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Calabi-Yaus and supersymmetric gauge theory

- Outline: 1) Introduction and overview
  2) Gauged linear sigma model (GLSM).
     ↳ definitions and examples
  3) D-branes in the GLSM & hemisphere partition function
  4) Outlook

- Punchline: Physical mathematics: use physics methods (SUSY gauge theory) to study the mathematical properties of CYs

1) Introduction and overview

- setting: • Calabi-Yau compactification of type II string theory with D-branes
  • study compact CY threefolds X and their moduli spaces
  • moduli space $\mathcal{M} = \boxed{\mathcal{M}_K} \times \mathcal{M}_C$

Kähler moduli $h^{1,1}(X)$ "size"

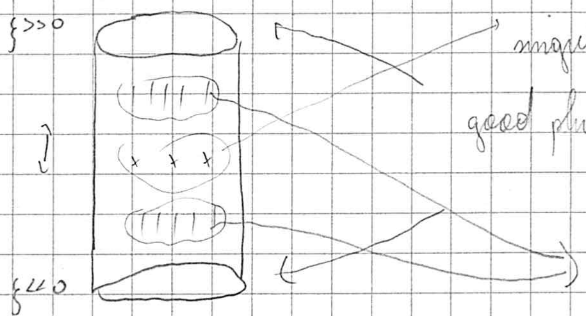
complex structure moduli $h^{2,1}(X)$ "shape"

• this talk: dim $\mathcal{M}_K = 1$ ($h^{1,1}(X) = 1$) $t \in \mathcal{M}_K$ $t = \xi - i\theta$

↳ 2π-periodic

(dim $\mathcal{M}_K > 1$ is interesting but quite challenging)

• $\mathcal{M}_K$ has a non-trivial structure



singular points ("conifold singularities")

good physical / mathematical description
e.g: • non-linear σ-model / geometry
• Landau Ginzburg model / FJRW
• hybrid

no good effective description

↳ the CY can undergo topological transitions as one moves between regions in $\mathcal{M}_K$ (generalised flops, blow-ups, blow-downs)

↳ study problems of analytic continuation in $\mathcal{M}_K$

- D-branes: × choose boundary conditions that half of SUSY is broken
  - × A-branes: e.g. Lagrangian submanifolds
  - × $\boxed{\text{B-branes}}$ coherent sheaves (geometry), matrix factorisations (LG)

↳ math: D-branes fit into the abstract mathematical framework of categories
  - × D-branes ↔ objects
  - × open strings ↔ morphisms

⇒ Q: Can we learn about CYs, their moduli spaces & D-branes via a physics approach

↳ Yes, in particular for $\mathcal{M}_K$ & B-branes

- physics realisation: gauged linear σ-model (GLSM) Witten hep-th/9301042
  - × SUSY ($\mathcal{N}=(2,2)$) gauge theory in 2 dimensions
    - ↳ "boring" physics because there are no propagating degrees of freedom (no photon), although can study confinement
  - × solutions of equations of motion constrain the configurations of scalar fields $\phi \in V$ (V... complex vector space) → "vacua"
  - × for suitably chosen GLSM data (field content, gauge charges) the vacua describe a CY space: scalars ↔ coordinates
  - × FI-θ parameters (coupling constants) can be identified with the Kähler parameters $t \in \mathcal{M}_K$ of X
    - ⇒ by studying the GLSM at different couplings, we can probe $\mathcal{M}_K$
  - × different solutions of the equations of motion, depending on t
    - → "phases" of the GLSM

$$\begin{tikzcd} \text{UV} & & \text{GLSM} \arrow[dl, "\xi \ll 0"'] \arrow[dr, "\xi \gg 0"] & \\ \text{IR} & \text{phase 1} & & \text{phase 2} \\ G=U(1): & \text{LG} \arrow[rr, leftrightarrow, dashed] & & \text{CY hypersurface} \end{tikzcd}$$

LG/CY correspondence

- B-branes:
  - × B-branes are matrix of the GLSM superpotential $W(\phi)$
    - ↳ factorise W over the polynomials, i.e. ∃ square matrix Q s.t.

$$Q^2 = W \cdot 1\!\!1$$

  - × D-brane transport

$$\begin{tikzcd} \text{UV} & & \text{GLSM (MF)} \arrow[dl, "\xi \ll 0"'] \arrow[dr, "\xi \gg 0"] & \\ \text{IR} & D(\text{phase 1}) & & D(\text{phase 2}) \\ G=U(1) & MF(W_{LG}) \arrow[rr, leftrightarrow, dashed] & & D^b Coh(X) \end{tikzcd}$$

Orlov '05

Herbst-Hori-Page 0803.2045 [hep-th]

  - ↳ GLSM establishes equivalence of two categories
  - ↳ non-trivial due to singularities in $\mathcal{M}_K$

- recent developments:
  - × GLSMs with non-abelian gauge groups
    - ↳ all CY complete intersections in toric ambient spaces can be formulated in terms of GLSMs with <u>abelian</u> gauge groups.
    - ↳ non-abelian GLSMs give more exotic CYs: e.g. complete intersections in Grassmannians, determinantal varieties

      Hori-Tong: hep-th/0609032: Pfaffian-Grassmannian correspondence

      more examples: Jockers et al, Hori/JK, Hosono-Takagi,
    - ↳ strongly coupled phases: (→ asymptotic series & resurgence)
    - ↳ new non-abelian duality (2D version of Seiberg duality)

Hori: 1104.2853 [hep-th]

"GLSMs with different gauge groups lead to same CYs

↳ strong/weak coupling duality

× SUSY localization in GLSMs

→ new methods to compute quantum corrections in string compactifications

→ applies (in principle) to non-abelian GLSMs & strongly coupled/exotic phases

→ elliptic genus Z_{T^2}: Benini, Eager, Hori, Tachikawa

1305.0533 [hep-th], 1308.4896 (hep-th)

→ sphere partition function Z_{S^2}: Benini-Cremonesi 1206.2356 (hep-th)

Doroud, Gomis, Le Floch, Lee 1206.2606 [hep-th)

Jockers, Kumar, Lapan, Morrison, Romo 1208.6244 (hep-th)

$Z_{S^2} = e^{K(t,\bar{t})}$ K... quantum corrected Kähler potential on M_K

→ $\boxed{\text{hemisphere partition function } Z_{D^2}}$

Sugishita-Terashima 1308.1973 (hep-th]

Honda-Okuda 1308.2217 [hep-th]

Hori-Romo 1308.2438 [hep-th)

$Z_{D^2} = Z(B)$ quantum-corrected central charge of a B-brane

→ A-twisted sphere partition functions

Closset, Cremonesi, Park 1504.06308 [hep-th]

↳ Yukawa couplings

× GLSM in math: variation of GIT quotients

Ballard, Favero, Katzarkov '12, E. Segal et al

2) GLSM: definitions and examples

- data:
  - × G … compact Lie group (gauge group)
  - × V … complex vector space "space of chiral fields" $\phi \in V$
  - × $\rho_V : G \to GL(V)$ faithful representation (weights ↔ gauge charges)
  - × $R: U(1) \to GL(V)$ "R-symmetry"
    - (R & ρ_V commute)
  - × $W \in \mathrm{Sym}(V^*)$: holomorphic, G-invariant function of R-charge 2
    - "GLSM superpotential"
  - × $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{t} = \mathrm{Lie}(T)$ T… max. Torus of G
  - × $\sigma \in \mathfrak{g}_{\mathbb{C}}$ "twisted chiral fields" (scalars of gauge multiplets)
  - × $t \in \mathfrak{g}_{\mathbb{C}}^*$ FI-θ parameters $t = \xi - i\theta$
- → notation:
  - × $Q_i^a \in \mathfrak{t}_{\mathbb{C}}^*$ charges of ϕ_i
  - × R_i .. weights of R: "R-charges"
- → extra conditions:
  - × $R(e^{i\pi}) = \rho_V(J)$ $J \in G$ "charge integrality"
    - (compatibility with A-twist)
  - × $\rho_V: G \to SL(V)$ CY condition

- equations of motion
  - → vacuum: minimise the classical potential

$$V = \frac{1}{8e^2}|[\sigma,\bar\sigma]|^2 + \frac{1}{2}\left(|\sigma\phi|^2 + |\bar\sigma\phi|^2\right) + \frac{e^2}{2}D^2 + |F|^2$$

  - × e… gauge coupling
  - × D-term: $\mu(\phi) - \xi$ $\mu: V \to i\mathfrak{g}^*$: moment map
  - × F-term: $dW(\phi)$
  - → Higgs branch: $\sigma = 0 \Rightarrow D = 0,\ F = 0$

vacuum $X_\xi = \{dW^{-1}(0)\} \cap \mu^{-1}(\xi)/G$ with $\mu^{-1}(\xi)/G \simeq (V - \overset{\text{deleted set}}{\overset{\uparrow}{F_\xi}})/G_{\mathbb{C}}$

↳ for abelian GLSMs one recovers the definition of a toric variety as a symplectic quotient

↳ gauge symmetry broken to a subgroup

→ Coulomb branch: $\phi = 0$, $\sigma \in t_{\mathbb{C}}$, $\xi = 0$

↳ gauge group broken to T

↳ σ classically unconstrained

↳ one-loop corrections generate an effective potential

$$\tilde{W}_{eff}(\sigma) = -t(\sigma) - \sum_i Q_i(\sigma)\left[\log(Q_i(\sigma)) - 1\right] + \pi i \sum_{\alpha>0} \alpha(\sigma)$$

$\alpha > 0$ ... positive roots

↳ Coulomb branch lifted except at points, lines etc → discriminant

↳ one can smoothly interpolate between phases

→ mixed branches: if $\mathrm{rk}\, T > 1$.

- example: quintic in $\mathbb{P}^4$

× $G = U(1)$, $t = \xi - i\theta$, $\sigma \in g_{\mathbb{C}}$, $V = \mathbb{C}(-5) \oplus \mathbb{C}(1)^{\oplus 5}$

× chiral matter:

ϕ	$x_1 \ldots x_5$	p
Q_i	1	-5
R_i	0	2

CY: $\sum Q_i = 0$

× superpotential $W = p\, G_5(x_{1,\ldots,5})$ G_5: homogeneous degree 5 polynomial

× D-terms: $-5|p|^2 + \sum_i |x_i|^2 = \xi$

× F-terms: $dW = 0$

× phases: $\xi >> 0$ $X^{\xi >> 0} = \{G_5 = 0\} \subset \mathbb{P}^4$ $F_\xi = (x_1, \ldots, x_5)$

"geometric phase"

$\xi << 0$ $X^{\xi << 0} = \left\{p = \sqrt{-\frac{\xi}{5}}\right\}$ $F_\xi = (p)$ point!

↳ classical fluctuation of the ϕ_i around the vacuum

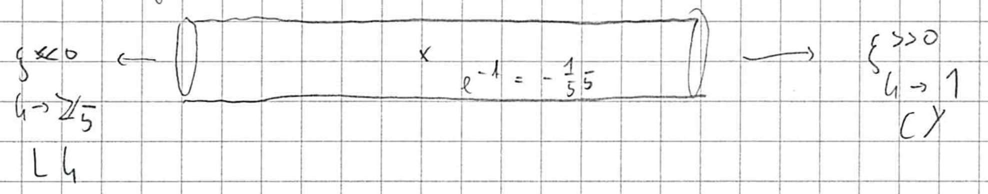
↳ get $[\mathbb{C}^5/\mathbb{Z}_5]$ with superpotential $\tilde{W} = G_5(x)$

"Landau-Ginzburg model"

- Coulomb branch $\tilde{W}_{eff} = -t\sigma - [(-5\sigma)(\log(-5\sigma) - \cancel{1}] - 5\sigma[\log\sigma - \cancel{1}]$

$$\frac{\partial \tilde{W}_{eff}}{\partial\sigma} = 0 \quad \rightsquigarrow \quad e^{-t} = -\frac{1}{5^5}$$

- moduli space:



- example: Rødland model

  - $G = U(2)$ $\quad t = \xi - i\theta \quad \sigma_{1,\sigma} \in \mathfrak{g}_{\mathbb{C}} \quad S = \mathbb{C}^2$

  - matter $(p^1, \ldots, p^7, x_1, \ldots, x_7) \in (\det^{-1} S)^{\oplus 7} + S^{\oplus 7}$

  - superpotential

$$W = \sum_{i,j,k=1}^{7} \sum_{a,b=1}^{7} A_k^{ij} p^k x_i^a \varepsilon_{ab} x_j^b = \sum_{i,j=1}^{7} A^{ij}(p)[x_i x_j]$$

  - D-terms: $-\sum_{i=1}^{7} |p^i|^2 + \sum_{j=1}^{7} \|x_j\|^2 = \xi$

$$x x^\dagger - \frac{1}{2}\|x\|^2 \mathbb{1}_{(2\times 2)} = 0$$

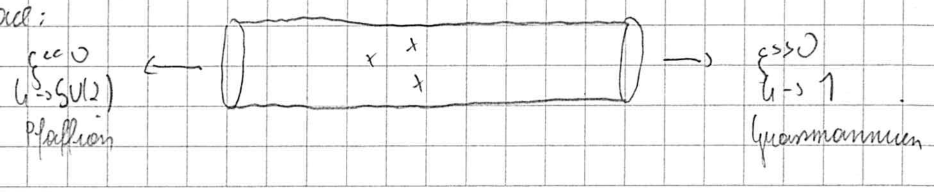
  - phases: $\xi \gg 0$: complete intersection of codimension 7 in $G(2,7)$

$$X^{\xi\gg 0} \quad \sum_{i,j=1}^{7} A_k^{ij}(x_i x_j) = 0 \subset G(2,7) \quad k = 1, \ldots, 7$$

$$\xi \ll 0 \qquad X^{\xi \ll 0} \quad \{ p \in \mathbb{P}^6 \mid \text{rk}\, A(p) = 4 \} \qquad \text{Pfaffian CY}$$

→ strongly coupled phase

- moduli space:



↳ $X^{\xi>>0}$ & $X^{\xi<<0}$ are NOT birational, but their derived categories are equivalent

3) B-branes in the GLSM and hemisphere partition function

- B-branes: G-invariant matrix factorisations W with R-charge 1

→ data: × $M = M^0 \oplus M^1$ $\mathbb{Z}_2$-graded S-module with $S = \mathrm{Sym}\, V^* = \mathbb{C}[\phi]$

"Chan-Paton space"

× $Q \in \mathrm{End}^1_S(M)$ such that $Q^2 = W \cdot \mathrm{id}_M$ matrix factorisation

× $\rho : G \to GL(M)$ such that

$$\rho(g)^{-1} Q(g\phi)\, \rho(g) = Q(\phi) \qquad g \in G$$

× $r_* : u(1)_R \to gl(M)$ with

$$\lambda^{r_*} Q(\lambda^R \phi)\, \lambda^{-r_*} = \lambda\, Q(\phi)$$

→ examples on the quintic: choose a Clifford basis satisfying

$$\{\eta_i, \bar\eta_j\} = \delta_{ij} \qquad \{\eta_i, \eta_j\} = \{\bar\eta_i, \bar\eta_j\} = 0$$

× $Q_1 = \begin{pmatrix} 0 & p \\ G_5 & 0 \end{pmatrix} = p\eta + G_5 \bar\eta$

× $Q_2 = \sum_{i=1}^{5} x_i \eta_i + \frac{p}{5} \frac{\partial G_5}{\partial x_i} \bar\eta_i$

× Fermat point $G_5 = \sum_i x_i^5$

$$Q_3 = (x_1 + x_2)\eta_1 + (x_1^4 \pm \ldots \pm x_2^4)\bar\eta_1 + \sum_{i=3}^{5} x_i \eta_{i-1} + x_i^4 \bar\eta_{i-1}$$

- hemisphere partition function Z_{D^2}

$$Z_{D^2}(B) = C \int_{\gamma \in \Gamma_\sigma} d^{\mathrm{rk}G}\sigma \prod_{\alpha>0} \alpha(\sigma) \sinh(\pi\alpha(\sigma)) \prod_i \Gamma\left(i Q_i(\sigma) + \frac{R_i}{2}\right) \times e^{it(\sigma)} f_B(\sigma)$$

× $\gamma \in \Gamma_\sigma$: integration contour : Lagrangian $\subset T$, s.t. poles are avoided and $Z_{D^2}(B)$ converges

⑨

- $f_B(\sigma)$ ... "brane factor"

$$f_B(\sigma) = \mathrm{tr}_M \left(e^{i\pi r_*} e^{2\pi \rho(\sigma)} \right)$$

- example: Z_{D^2} on the quintic

$$Z_{D^2}(B) = \int_{-\infty}^{\infty} d\sigma \, \Gamma(i\sigma)^4 \, \Gamma(1-5i\sigma) \, e^{it\sigma} \, f_B(\sigma)$$

↳ evaluate for $\xi \gg 0$: close contour in upper half plane

- $f_{B_1} = (1 - e^{-10\pi\sigma})$
- $f_{B_2} = (1 - e^{2\pi\sigma})^5$
- $f_{B_3} = (1 - e^{2\pi\sigma})^4$

- $$Z_{D^2}^{\xi\gg0}(B_1) = \frac{\overset{\nearrow H^3}{\boxed{5}}}{3!}\left(\frac{it}{2\pi}\right)^3 + \frac{\overset{\nearrow c_2\cdot H}{\boxed{50}}}{24}\left(\frac{it}{2\pi}\right) + \frac{i\zeta(3)}{(2\pi)^3}\overset{\nearrow \chi(X)}{\boxed{(-200)}} + \mathcal{O}(e^{-t})$$

$B_1 \leftrightarrow \mathcal{O}_X$ structure sheaf $\qquad (Z_{D^2}^{\xi\ll0}(B_1) = 0)$

- $Z_{D^2}^{\xi\gg0}(B_2) = 0$ does not exist in CY phase
- $Z_{D^2}^{\xi\gg0}(B_3) = 1 + \mathcal{O}(e^{-t})$ D0-brane

generally: $$Z_{D^2}^{\xi\gg0}(B) = \sum_{n=0}^{\infty} e^{-nt} \int_X \hat{\Gamma}(n) \, e^{B - \frac{i}{2\pi}\overset{\nearrow \text{complexified Kähler class}}{\omega}} \, \mathrm{ch}(B^{LV})$$

↑ Γ-class $\Gamma\Gamma^* = \hat{A}_X$

$\hat{\Gamma}(x) = \Gamma(1 - \frac{x}{2\pi i})$

↳ Chern character

→ Z_{D^2} can be expressed in terms of periods of the mirror CY & satisfies a Picard-Fuchs equation

→ works (in principle) for non-abelian GLSMs, strongly coupled phases, (→ asymptotic series), exotic phases etc.

4) Outlook:

- systematic construction of exotic CY's via GLSMs (representation theory)

- × central charge in non-geometric phases
- × hemisphere partition function as a "global" stability condition
- × better understanding of hybrid phases
- × D-branes & D-brane transport in non-abelian GLSMs
- × mirror symmetry for exotic CYs