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(particles at or near c) SR ↘ QFT ↙ QM (small length scales)

Central principle of the theory of relativity is Nahwirkungsprinzip = no measurable effect propagates faster than the speed of light.

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Rigid bodies complicated to describe

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Pointlike localized physical observables

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Principle of QM is the uncertainty relation: Any localization in position space is incompatible with localization in mom.-space. So observables that are pointlike localized in space-time must show divergences in their high-energy behaviour.

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math. consistent <u>description of QFT</u> is enormously difficult.

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main ingredient = observables

how to model them?

↙ QT = operators on a Hilbert space

↘ R = associate them with space-time points.

So one considers maps

$$x \longmapsto \Phi(x)$$

x a point in Minkowski Space

$\Phi(x)$ an operator depending on x.

These maps are the "quantum fields"!

Require : $[\phi(x), \phi(y)] = 0$ if x and y are spacelike separated.

Problem : $\hookrightarrow$ ϕ not bounded op. , not unbounded , unbounded quadr. forms

$\rightsquigarrow$ how to form well-defined products , commut. , correl. fcts.

Wightman framework / Haag – Kastler framework . (to circumvent the above problem.)

③ <u>Wightman framework.</u>

$\phi(x)$ are operator-valued distributions

$$\phi(f) = \int d^4x \; f(x) \, \phi(x)$$

lin. map $f \longmapsto \phi(f)$

$\mathcal{S}(\mathbb{R}^4) \longrightarrow$ space of (unbounded) lin. operators on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$.

Restrict to one scalar Bose field.

Axioms from physics

If $f, g \in \mathcal{S}(\mathbb{R}^4)$ s.t. supp f & supp g are s l s , then

$$[\phi(f), \phi(g)] = 0 \quad \text{on } \mathcal{D}.$$

Symmetry group of Mink. space $\mathcal{P}$

$$U(x,\Lambda)\,\phi_i(f)\,U(x,\Lambda)^* = \sum_{j=1}^{k} \cancel{S_{ij}(\Lambda^{-1})}\, \phi_j(f_{x,\Lambda})$$

$$f_{\Lambda,x}(y) = f(\Lambda^{-1}(y-x))$$

finite dim repres. of ~~Lorentz~~ Lorentz group
vector fields
fields with spin

$$\left(\phi_i = \sum_{j=1}^{k} S_{ij} \, \phi_j \right)$$

if S is an irred. repres. then $\{\phi_1 \dots \phi_k\}$

spin.

- Spectrum condition: the joint spectrum of the generators of translations P_μ is contained in the closed forward light cone. $\Rightarrow$ $H \geq 0$ in every Lorentz frame.

- There exists a unit vector $\Omega \in \mathcal{D}$, unique up to a phase, which is invariant under all $U(x,\Lambda)$. (vacuum vector)

$\mathcal{H}$

<u>Example</u>: A real scalar free field of mass $m \geq 0$.

mass hyperboloid $p^2 = m^2$

$$\mathcal{H} = \text{Symm. Fock space over } \mathcal{H}_1 = L^2\left(\mathbb{R}^3, \frac{d^3p}{2\sqrt{\vec{p}^2+m^2}}\right)$$

$$\phi(f) := a^+(\tilde{f}) + a(\tilde{f})$$

$\int dx \, f(x) \int dp \, e^{ipx} a^+(p) + \dots$

$\int dp \, \tilde{f}(p) a^+(p) + \dots$

$\mathcal{D}$ = space of vectors of finite particle number that decay rapidly in mom. space (faster than any power).

$$[\phi(f), \phi(g)] = i\,\sigma(f,g)\,\mathbb{1} \quad \longrightarrow 0 \text{ if } \operatorname{supp} f \times \operatorname{supp} g$$

$$\sigma(f,g) = \int d^4x \, f(x) \, (Eg)(x)$$

"E is the propagator of KG op. $(\Box + m^2)$ $\to$ it assigns to g a sol. of KG eq. that has supp. in causal shadow of supp. g.

ret. - adv.

$$(U_{\mathcal{H}_1}(x, \mathbb{1})\tilde{f})(\vec{p}) = e^{i(\sqrt{\vec{p}^2+m^2}\,x^0 - \vec{p}\cdot\vec{x})} \tilde{f}(\vec{p})$$

$\mathcal{H}_1$ 2nd quantiz.

~~S trivial repres.~~ boost? complicated!

Results of Wightman QFT: conclusions indep. of specific models

* Haag - Ruelle Scattering theory: construction of well-defined asymptotic Scattering states, scattering matrix (relate asympt. particle config to localized fields)

* PCT theo: $\exists$ antiunit. op. J s.t.

"charge conjugation"

$$J \phi(f) J = \phi^*(f_J), \quad f_J(x) = f(-x)$$

$$J U(x,\Lambda) J = U(-x, \Lambda)$$

PCT symmetry

* spin - statistics theo: Fermi fields

$$[\ ,\ ]_+$$

Lorentz gr. $\rightarrow$ $SL(2, \mathbb{C})$ $\rightarrow$ allowing
its covering. also repres.s with half-integer Spin for $S(\Lambda)$.

One can then prove that Fermi fields can transf. only under half-integer spin repr.s and Bose fields only under integer spin.

5) Haag-Kastler framework

* minimize the necessary structures on the side of physics (spin of quantum fields, by hand Fermi fields, ...)

* abstraction and simplification on mathematical side.
(physical observables are generated by products and adjoints of the

set of fields ϕ ("basis"). So the $*$-algebra of observables is a more fundamental object.) vector space of fields or rather

$*$ Fields are unbounded operator and on a technical mathem. side are difficult to analyse. Prefer to work with bounded operators, which are more accessible to the standard tools of functional analysis.

map $\mathcal{O} \longmapsto \mathcal{U}(\mathcal{O})$ assigns to each (bounded, open) region $\mathcal{O}$ in Mink. a C^*-alg. $\mathcal{U}(\mathcal{O})$. "net of algebras"

$\underset{\mathbb{R}^4}{\cap}$

of lin bounded ops

$\mathcal{U}(\mathcal{O})$ generated by bounded fcts of $\phi(f)$'s with $\operatorname{supp} f \subset \mathcal{O}$.

their selfadj. elements are interpreted as the set of physical observables that can be measured in $\mathcal{O}$.

Axioms from physics

$\bigcup_{\mathcal{O}} \mathcal{U}(\mathcal{O})$

$*$ $\mathcal{U}(\mathcal{O}_1) \subset \mathcal{U}(\mathcal{O}_2)$ if $\mathcal{O}_1 \subset \mathcal{O}_2$

$*$ $[A_1, A_2] = 0$ if $\mathcal{O}_1 \times \mathcal{O}_2$, $A_i \in \mathcal{U}(\mathcal{O}_i)$

$*$ $\exists$ repres. $(x, \Lambda) \mapsto \alpha_{x,\Lambda}$ of $\mathcal{P}$ as autom. of $\mathcal{U}$ s.t.

$\alpha_{x,\Lambda}\, \mathcal{U}(\mathcal{O}) = \mathcal{U}(\Lambda\mathcal{O} + x) \quad \forall \mathcal{O} \quad \forall (x,\Lambda) \in \mathcal{P}$. (no reference to basis, spin)

$\mathcal{H}$

vacuum sector

$*$ $\exists$ strongly cont. unit. repres. $(x,\Lambda) \mapsto \mathbf{U}(x,\Lambda)$ of $\mathcal{P}$ on $\mathcal{H}$ s.t. $\alpha_{x,\Lambda} = \operatorname{ad} \mathbf{U}(x,\Lambda)$.

$*$ Positivity of en.
$*$ uniqueness of vacuum
$\Big\}$ as before.

Example: real scalar free field. (relation with Wightman framework)

$\phi(f)$ have unique selfadj extensions

$\rightsquigarrow$ define unit. ops. $W(f) := e^{i\phi(f)}$ Weyl ops.

$\rightsquigarrow$ consider the net generated by them:

$$\underbrace{\mathcal{U}(\mathcal{O})}_{\text{Weyl algebras}} := \{W(f) \mid \operatorname{supp} f \subset \mathcal{O}\}''$$

vN double commutant

$$W(f)\,W(g) = e^{i\sigma(f,g)}\,W(f+g)$$

⑥ Results of AQFT:

$\mathcal{H}$ Ω already there in Haag-Kastler
We have already the output of GNS
$\rightsquigarrow$ no need of GNS for vacuum state

* $(\mathcal{U}, \omega$ state$)$ start

GNS $\exists$ Ω, π repr. of $\mathcal{U}$ on $\mathcal{H}$ s.t. $\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$

define

ref. state ρ needs not to be the vacuum.
charge state ρ

e.g. $\rightsquigarrow$ $\pi_\rho \not\cong \pi_\omega$

$\rightsquigarrow$ not vector states on "same" Hilbert space

$\rightsquigarrow$ cannot be coherently superposed

$\rightsquigarrow$ superselection rules.

* In an extension of the loc. alg.s $\mathcal{U}(\mathcal{O})$, one finds algebras of "charge-carrying" operators $\mathcal{F}(\mathcal{O}) \subset \mathcal{U}(\mathcal{O})$ that intertwine the representation space $\mathcal{H}_\rho$ and $\mathcal{H}_\omega$.

$\rightarrow$ not observables, not fulfill causal comm. rel.

$\rightarrow$ fulfill either ~~comm.~~ / anticomm. rel.
graded comm. rel.

$\rightsquigarrow$ Bose-Fermi alternative 3+1

fields are either Bose or Fermi

* global gauge group ; gauge fields can be constructed from a given net of algebras.

* thermodynamic states , both equil. and nonequil.

* Haag - Ruelle scattering theory.

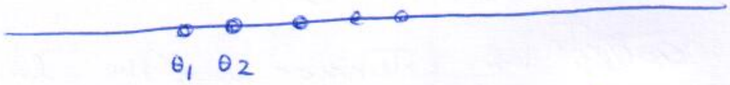
* Work directly with ; C^* and W^* algebras and on the level of bounded involution?
operators : Banach spaces
locally convex spaces
functional analysis
Tomita - Takesaki theory

- (7) <u>Quantum integrable models</u> (no need of a Lagr.) (Advantage of HK framework) → In presence of interact. pointlike local observ. have a complicate structure.

• Bosons (no spin , $m>0$) in 1+1 dimens. Mink.

• Two - momentum and rapidity

$$p = p(\theta) = m\,(\mathrm{ch}\,\theta, \mathrm{sh}\,\theta)$$



• Two particle scattering allows exchange of a phase factor.
⇝ 2-particle scattering matrix $S(\theta_1 - \theta_2)$

• multi-particle scattering matrix - product of 2-particle scattering matrices "factorizing S matrix". (integrable (def)

• 2-particle scatt. fct. S is
⇝ an analy. fct. in strip $0 < \mathrm{Im}\,\theta < \pi$
⇝ with certain symm. prop.s (cross. sym, Bootstrap, Yang Baxter
⇝ $S=1$ free field , $S=-1$ Ising model ,

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sinh-Gordon $\quad S(\zeta) = \dfrac{\operatorname{sh}\zeta - i\sin\frac{B\pi}{2}}{\operatorname{sh}\zeta + i\sin\frac{B\pi}{2}} \qquad 0<B<2$

$O(N)$ nonlin σ models (matrix-valued fct.)

<u>Task</u> : Given ~~the~~ a fct. S, construct a corresponding QFT.

The theory is constructed as a deformation of a free field :

- ZF algebra

$z(\theta)$, $z^+(\theta)$

$$z(\theta_1)\, z(\theta_2) = S(\theta_1-\theta_2)\; z(\theta_2)\, z(\theta_1)$$
$$z^+(\theta_1)\, z^+(\theta_2) = S(\theta_1-\theta_2)\; z^+(\theta_2)\, z^+(\theta_1)$$
$$z(\theta_1)\, z^+(\theta_2) = S(\theta_2-\theta_1)\; z^+(\theta_2)\, z(\theta_1) + \delta(\theta_1-\theta_2)\cdot \mathbb{1}$$

These act on "S-symm." Fock space.

- Repres. of $\mathcal{P}$, including of J.

- Define $\quad \phi(f) := \int dx\, f \int d\theta \quad e^{ip(\theta)\cdot x}\, z^+(\theta) + e^{-ip(\theta)\cdot x}\, z(\theta)$

- the field is <u>not</u> local :

$[\phi(f), \phi(g)] \neq 0 \quad$ even if $\operatorname{supp} f$ spacelike sep. from $\operatorname{supp} g$

- <u>But</u> with $\phi'(f) := U(j)\, \phi(f_j)\, U(j)^* :$

$[\phi(f), \phi'(g)] = 0 \quad$ if $\operatorname{supp} f$ spacelike sep. to the left of $\operatorname{supp} g$

$\rightsquigarrow$ this assumes that S is analytic in the "physical strip" $0 < \operatorname{Im}\zeta < \pi$.

$\rightsquigarrow$ Interpretation : $\phi(f)$ is localized in the wedge region $\operatorname{supp} f \subset W_L + x$, and $\phi'(g)$ is localized in the wedge region $\operatorname{supp} g \subset W_R - y$.

- Further wedge-local observables by relative locality / associated vN algebras:

$$\mathcal{A}(W_L + x) = \{ e^{i\phi(f)} \mid \operatorname{supp} f \subset W_L + x \}''$$

- Observables localized in bounded regions are obtained as intersections of vN algebras.

$$\mathcal{A}(\mathcal{O}) := \mathcal{A}(W_L + x) \cap \mathcal{A}(W_R - y) \quad \text{where}$$

$$\mathcal{O} = W_L + x \cap W_R - y$$

- Result: Such observables <u>exist</u> for a large class of S.
