + \mathrm{Re}\,\tau \int_{\mathbb{R}_4} \mathrm{Tr}\,(F \wedge F)$$

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- *N* =4 SYM possess an exact duality invariance which contains the electro-magnetic duality S

(Montonen-Olive ‘77, Vafa-Witten ‘94, Sen ’94…)

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- If the algebra $\mathfrak{g}$ of the gauge group G is simply laced (ADE) the modular group is $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, whose generators are:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ ; \quad S^2 = -1 \ , \quad (ST)^3 = -1$$

- $S(\tau) = -1/\tau$ and $T(\tau) = \tau + 1$ $(\theta \to \theta + 2\pi)$

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- If the algebra $\mathfrak{g}$ of the gauge group G is simply laced (ADE) the modular group is $\Gamma = \mathrm{SL}(2,\mathbb{Z})$

  - It is a weak/strong duality, acting on the coupling by

$$S(\tau) = -1/\tau$$

  - S maps the theory to itself but with electric and magnetic states exchanged

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  (Montonen-Olive ‘77, Vafa-Witten ‘94, Sen ’94…)

- If the algebra $\mathfrak{g}$ of the gauge group G is non-simply laced (BCFG) duality relation still exist, but they are more involved… (see Billò et al. ‘15 and Ashok et al.‘16)

- For simplicity we will only describe the case of simply laced algebra $\mathfrak{g}$, but all the arguments can be generalized to include also the non-simply laced cases

N=4 SYM as a *N*=2 theory

Let us decompose the *N*=4 multiplet into

- one *N*=2 vector multiplet

2 Weyl fermions

1 vector $\rightarrow$ $A \, , \quad \psi^I \, , \quad \phi$ $\leftarrow$ 1 complex scalar

- one *N*=2 hypermultiplet

2 Weyl fermions

$$q \, , \quad \tilde{q} \, , \quad \chi \, , \quad \tilde{\chi}$$

2 complex scalars

N=4 SYM as a *N*=2 theory

Let us decompose the *N*=4 multiplet into

- one *N*=2 vector multiplet
- one *N*=2 hypermultiplet

By introducing the v.e.v.

$$\langle \phi \rangle = a = \mathrm{diag}(a_1, ..., a_n)$$

- we break the gauge group $G \to U(1)^n$
- we spontaneously break conformal invariance
- we can describe the dynamics in terms of a holomorphic prepotential $\mathcal{F}(a)$, as in *N*=2 theories

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- we break the gauge group $G \to U(1)^n$
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- we can describe the dynamics in terms of a holomorphic prepotential $\mathcal{F}(a)$, as in *N*=2 theories
- the magnetic variable and the effective gauge coupling are:

$$a_u^{\mathrm{D}} \equiv \frac{1}{2\pi \mathrm{i}} \frac{\partial \mathcal{F}}{\partial a_u} \qquad \tau_{uv} \equiv \frac{1}{2\pi \mathrm{i}} \frac{\partial^2 \mathcal{F}}{\partial a_u \partial a_v}$$

N=4 SYM as a *N*=2 theory

- The prepotential of the *N*=4 theory is simply

$$\mathcal{F} = \mathrm{i}\,\pi\tau\,a^2$$

- The dual variables are

$$a^{\mathrm{D}} = \frac{1}{2\pi\mathrm{i}}\frac{\partial\mathcal{F}}{\partial a} = \tau a$$

- S-duality relates the electric variable a to the magnetic variable a_D :

$$S\begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} -a \\ a_D \end{pmatrix}$$

$\mathcal{N}$=4 SYM as a $\mathcal{N}$=2 theory

- Let's find the S-dual prepotential:

$$S(\mathcal{F}) = i\,\pi\Big(-\frac{1}{\tau}\Big)\,(a_D)^2 = -i\,\pi\,\frac{1}{\tau}\,a_D^2$$

- S-duality exchanges the description based on a with its Legendre-transform, based on a_D :

$$\mathcal{L}(\mathcal{F}) = \mathcal{F} - a\,\frac{\partial\mathcal{F}}{\partial a} = i\,\pi\,\tau\,a^2 - 2\pi\,i\,a\,a_D$$

$$= -i\,\pi\,\frac{1}{\tau}\,a_D^2$$

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- Thus

$$S(\mathcal{F}) = \mathcal{L}(\mathcal{F})$$

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- Thus

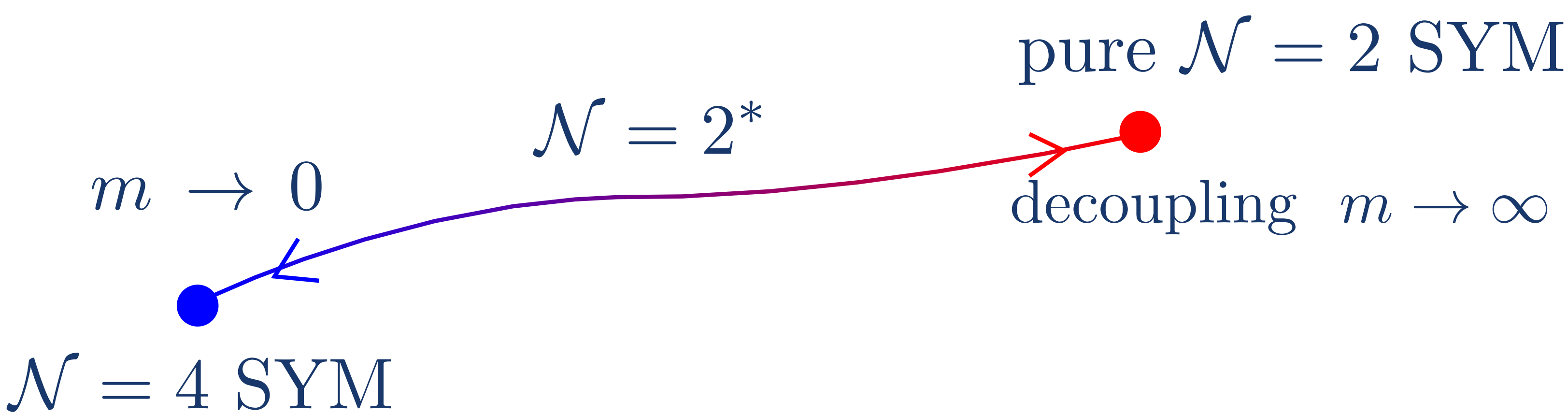
$$S(\mathcal{F}) = \mathcal{L}(\mathcal{F})$$

- This structure is present also in $\mathcal{N}$=2 theories and has important consequences on their strong coupling dynamics!

$\mathcal{N}$=2* SYM

The *N*=2* set-up

- The *N*=2* theory is a mass deformation of the *N*=4 SYM
- Field content:
  - one *N*=2 vector multiplet for the algebra $\mathfrak{g}$
  - one *N*=2 hypermultiplet in the adjoint rep. of $\mathfrak{g}$ with **mass *m***
- Half of the supercharges are broken, and we have *N*=2 SUSY
- The β-function still vanishes, but the superconformal invariance is explicitly broken by the mass ***m***



Structure of the *N*=2* prepotential

- The *N*=2* prepotential contains classical, 1-loop and non-perturbative terms

$$\mathcal{F} = \mathrm{i}\,\pi\,\tau\,a^2 + f \quad \text{with} \quad f = \mathcal{F}_{1-loop} + \mathcal{F}_{inst}$$

- The 1-loop term reads

$$\frac{1}{4}\sum_{\alpha\in\Psi_{\mathfrak{g}}}\left[-(\alpha\cdot a)^2\log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2 + (\alpha\cdot a+m)^2\log\left(\frac{\alpha\cdot a+m}{\Lambda}\right)^2\right]$$

  - $\Psi_{\mathfrak{g}}$ is the set of the roots α of the algebra
  - $\alpha\cdot a$ is the mass of the W-boson associated to the root α

Structure of the *N*=2* prepotential

- The *N*=2* prepotential contains classical, 1-loop and non-perturbative terms

$$\mathcal{F} = \mathrm{i}\,\pi\,\tau\,a^2 + f \quad \text{with} \quad f = \mathcal{F}_{1-loop} + \mathcal{F}_{inst}$$

- The non-perturbative contributions come from all instanton sectors and and can be explicitly computed using localization for all classical algebras (Nekrasov '02, Nekrasov-Okounkov '03, ..., Billò et al 15, ...)

$$\log Z_{inst} = -\frac{\mathcal{F}_{inst}}{\epsilon_1\epsilon_2} \qquad \mathcal{F}_{inst} = \sum_k q^k\,\mathcal{F}_k$$

- $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ is the instanton counting parameter:

$$q^k = \mathrm{e}^{-S_{inst}} \qquad S_{inst} = -2\pi\mathrm{i}\,\tau\,\frac{1}{8\pi^2}\int_{\mathbb{R}_4}\mathrm{Tr}\,(F\wedge F) = -2\pi\mathrm{i}\,\tau\,k$$

S-duality and the prepotential

- The dual variables are defined as

$$a_D \equiv \frac{1}{2\pi i}\frac{\partial \mathcal{F}}{\partial a} = \tau\left(a + \frac{1}{2\pi i\tau}\frac{\partial f}{\partial a}\right)$$

- Applying S-duality we get

$$S(\mathcal{F}) = i\,\pi\Big(-\frac{1}{\tau}\Big)a_D^2 + f\Big(-\frac{1}{\tau}, a_D\Big)$$

- Computing the Legendre transform we get

$$\mathcal{L}(\mathcal{F}) = \mathcal{F} - 2i\pi\, a\cdot a_D$$
$$= i\,\pi\Big(-\frac{1}{\tau}\Big)a_D^2 + f(\tau, a) + \frac{1}{4i\pi\tau}\left(\frac{\partial f}{\partial a}\right)^2$$

S-duality and the prepotential

- Requiring

$$S(\mathcal{F}) = \mathcal{L}(\mathcal{F})$$

implies

$$f\left(-\frac{1}{\tau}, a_D\right) = f(\tau, a) + \frac{1}{4i\pi\tau}\left(\frac{\partial f}{\partial a}\right)^2$$

Modular anomaly equation!

- This constraint has very deep implications!

Modular anomaly equation

- We organize the quantum prepotential f in a mass expansion

$$f(\tau, a) = \sum_{n=1} f_n(\tau, a) \quad \text{with} \quad f_n \propto m^{2n}$$

- From explicit calculations, one sees that:

  - f_1 is only 1-loop and thus τ-independent

$$f_1(a) = \frac{m^2}{4} \sum_{\alpha \in \Psi_{\mathfrak{g}}} \log \left(\frac{\alpha \cdot a}{\Lambda} \right)^2$$

  - $f_n \; (n \geq 2)$ are both 1-loop and non-perturbative and, since the prepotential has mass dimension 2, they are homogeneous functions of the a's of weight 2-2n

Modular anomaly equation

- In order to solve the modular anomaly equation

$$f\Big(-\frac{1}{\tau}, a_D\Big) = f(\tau, a) + \frac{\delta}{24}\left(\frac{\partial f}{\partial a}\right)^2 \quad , \quad \delta = \frac{6}{i\pi\tau}$$

we must have

$$f_n\Big(-\frac{1}{\tau}, a_D\Big) = f_n\Big(-\frac{1}{\tau}, \tau(a+\cdots)\Big) = f_n(\tau, a) + \cdots$$

Modular anomaly equation

- In order to solve the modular anomaly equation

  we must have

$$f_n\Big(-\frac{1}{\tau}, a_D\Big) = f_n\Big(-\frac{1}{\tau}, \tau(a+\cdots)\Big) = f_n(\tau, a) + \cdots$$

- Since the $f_n^{\mathfrak{g}}$ $(n \geq 2)$ are homogeneous functions of the a's of weight 2-2n:

$$f_n\Big(-\frac{1}{\tau}, a_D\Big) = f_n\Big(-\frac{1}{\tau}, \tau(a+\cdots)\Big) = \tau^{2-2n}\, f_n\Big(-\frac{1}{\tau}, a+\cdots\Big)$$

- to compensate for the factors of τ the f_n $(n \geq 2)$ must be (quasi) modular of weight 2n-2:

$$f_n\Big(-\frac{1}{\tau}, a+\cdots\Big) = \tau^{2n-2}\, f_n\Big(\tau, a+\cdots\Big)$$

Modular anomaly equation

- The f_n 's must be (quasi) modular of weight 2n-2:

$$f_n\Big(-\frac{1}{\tau}, a+\cdots\Big) = \tau^{2n-2}\, f_n\Big(\tau, a+\cdots\Big)$$

- thus we must require that they depends on τ through "modular" functions, *i.e*

$$f_n(\tau,a) = f_n\Big(E_2(\tau), E_4(\tau), E_6(\tau), a\Big)$$

where $E_2(\tau), E_4(\tau), E_6(\tau)$ are the Eisenstein series.

Eisenstein series

- The Eisenstein series are “modular” forms with a well-known Fourier expansion in $q = e^{2i\pi\tau}$:

$$E_2(\tau) = 1 - 24q - 72q^2 - 96q^3 - 168q^4 + \cdots$$

$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \cdots$$

$$E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \cdots$$

- E_4 and E_6 are truly modular forms of weight 4 and 6

$$E_4\left(-\frac{1}{\tau}\right) = \tau^4\, E_4(\tau) \quad , \quad E_6\left(-\frac{1}{\tau}\right) = \tau^6\, E_6(\tau)$$

- E_2 is quasi-modular of weight 2

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2\left[E_2(\tau) + \delta\right] \quad , \quad \delta = \frac{6}{i\pi\tau}$$

Recursion relation

- S-duality

$$f\Big(-\frac{1}{\tau},a_D\Big)=f\left(E_2(-\frac{1}{\tau}),E_4(-\frac{1}{\tau}),E_6(-\frac{1}{\tau}),\tau\big(a+\frac{\delta}{12}\frac{\partial f}{\partial a}\big)\right)$$

$$=f\left(E_2+\delta,E_4,E_6,\big(a+\frac{\delta}{12}\frac{\partial f}{\partial a}\big)\right)$$

$$=f\left(\tau,a\right)+\delta\left[\frac{\partial f}{\partial E_2}+\frac{1}{12}\left(\frac{\partial f}{\partial a}\right)^2\right]+\mathcal{O}(\delta^2)$$

- Modular anomaly equation

$$f\Big(-\frac{1}{\tau},a_D\Big)=f(\tau,a)+\delta\left[\frac{1}{24}\left(\frac{\partial f}{\partial a}\right)^2\right]$$

Recursion relation

- S-duality

$$f\left(-\frac{1}{\tau}, a_D\right) = f\left(E_2(-\frac{1}{\tau}), E_4(-\frac{1}{\tau}), E_6(-\frac{1}{\tau}), \tau(a + \frac{\delta}{12}\frac{\partial f}{\partial a})\right)$$

$$= f\left(E_2 + \delta, E_4, E_6, (a + \frac{\delta}{12}\frac{\partial f}{\partial a})\right)$$

$$= f(\tau, a) + \delta\left[\frac{\partial f}{\partial E_2} + \frac{1}{12}\left(\frac{\partial f}{\partial a}\right)^2\right] + \mathcal{O}(\delta^2)$$

- Modular anomaly equation

$$f\left(-\frac{1}{\tau}, a_D\right) = f(\tau, a) + \delta\left[\frac{1}{24}\left(\frac{\partial f}{\partial a}\right)^2\right]$$

Recursion relation

- We thus obtain

$$\frac{\partial f}{\partial E_2} + \frac{1}{24}\left(\frac{\partial f}{\partial a}\right)^2 = 0$$

which implies the following recursion relation

(Minahan et al ’97)

$$\frac{\partial f_n}{\partial E_2} = -\frac{1}{24}\sum_{\ell=1}^{n-1} \frac{\partial f_\ell}{\partial a}\frac{\partial f_{n-\ell}}{\partial a}$$

  - This allows us to determine f_n from the lower coefficients up to E_2-independent terms. These are fixed by comparison with the perturbative expressions (or the first instanton corrections).

- Once this is done, <u>the result is valid to all instanton orders.</u>

Exploiting the recursion

- Using this recursive procedure we find

$$f_2 = -\frac{m^4}{24} E_2\, C_2^{\mathfrak{g}}$$

$$f_3 = -\frac{m^6}{720}\,(5E_2^2 + E_4)C_2^{\mathfrak{g}} - \frac{m^6}{576}\,(E_2^2 - E_4)C_{2;1,1}^{\mathfrak{g}}$$

where $C_2^{\mathfrak{g}}$ and $C_{2;1,1}^{\mathfrak{g}}$ are root lattice sums of $\mathfrak{g}$ defined as

$$C_2^{\mathfrak{g}} = \sum_{\alpha\in\Psi_{\mathfrak{g}}} \frac{1}{(\alpha\cdot a)^2}$$

$$C_{2;1,1}^{\mathfrak{g}} = \sum_{\alpha\in\Psi_{\mathfrak{g}}} \sum_{\beta_1\neq\beta_2\in\Psi_{\mathfrak{g}}(\alpha)} \frac{1}{(\alpha\cdot a)^2(\beta_1\cdot a)(\beta_2\cdot a)}$$

with $\Psi_{\mathfrak{g}}(\alpha) = \{\beta\in\Psi_{\mathfrak{g}}\ :\ \alpha\cdot\beta = 1\}$

Checks on the results

- This procedure uniquely determine the exact result to all instantons !
- It can be generalized to all algebras, even the non-simply laced ones (Billò et al '15)
- For the classical algebras A, B, C and D the integration of the moduli action over the instanton moduli spaces can be performed à la Nekrasov using localization techniques
- In principle straightforward; in practice computationally rather intense. Not many explicit results for the $\mathcal{N}=2^*$ theories in the literature

Checks on the results

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- For the classical algebras A, B, C and D the integration of the moduli action over the instanton moduli spaces can be performed à la Nekrasov using localization techniques
- We worked it out:
  - for A_n and D_n with $n<6$, up to 5 instantons;
  - for C_n with $n<6$, up to 4 instantons;
  - for B_n with $n<6$, up to 2 instantons.
- The results match the q-expansion of those obtained above

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- For the classical algebras A, B, C and D the integration of the moduli action over the instanton moduli spaces can be performed à la Nekrasov using localization techniques
- For the exceptional algebras, where no ADHM construction is known, our results are predictions!

Chiral correlators

- Other observables of the theory are the chiral correlators

$$< Tr\phi^n >= \sum_{i=1}^{N} a_i^n + \cdots$$

- They can be computed using equivariant localization

(Bruzzo et al. 03, Losev et al. 03, Flume et al. 04, Billò et al. ’12)

- The results can be expressed in terms of modular functions and lattice sums (Ashok et al. ‘16)

Chiral correlators

- Using the explicit results for $< Tr\, \phi^n >$, it is possible to change basis and find the quantum symmetric polynomials in the a's

$$A_n(\tau, a) = \sum_{i_1 < i_2 < \cdots < i_n} a_{i_1} a_{i_2} \cdots a_{i_n} + \cdots$$

that transform as modular form of weight n $S(A_n) = \tau^n \, A_n$

$$A_1 = \sum_{i_i} a_{i_1}$$

$$A_2 = \sum_{i_i < i_2} a_{i_1} a_{i_2} + \binom{N}{2} \frac{m^2}{12} E_2 + \frac{m^4}{288} (E_2^2 - E_4) C_2 + \cdots$$

Chiral correlators

- These expressions

$$A_1 = \sum_{i_i} a_{i_1}$$

$$A_2 = \sum_{i_i < i_2} a_{i_1} a_{i_2} + \binom{N}{2} \frac{m^2}{12} E_2 + \frac{m^4}{288} (E_2^2 - E_4) C_2 + \cdots$$

coincide with the solution of the modular anomaly equation satisfied by the A_n's

$$\frac{\partial A_n}{\partial E_2} + \frac{1}{24} \frac{\partial A_n}{\partial a} \frac{\partial f}{\partial a} = 0$$

that can be obtained directly from its S-duality properties!

Surface operators in $\mathcal{N}$=2* SYM

Surface operators

- We study SU(N) *N*=2* SYM in presence of surface operators, *i. e.* non local defects D supported on a 2d plane in $\mathbb{R}_4$:

$$\mathbb{R}_4 : (w_1, w_2) \qquad D : (w_1, 0)$$

- When describing the 4d SYM theory as the world sheet theory of M_5 branes wrapped on a Riemann surface, the 2d defects correspond to M_5' or to M_2 branes:

M_5	x	x	x	x	-	-	x	-	-	-	x	
M_5'	x	x	-	-	x	x	x	-	-	-	x	cod 2
M_2	x	x	-	-	-	-	-	x	-	-	-	cod 4

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- In the 4d-2d correspondence:

cod 2 defects ⟹ conformal blocks of affine sl(N) theories

(Alday, Tachikawa '10, Kozcaz, Pasquetti, Passerini, Wyllard '11, …)

cod 4 defects ⟹ conformal blocks of Toda theory

(Alday, Gaiotto, Gukov, Tachikawa, Verlinde '10, …)

Description of surface operators

- The presence of the defect induce a singular behavior in the 1-form gauge connection:

$$A = -\,\mathrm{diag}\,(\underbrace{\gamma_1, \cdots, \gamma_1}_{n_1}, \underbrace{\gamma_2, \cdots, \gamma_2}_{n_2}, \cdots, \cdots, \underbrace{\gamma_M, \cdots, \gamma_M}_{n_M})\; d\theta$$

(Gukov, Witten '06, '08)

$$\sum_{I=1}^{M} n_I = N \qquad w_2 = \rho \mathrm{e}^{\mathrm{i}\theta}$$

- The vector $\vec{n} = (n_1, n_2, \cdots, n_M)$ characterizes the defect and describes the breaking of the gauge group:

$$\mathrm{SU}(N) \to \mathrm{S}\big[\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \cdots \times \mathrm{U}(n_M)\big]$$

Description of surface operators

- In presence of the defect quantized magnetic fluxes are allowed for each group factor:

$$\frac{1}{2\pi}\int_D \mathrm{Tr}\, F_{U(n_I)} = m_I$$

- The instanton action becomes:

$$S_{inst}[\vec{n}] = -2\pi \mathrm{i}\tau\, k - 2\pi \mathrm{i}\sum_{I=1}^{M}\big(\eta_I + \tau\,\gamma_I\big)m_I = -2\pi \mathrm{i}\tau\, k - 2\pi \mathrm{i}\,\vec{t}\cdot\vec{m}$$

- The electric and magnetic parameters γ_I and η_I are combined in the complex M-dimensional vector

$$\vec{t} = \{t_I\} = \{\eta_I + \tau\gamma_I\}$$

that describes the charges of the defect.

Twisted chiral superpotential

- SU(N) *N*=2* SYM in presence of a cod 2 defect has been shown to be equivalent to the world volume theory of fractional D3-branes in the orbifold (Kanno, Tachikawa '11)

$$\mathbb{C} \times \mathbb{C} \times \mathbb{C}^2/Z_N \times \mathbb{C}$$

- In this case the computation of the non perturbative contribution can be performed via localization.
- In the N-S limit $\epsilon_2 \to 0$ we have

$$\log Z_{inst} = -\frac{F}{\epsilon_1 \epsilon_2} + \frac{W_{inst}}{\epsilon_1}$$

- $W_{inst} = W_{inst}(q, m, \epsilon_1, t_I)$ is the twisted chiral superpotential governing the 2d dynamic on the defect.

S-duality properties of W_{inst}

- In the N=4 case

$$W \equiv W_{class} = 2\pi\mathrm{i}\vec{t}\cdot\vec{a} = 2\pi\mathrm{i}\sum_{I=2}^{N} z_I a_I$$

where $z_I = t_{I+1} - t_1$.

Therefore

$$W^{(I)} \equiv \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial z_I}W = a_I$$

and we have (since $S(a_I) = a_I^D = \tau a_I$)

$$S(W^{(I)}) = \tau W^{(I)}$$

$W^{(I)}$ transforms as a weight 1 modular form!

S-duality properties of W_{inst}

- In the N=2 case

$$W = W_{class} + W_{1-loop} + W_{inst}$$

and $W^{(I)}$ can be written as a mass (and ϵ_1) expansion

$$W^{(I)} = a_I + (W_{1-loop} + W_{inst})^{(I)} = a_I + \sum_{\ell=1}^{\infty} w_\ell^{(I)}$$

with $w_\ell^{(I)}$ homogenous functions of weight $1-\ell$ in the a_I

The requirement that also in this case

$$S(W^{(I)}) = \tau W^{(I)}$$

implies that the $w_\ell^{(I)}$ are combinations of elliptic functions and (quasi) modular forms with weight ℓ .

Modular anomaly eq. for W_{inst}

- The requirement

$$S(W^{(I)}) = \tau W^{(I)}$$

implies that $W^{(I)}$ satisfies a modular anomaly equation

$$\frac{\partial W^{(I)}}{\partial E_2} + \frac{1}{12}\frac{\partial W^{(I)}}{\partial \vec{a}}\frac{\partial f}{\partial \vec{a}} = 0$$

and that $w_\ell^{(I)}$ satisfy a recursion relation

$$\frac{\partial w_\ell^{(I)}}{\partial E_2} + \frac{1}{12}\sum_{n=0}^{\ell-1}\frac{\partial w_n^{(I)}}{\partial \vec{a}}\frac{\partial f_{\ell-n}}{\partial \vec{a}} = 0$$

This allows to completely determine them, given the initial conditions.

Explicit results for W_{inst}

- For instance in the simple case of a (1,1) defect in SU(2) (where we have a single z):

$$w_1' = \left(m - \frac{\epsilon_1}{2}\right)\left(h_1 + \frac{1}{2}\right)$$

$$w_2' = \frac{1}{24a}\left(m^2 - \frac{\epsilon_1^2}{4}\right)\left(E_2 + 12\,\widetilde{\wp}\right)$$

$$w_3' = \frac{\epsilon_1}{4a^2}\left(m^2 - \frac{\epsilon_1^2}{4}\right)\widetilde{\wp}'$$

$$w_4' = \frac{1}{1152a^3}\left(m^2 - \frac{\epsilon_1^2}{4}\right)\left[\left(m^2 - \frac{\epsilon_1^2}{4}\right)\left(2E_2^2 - E_4 + 24E_2\,\widetilde{\wp} + 144\widetilde{\wp}^2\right) + 6\,\epsilon_1^2\left(E_4 - 144\widetilde{\wp}^2\right)\right]$$

$$w_5' = \frac{\epsilon_1}{48a^4}\left(m^2 - \frac{\epsilon_1^2}{4}\right)\left[\left(m^2 - \frac{\epsilon_1^2}{4}\right)\left(E_2 + 12\widetilde{\wp}\right)\widetilde{\wp}' - 36\,\epsilon_1^2\,\widetilde{\wp}\,\widetilde{\wp}'\right]$$

where $\quad \widetilde{\wp} = \dfrac{\wp}{4\pi^2} \qquad h_1(z|\tau) = \dfrac{1}{2\pi\mathrm{i}}\dfrac{\partial}{\partial z}\log\theta_1(z|\tau)$

Explicit results for W_{inst}

- For instance in the simple case of a (1,1) defect in SU(2) (where we have a single z):

$$w_2' = \frac{1}{2a}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)h_1' \ ,$$

$$w_3' = \frac{\epsilon_1}{4a^2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)h_1'' \ ,$$

$$w_4' = \frac{1}{48a^3}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)(E_2\, h_1 - h_1'') + 6\,\epsilon_1^2\, h_1''\Big]' \ ,$$

$$w_5' = \frac{\epsilon_1}{8a^4}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)(h_1')^2 + \frac{\epsilon_1^2}{2}(E_2 - 6h_1')h_1'\Big]'$$

where $\widetilde{\wp} = \dfrac{\wp}{4\pi^2}$ $\qquad h_1(z|\tau) = \dfrac{1}{2\pi \mathrm{i}}\dfrac{\partial}{\partial z}\log\theta_1(z|\tau)$

Explicit results for W_{inst}

- For instance in the case of a (p,N-p) defect in SU(N) (one z):

$$w_2' = \Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{\vec\alpha\in\Psi} \frac{h_1'(-\vec\alpha\cdot\vec t)}{\vec\alpha\cdot\vec a} = \Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{\vec\alpha\in\Psi} \frac{h_1'(\vec\alpha\cdot\vec t)}{\vec\alpha\cdot\vec a}$$

$$w_3' = -\epsilon_1 \Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{\vec\alpha\in\Psi} \frac{h_1''(\vec\alpha\cdot\vec t)}{(\vec\alpha\cdot\vec a)^2} - \frac{1}{2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big) \sum_{\vec\alpha\in\Psi} \sum_{\vec\beta\in\Psi(\vec\alpha)} \frac{h_1''(\vec\alpha\cdot\vec t)}{(\vec\alpha\cdot\vec a)\,(\vec\beta\cdot\vec a)}$$

$$\begin{aligned}
w_4' = \frac{1}{6}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{\vec\alpha\in\Psi} \frac{E_2\, h_1'(\vec\alpha\cdot\vec t) - h_1'''(\vec\alpha\cdot\vec t)}{(\vec\alpha\cdot\vec a)^3} + 6\,\epsilon_1^2 \sum_{\vec\alpha\in\Psi} \frac{h_1'''(\vec\alpha\cdot\vec t)}{(\vec\alpha\cdot\vec a)^3}\Big] \\
+ \epsilon_1 \Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big) \sum_{\vec\alpha\in\Psi} \sum_{\vec\beta\in\Psi(\vec\alpha)} \frac{h_1'''(\vec\alpha\cdot\vec t)}{(\vec\alpha\cdot\vec a)^2\,(\vec\beta\cdot\vec a)} \\
+ \frac{1}{4}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big)^2 \Big[\sum_{\vec\alpha\in\Psi} \sum_{\vec\beta\neq\vec\gamma\in\Psi(\vec\alpha)} \frac{h_1'''(\vec\alpha\cdot\vec t)}{(\vec\alpha\cdot\vec a)\,(\vec\beta\cdot\vec a)\,(\vec\gamma\cdot\vec a)} \\
- \frac{1}{3} \sum_{\vec\alpha\in\Psi} \sum_{\vec\beta\in\Psi(\vec\alpha)} \sum_{\vec\gamma\in\Psi(\vec\alpha,\vec\beta)} \frac{h_1'''(\vec\alpha\cdot\vec t)}{(\vec\alpha\cdot\vec a)\,(\vec\beta\cdot\vec a)\,(\vec\gamma\cdot\vec a)}\Big] .
\end{aligned}$$

Explicit results for W_{inst}

In summary:

- The recursion relation allows to exactly determine all terms in the mass expansion of $W^{(I)}$ knowing only the perturbative or the first non-perturbative contributions
- From $W^{(I)}$ we can reconstruct W, that turns out not to have definite transformation properties
- Our results agree with the explicit (or implicit!) results that are present in the literature, surprisingly also with the results relative to cod 4 surface operators!

(Kashani-Poor, Troost ’12, Gaiotto, Gukov, Seiberg ‘13)

- This seems to support the evidence of a duality between the two kinds of defects

(Frenkel, Gukov, Teschner ’15, Wyllard ‘13)

Conclusions

Conclusions

- The requirement that the duality group acts simply as in the *N*=4 theories also in the mass-deformed cases leads to a modular anomaly equations

- This allows one to efficiently reconstruct the mass-expansion of the prepotential, the chiral correlators and the twisted chiral superpotential, resumming all instanton corrections into (quasi-)modular forms of the duality group

- The existence of such modular anomaly equations seems to be a rather general feature for all the observables that have well defined transformation properties under S-duality!

Conclusions

- The requirement that the duality group acts simply as in the N=4 theories also in the mass-deformed cases leads to a modular anomaly equation
- This allows one to efficiently reconstruct the mass-expansion of the prepotential, the chiral correlators and the twisted chiral superpotential, resumming all instanton corrections into (quasi-)modular forms of the duality group
- A similar pattern (although a bit more intricate) arises in N=2 conformal SQCD theories, where it has been possible to describe the structure of the low energy effective theory at the special vacuum (Ashok et al. '15 and '16)

Conclusions

- This approach can be profitably used in other contexts to study the consequences of S-duality on:

  - theories formulated in curved spaces (*e.g.* S^4)
  - correlation functions of chiral and anti-chiral operators
  - other observables (e.g. Wilson loops, cusp anomaly, …)
  - more general extended observables (intersecting surface operators, ...)
  - …

  with the goal of studying the strong-coupling regime

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Thank you for your attention