Worldsheet string theory in AdS/CFT and lattice

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Junior Research Group “Gauge fields from strings”

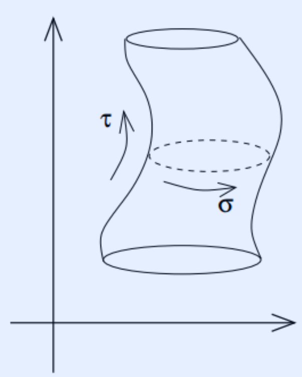
Women at the intersection of Mathematics and High Energy Physics,

Mainz Institute for Theoretical Physics, **March 8** 2017

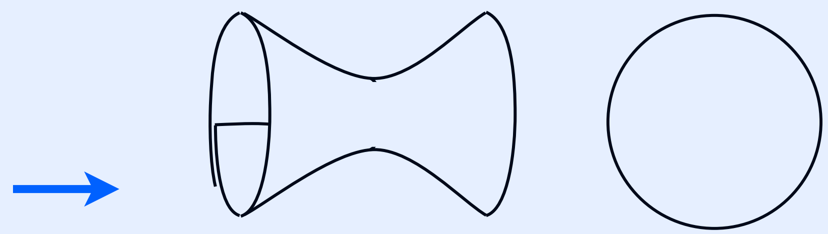
Framework

String/gauge correspondence, addresses together

- understanding gauge theories at all values of the coupling
- understanding string theories in non-trivial backgrounds



$\mathbb{R} \times S^1$

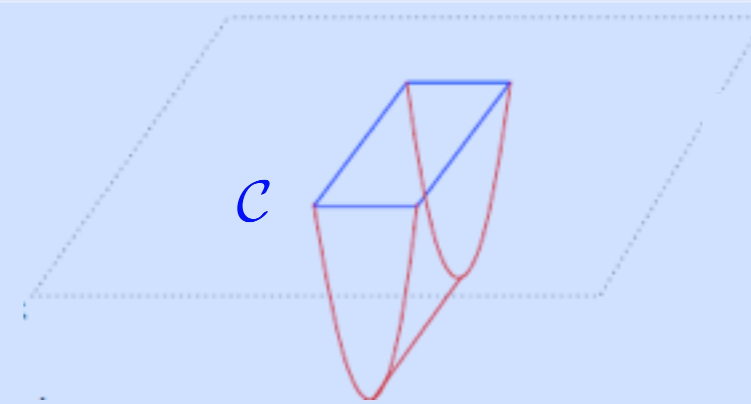


“Quark-antiquark” potential

Type IIB strings in $AdS_5 \times S^5$
(g_S, R)

$$Z_{\text{string}}\big|_{\mathcal{C}} = \int DX\, D\Psi\, e^{-S_{\text{string}}} \sim e^{-\text{Area}}$$

AdS/CFT

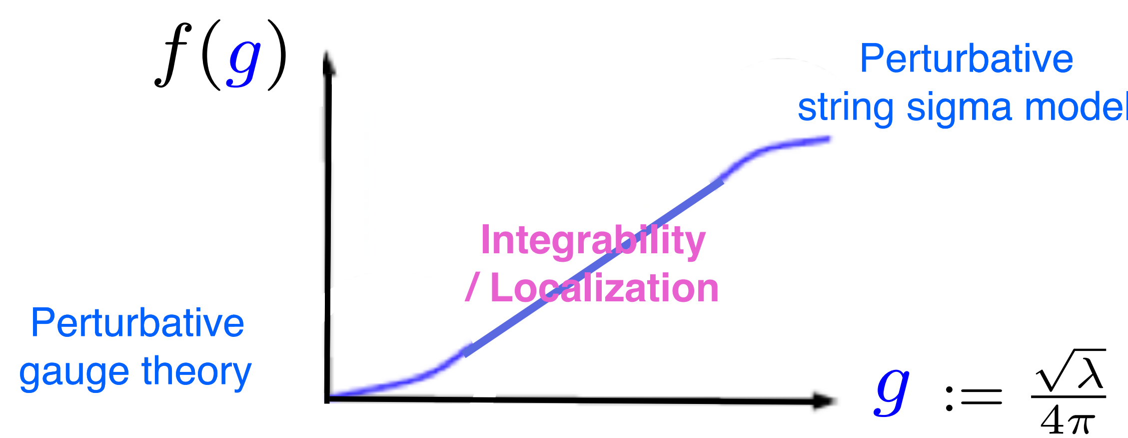


$\mathcal{N} = 4$ super Yang-Mills in 4d
(g_{YM}, N)

$$\langle W[\mathcal{C}] \rangle = \frac{1}{N} \mathrm{Tr} \mathcal{P}\, e^{\oint (i A_\mu\, \dot{x}^\mu + \Phi_i\, \dot{y}^i)\, ds}$$

Motivation

Beautiful progress in obtaining exact results within AdS/CFT



$$f(g) = c\,g + d + \frac{e}{g} + \dots$$

$$f(g) = a\,g^2 + b\,g^4 + \dots$$

$$g := \frac{\sqrt{\lambda}}{4\pi} = \frac{\sqrt{g_{YM}^2 N}}{4\pi} \equiv \frac{R^2}{4\pi\alpha'}$$

- from integrability
- from supersymmetric localization

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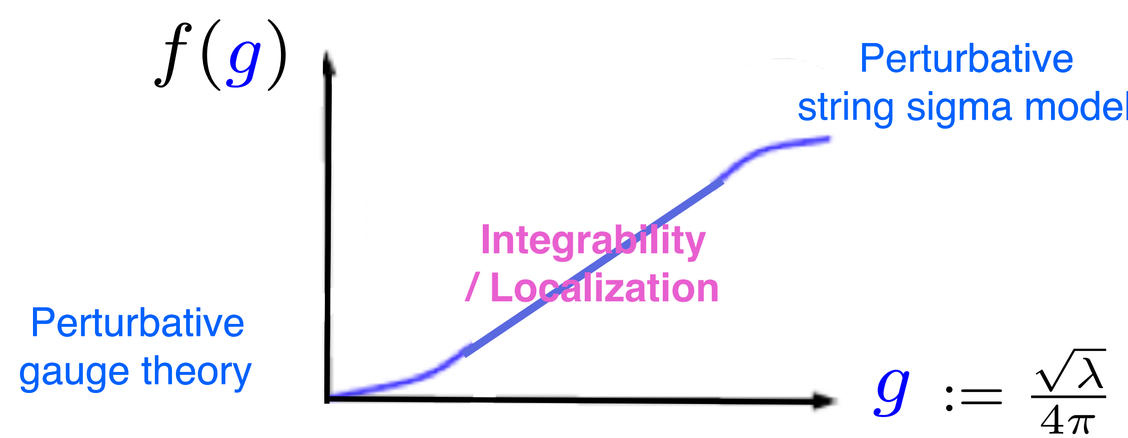

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- from integrability (assumed)
- from supersymmetric localization (supersymmetric observables)

In the world-sheet string theory integrability only classically, localization not formulated.

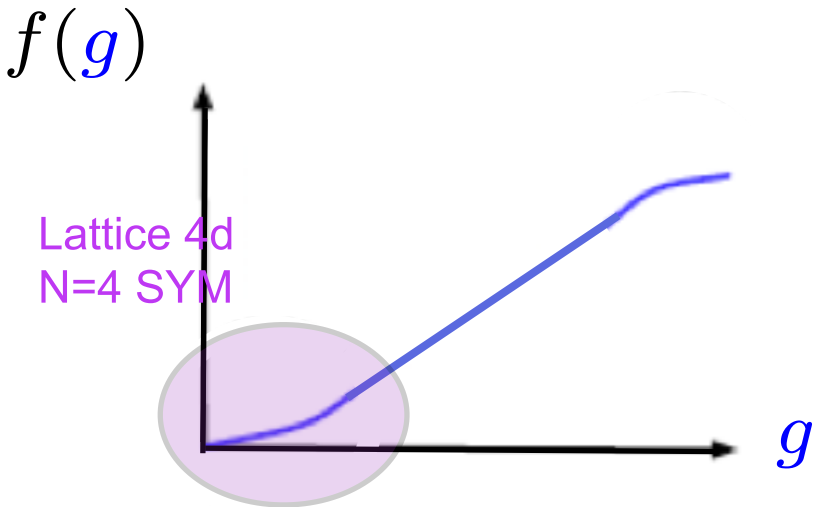
The relevant string sigma-model (Green-Schwarz superstrings in AdS backgrounds with RR-fluxes) is a complicated interacting 2d field theory which has subtleties also perturbatively.

Call for genuine 2d QFT to cover the finite-coupling region.

Lattice techniques in AdS/CFT

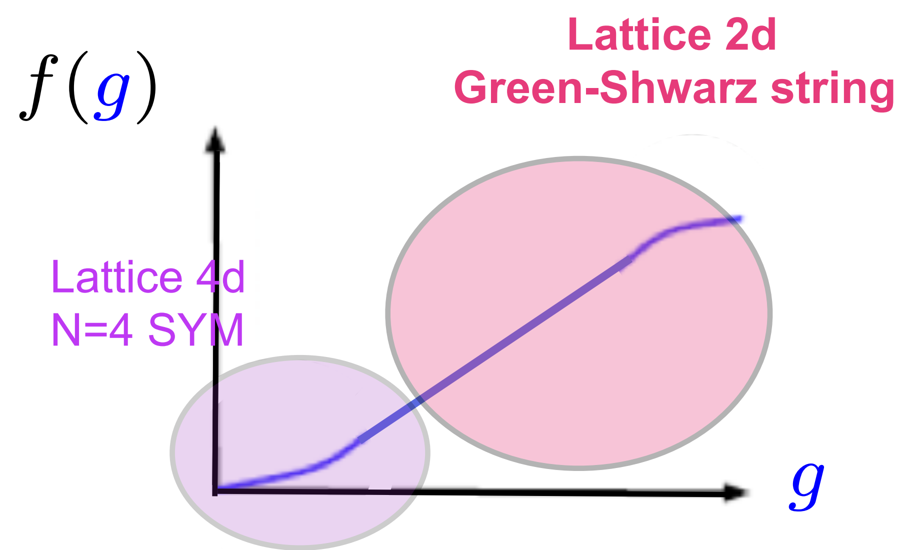
Exciting program on the
4d susy CFT side,
subtleties with supersymmetry.

[Catterall et al.]



Lattice techniques in AdS/CFT

Lattice for superstring world-sheet in $AdS_5 \times S^5$



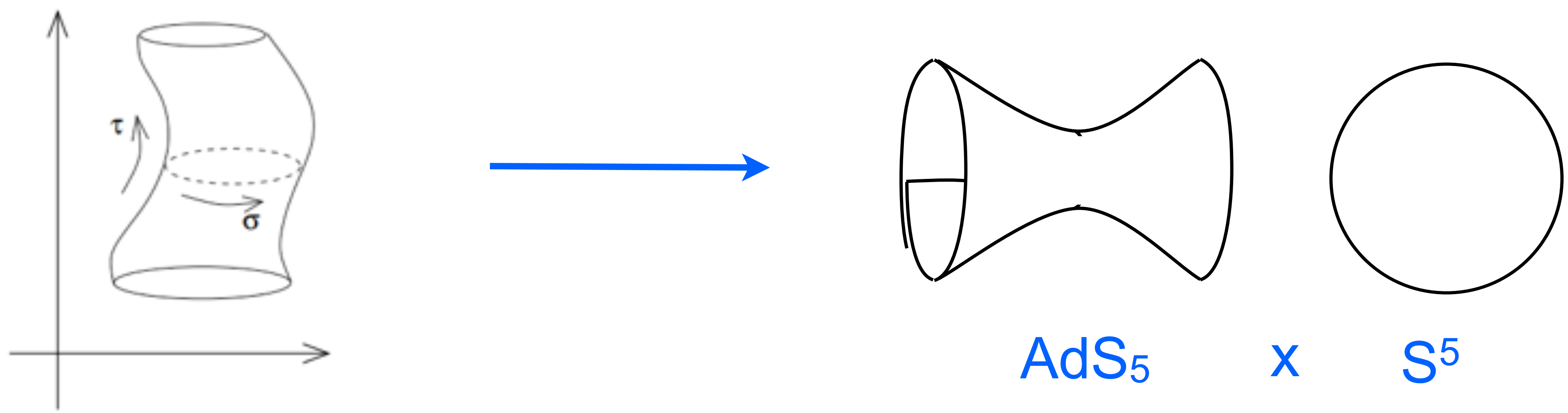
[previous study: Roiban McKeown 2013]

- 2d: computationally cheap
- no supersymmetry (Green-Schwarz formulation)
- all local (diffeo, κ) symmetries are fixed, only scalar fields (some of which Graßmann-valued)

Non-trivial 2d qft with strong coupling analytically known, finite-coupling (numerical) prediction.

The model in perturbation theory

Green-Schwarz string in $AdS_5 \times S^5$ + RR flux



[Metsaev Tseytlin 1998]

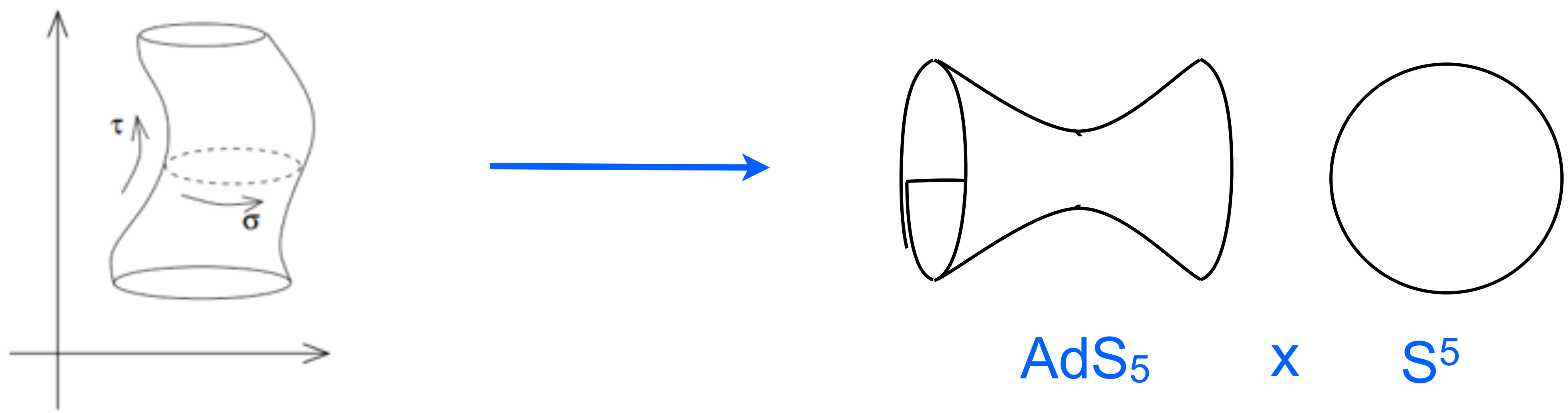
$$S = g \int d\tau d\sigma \left[\partial_a X^\mu \partial^a X^\nu G_{\mu\nu} + \bar\theta\, \Gamma \left(D + F_5\right) \theta\, \partial X + \bar\theta\, \partial\theta\, \bar\theta\, \partial\theta + \dots \right]$$

Symmetries:

- global $PSU(2,2|4)$, local bosonic (diffeomorphism) and fermionic (κ-symmetry)
- classical integrability

manifest when written as sigma-model action on $G/H = \frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$.

Green-Schwarz string in $AdS_5 \times S^5$ + RR flux



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Highly non-linear, to quantize it use semiclassical methods

$$X = X_{\rm cl} + \tilde{X} \quad \longrightarrow \quad \Gamma = g \left[\Gamma_0 + \frac{\Gamma_1}{g} + \frac{\Gamma_2}{g^2} + \ldots\right]$$

Green-Schwarz string in $AdS_5 \times S^5$ + RR flux perturbatively

Highly non-linear, to quantize it use semiclassical methods

$$X = X_{\rm cl} + \tilde{X} \qquad \longrightarrow \qquad E = g\left[E_0 + \frac{E_1}{g} + \frac{E_2}{g^2} + \dots\right], \qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

- General analysis of fluctuations in terms of background geometry
  e.g. $\mathrm{Tr}(\mathcal{M}) = a\,{}^{(2)}\!R + b\,\mathrm{Tr}(K^2)$.
  [Alvarez-Gaume, Freedman, Mukhi, 81]
  [Drukker Gross Tseytlin 00] [**VF** Giangreco Griguolo Seminara Vescovi 15]

- Explicit analytic form of one-loop partition function $Z = \det O_F / \sqrt{\det O_B}$ for a class of effectively one-dimensional problems.
  Non-trivial differential operators, e.g. elliptic-function potentials:
  $\mathcal{O} = -\partial_\sigma^2 + \omega^2 + k^2 \mathrm{sn}^2(\sigma, k^2)$. Then use Gelf'and-Yaglom method:

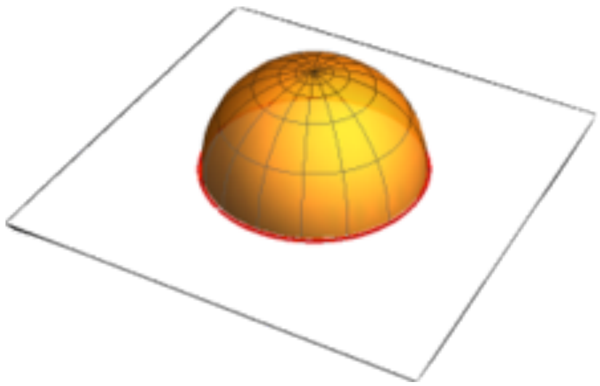
$$\mathcal{O}\,\phi(x) = \lambda\,\phi(x)\,, \qquad \phi(0) = \phi(L) = 0 \qquad\qquad \frac{\det \mathcal{O}}{\det \mathcal{O}_{\rm free}} = \frac{u(L)}{u_{\rm free}(L)}$$

where u are solutions of auxiliary boundary value problem, $u(0) = 0\,, u'(0) = 1$.

Several configurations (GKP string, quark-antiquark potential, generalized cusp) have been “solved” this way at one loop, and agree with predictions.

1/2 BPS circular Wilson loop

[Erickson, Semenoff, Zarembo 00]
[Drukker Gross 00]
[Pestun 07]



$$\log\langle\mathcal{W}(\lambda)\rangle = \log \tfrac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) = \sqrt{\lambda} - \frac{3}{4}\log\lambda + \frac{1}{2}\log\frac{2}{\pi} + \mathcal{O}(\lambda^{-\frac{1}{2}})$$

The 1-loop disc partition function $\log Z = \log\langle W\rangle$ differs: $\frac{1}{2}\log\frac{1}{2\pi}$

[Kruczenski Tirziu 08]
[Buchbinder Tseytlin 14]

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[Drukker Giombi Ricci Trancanelli 07]
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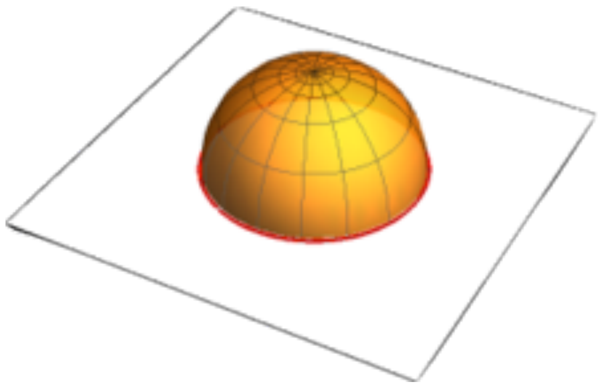

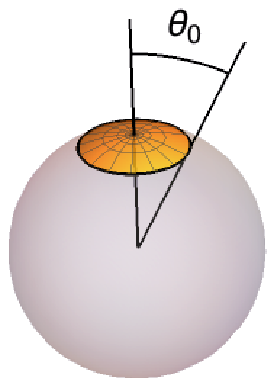

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To avoid measure ambiguities, consider ratio of Zs for surfaces of the same topology: $\log\lambda$ factors and UV divergences ($\sim\chi$ [Drukker Gross Tseytlin 00]) should cancel out.

E.g. family of 1/4-BPS "latitudes", parametrized by θ_0 in $S^2 \in S^5$ ($\lambda' = \lambda\cos^2\theta_0$).

$$\log\frac{\langle\mathcal{W}(\lambda,\theta_0)\rangle}{\langle\mathcal{W}(\lambda,0)\rangle} = \sqrt{\lambda}(\cos\theta_0 - 1) - \frac{3}{2}\log\cos\theta_0 + \mathcal{O}(\lambda^{-\frac{1}{2}})$$

1/2 BPS circular Wilson loop

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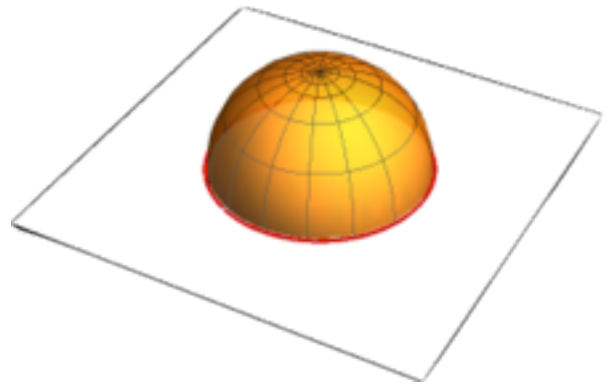

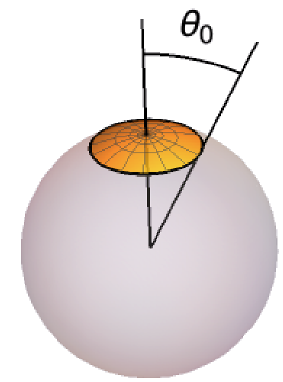

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$$= \sqrt{\lambda}(\cos\theta_0 - 1) + \frac{3}{4}\theta_0^2 + \mathcal{O}(\lambda^{-\frac{1}{2}})$$

Usual (Gelf'and Yaglom) method fails. Perturbative heat-kernel (near AdS_2 expansion) agrees.

{ > unphysical cutoff
> different regulariz. in τ and in σ }

[Forini Tseytlin Vescovi 17]

Green-Schwarz string in $AdS_5 \times S^5$ + RR flux perturbatively

Highly non-linear, to quantize it use semiclassical methods

$$X = X_{\rm cl} + \tilde{X} \quad \longrightarrow \quad \Gamma = g\left[\Gamma_0 + \frac{\Gamma_1}{g} + \frac{\Gamma_2}{g^2} + \dots\right]$$

2 loops is current limit: “homogenous” configs, “AdS light-cone” gauge-fixing

[Metsaev, Tseytlin] [Metsaev, Thorn, Tseytlin]

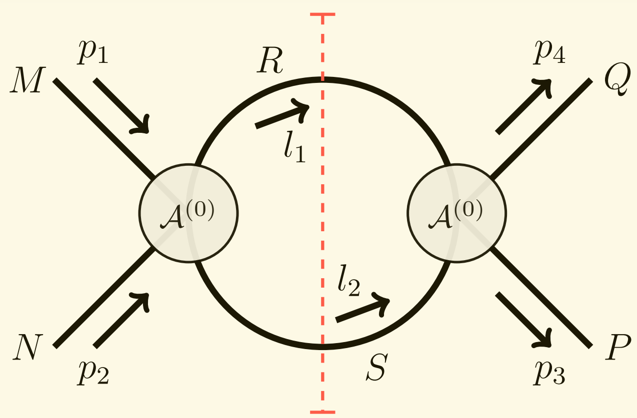
[Giombi Ricci Roiban Tseytlin 09] [Bianchi^2 Bres **VF** Vescovi 14]

UV divergences: set to zero power-divergent massless tadpoles (as in *dimreg*), all remaining log-divergent integrals cancel out in the sum (no need of reg. scheme).

Green-Schwarz string in $AdS_5 \times S^5$ + RR flux perturbatively

Unitarity cuts in $d = 2$, for worldsheet amplitudes (integrable S-matrix)

[Bianchi **VF** Hoare 13][Engelund, Roiban 13][Bianchi Hoare 14]



$$A^{(1)} = \sum \left(A^{(0)}\right)^2 I_{\text{bubble}} \,,$$

$$I_{\text{bubble}}(p) = \int \frac{d^2 q}{(2\pi^2)^2} \frac{1}{(q^2 - 1 + i\,\epsilon)\,((q-p)^2 - 1 + i\,\epsilon)}$$

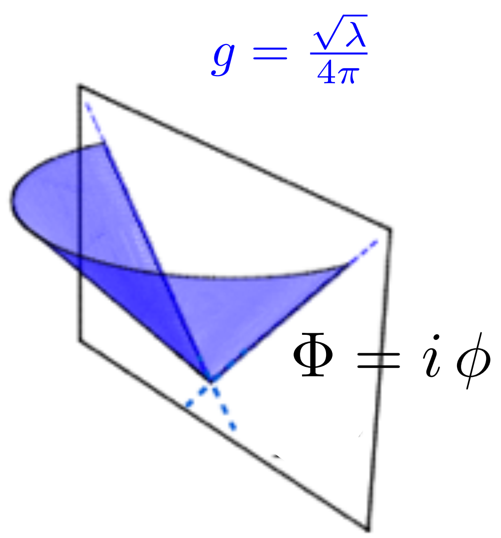
Inherently finite, bypasses any regularization issue: may miss rational terms.

A large class of 2-d models, relativistic and not (including string worldsheet models in AdS), appears to be cut-constructible.

Beyond perturbation theory

Based on 1601.04670, 1605.01726, 1702.02005, 1703.xxxxx
with L. Bianchi, M. S. Bianchi, B. Leder, P. Töpfer, E. Vescovi

The cusp anomaly of $\mathcal{N}=4$ SYM from string theory



Completely solved via integrability. [Beisert Eden Staudacher 2006]

Expectation value of a light-like cusped Wilson loop

$$\langle W[C_{\rm cusp}]\rangle \sim e^{-f(g)\,\phi\,\ln\frac{L_{\rm IR}}{\epsilon_{\rm UV}}}$$

AdS/CFT

$$Z_{\rm cusp} = \int [D\delta X][D\delta\theta]\, e^{-S_{\rm IIB}(X_{\rm cusp}+\delta X,\delta\theta)} = e^{-\Gamma_{\rm eff}}$$

String partition function with “cusp” boundary conditions

In Poincaré patch (boundary at z=0)

$$ds^2_{AdS_5} = \frac{dz^2 + dx^+\,dx^- + dx^*\,dx}{z^2} \qquad x^\pm = x^3 \pm x^0 \qquad x = x^1 \pm i\,x^2$$

classical solution (τ and σ vary from 0 to ∞) is a surface

$$z = \sqrt{\frac{\tau}{\sigma}} \qquad x^+ = \tau \qquad x^- = -\frac{1}{2\sigma}$$

bounded by a null cusp, since at the AdS$_5$ boundary it is $0 = z^2 = -2x^+x^-$.

[Giombi Ricci Roiban Tseytlin 2009]

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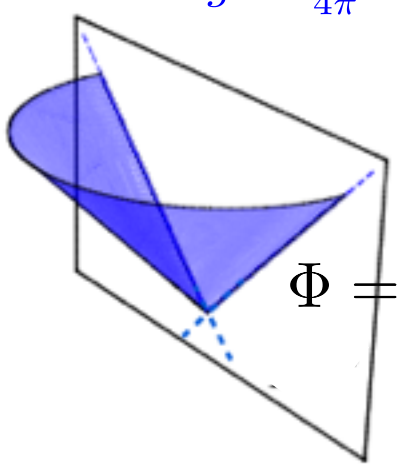

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String partition function with "cusp" boundary conditions, evaluated perturbatively

$$f(g)|_{g\to 0} = 8g^2\left[1-\frac{\pi^2}{3}g^2+\frac{11\,\pi^4}{45}g^4-\left(\frac{73}{315}+8\,\zeta_3\right)g^6+\ldots\right]$$ [Bern et al. 2006]

$$f(g)|_{g\to\infty} = 4g\left[1-\frac{3\ln 2}{4\pi}\frac{1}{g}-\frac{K}{16\pi^2}\frac{1}{g^2}+\ldots\right]$$ [Gubser Klebanov Polyakov 02] [Frolov Tseytlin 02][Giombi et al. 2009]

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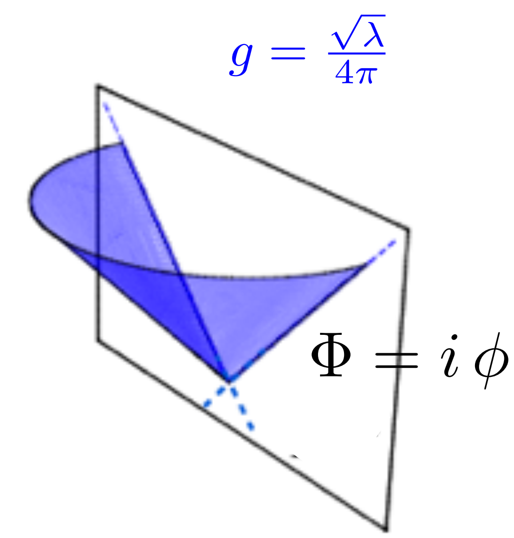

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String partition function with “cusp” boundary conditions, evaluated perturbatively

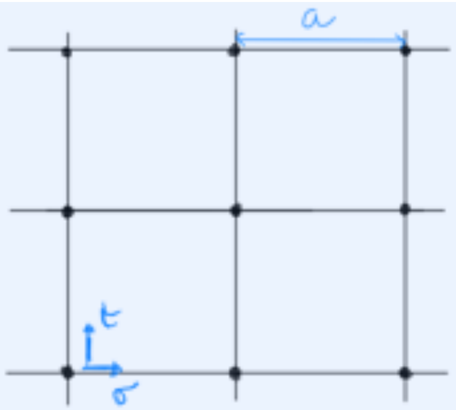
A lattice approach prefers expectation values

$$\langle S_{\rm cusp}\rangle = \frac{\int [D\delta X][D\delta\Psi]\, S_{\rm cusp}\, e^{-S_{\rm cusp}}}{\int [D\delta X][D\delta\Psi]\, e^{-S_{\rm cusp}}} = -g\,\frac{d\ln Z_{\rm cusp}}{dg} \equiv g\,\frac{V_2}{8}\,f'(g)$$

$$S_{\rm cusp} = g\int \mathcal{L}_{\rm cusp}$$

Simulations in lattice QFT

Spacetime grid with as lattice spacing a, size $L = N\,a$,
$\xi = (an_1, an_2) \equiv a\,n$ and fields $\phi \equiv \phi_n$
a) natural cutoff for the momenta, $-\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}$
b) path integral measure $[D\phi] = \prod_n d\phi_n$.



Then $\int \prod_n d\phi_n\, e^{-S_{\rm discr}}$ can be studied via Monte Carlo: generate an ensamble $\{\Phi_1, \ldots, \Phi_K\}$ of field configurations, each weighted by $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]}}{Z}$.

Ensemble average $\langle A \rangle = \int [D\Phi]\, P[\Phi]\, A[\Phi] = \frac{1}{K} \sum_{i=1}^{K} A[\Phi_i] + \mathcal{O}(\frac{1}{\sqrt{K}})$

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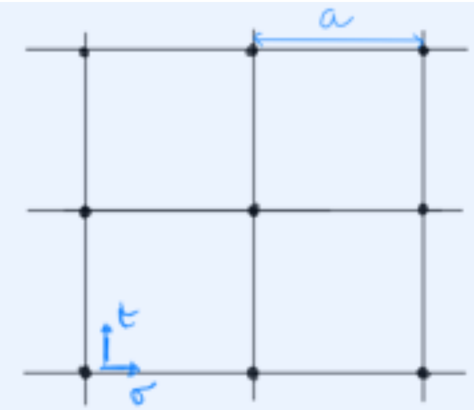

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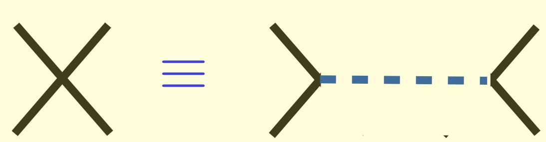
Graßmann-odd fields are formally integrated out: $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]} \det \mathcal{O}_F}{Z}$

- action must be quadratic in fermions :
  here, interactions at most quartic (AdS light cone gauge)



Introduce auxiliary fields
(complex bosons)

- determinant must be positive definite

$$\det O_F \longrightarrow \sqrt{\det(O_F^\dagger O_F)} \equiv \int D\zeta\, D\bar{\zeta}\; e^{-\int d^2\xi\, \bar{\zeta} (O_F^\dagger O_F)^{-\frac{1}{2}} \zeta}$$

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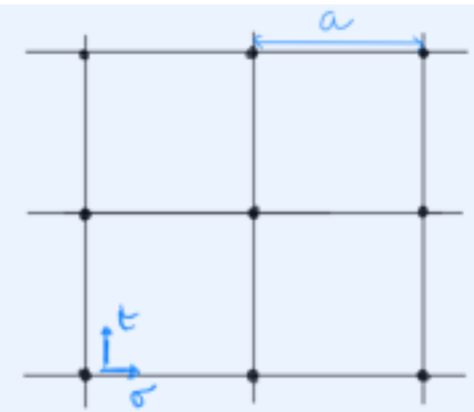

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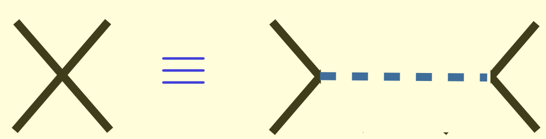


Introduce auxiliary fields
(complex bosons)

- determinant must be positive definite

$$\mathrm{Pf}\, O_F \longrightarrow (\det O_F^\dagger\, O_F)^{\frac{1}{4}} \equiv \int D\zeta\, D\bar{\zeta}\; e^{-\int d^2\xi\, \bar{\zeta}\, (O_F^\dagger O_F)^{-\frac{1}{4}} \zeta}$$

potential ambiguity!

Green-Schwarz string in the null cusp background

$$S_{\rm cusp} = g \int dt ds \mathcal{L}_{\rm cusp}$$

[Giombi Ricci Roiban Tseytlin 2009]

$$\mathcal{L}_{\rm cusp} = |\partial_t x + \tfrac{1}{2}x|^2 + \tfrac{1}{z^4}|\partial_s x - \tfrac{1}{2}x|^2 + \left(\partial_t z^M + \tfrac{1}{2}z^M + \tfrac{i}{z^2} z_N \eta_i \left(\rho^{MN}\right)^i{}_j \eta^j\right)^2 + \tfrac{1}{z^4}\left(\partial_s z^M - \tfrac{1}{2}z^M\right)^2$$
$$+ i\left(\theta^i \partial_t \theta_i + \eta^i \partial_t \eta_i + \theta_i \partial_t \theta^i + \eta_i \partial_t \eta^i\right) - \tfrac{1}{z^2}\left(\eta^i \eta_i\right)^2$$
$$+ 2i\Big[\tfrac{1}{z^3} z^M \eta^i \left(\rho^M\right)_{ij}\left(\partial_s \theta^j - \tfrac{1}{2}\theta^j - \tfrac{i}{z}\eta^j\left(\partial_s x - \tfrac{1}{2}x\right)\right) + \tfrac{1}{z^3} z^M \eta_i (\rho_M^\dagger)^{ij}\left(\partial_s \theta_j - \tfrac{1}{2}\theta_j + \tfrac{i}{z}\eta_j\left(\partial_s x - \tfrac{1}{2}x\right)^*\right)\Big]$$

- 8 bosonic coordinates: x, x^*, z^M $(M = 1, \cdots, 6)$, $z = \sqrt{z_M z^M}$;
- 8 fermionic variables, $\theta^i = (\theta_i)^\dagger$, $\eta^i = (\eta_i)^\dagger$, $i = 1, 2, 3, 4$ transforming in the fundamental of $SU(4)$
- ρ^M are off-diagonal blocks of SO(6) Dirac matrices $\gamma^M \equiv \begin{pmatrix} 0 & \rho_M^\dagger \\ \rho^M & 0 \end{pmatrix}$

Manifest global symmetry is $SO(6) \times SO(2)$.

Green-Schwarz string in the null cusp background

$$S_{\rm cusp} = g \int dt ds \mathcal{L}_{\rm cusp}$$

[Giombi Ricci Roiban Tseytlin 2009]

$$\mathcal{L}_{\rm cusp} = |\partial_t x + \tfrac{1}{2} x|^2 + \tfrac{1}{z^4} |\partial_s x - \tfrac{1}{2} x|^2 + \left(\partial_t z^M + \tfrac{1}{2} z^M + \tfrac{i}{z^2} z_N \eta_i \left(\rho^{MN}\right)^i{}_j \eta^j\right)^2 + \tfrac{1}{z^4} \left(\partial_s z^M - \tfrac{1}{2} z^M\right)^2$$
$$+ i \left(\theta^i \partial_t \theta_i + \eta^i \partial_t \eta_i + \theta_i \partial_t \theta^i + \eta_i \partial_t \eta^i\right) - \tfrac{1}{z^2} \left(\eta^i \eta_i\right)^2$$
$$+ 2i \Big[\tfrac{1}{z^3} z^M \eta^i \left(\rho^M\right)_{ij} \left(\partial_s \theta^j - \tfrac{1}{2} \theta^j - \tfrac{i}{z} \eta^j \left(\partial_s x - \tfrac{1}{2} x\right)\right) + \tfrac{1}{z^3} z^M \eta_i (\rho^\dagger_M)^{ij} \left(\partial_s \theta_j - \tfrac{1}{2} \theta_j + \tfrac{i}{z} \eta_j \left(\partial_s x - \tfrac{1}{2} x\right)^*\right)\Big]$$

- 8 bosonic coordinates: x, x^*, z^M $(M = 1, \cdots, 6)$, $z = \sqrt{z_M z^M}$;
- 8 fermionic variables, $\theta^i = (\theta_i)^\dagger$, $\eta^i = (\eta_i)^\dagger$, $i = 1, 2, 3, 4$ transforming in the fundamental of $SU(4)$
- ρ^M are off-diagonal blocks of SO(6) Dirac matrices $\gamma^M \equiv \begin{pmatrix} 0 & \rho^\dagger_M \\ \rho^M & 0 \end{pmatrix}$

Manifest global symmetry is $SO(6) \times SO(2)$.

Quartic fermionic interactions that can be linearized

$$\exp\Big\{-g \int dt ds \Big[-\tfrac{1}{z^2} \left(\eta^i \eta_i\right)^2 + \left(\tfrac{i}{z^2} z_N \eta_i {\rho^{MN}}^i{}_j \eta^j\right)^2\Big]\Big\}$$
$$\sim \int D\phi D\phi^M \exp\Big\{-g \int dt ds \left[\tfrac{1}{2} \phi^2 + \tfrac{\sqrt{2}}{z} \phi\, \eta^2 + \tfrac{1}{2} (\phi_M)^2 - i \tfrac{\sqrt{2}}{z^2} \phi^M \left(\tfrac{i}{z^2} z_N \eta_i {\rho^{MN}}^i{}_j \eta^j\right)\right]\Big\} .$$

Green-Schwarz string in the null cusp background

After linearization the Lagrangian reads ($m \sim P_+$)

$$\mathcal{L}_{\rm cusp} = |\partial_t x + \frac{m}{2} x|^2 + \frac{1}{z^4}|\partial_s x - \frac{m}{2} x|^2 + (\partial_t z^M + \frac{m}{2} z^M)^2 + \frac{1}{z^4}(\partial_s z^M - \frac{m}{2} z^M)^2 + \frac{1}{2}\phi^2 + \frac{1}{2}(\phi_M)^2 + \psi^T O_F \psi \,,$$

- $+7$ bosonic auxiliary fields ϕ, ϕ^M $(M = 1, \cdots, 6)$)
- formal variable $\psi \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$

$$O_F = \begin{pmatrix} 0 & i\partial_t & -\mathrm{i}\rho^M\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} & 0 \\ \mathrm{i}\partial_t & 0 & 0 & -\mathrm{i}\rho^\dagger_M\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} \\ \mathrm{i}\frac{z^M}{z^3}\rho^M\left(\partial_s - \frac{m}{2}\right) & 0 & 2\frac{z^M}{z^4}\rho^M\left(\partial_s x - m\frac{x}{2}\right) & i\partial_t - A^T \\ 0 & \mathrm{i}\frac{z^M}{z^3}\rho^\dagger_M\left(\partial_s - \frac{m}{2}\right) & \mathrm{i}\partial_t + A & -2\frac{z^M}{z^4}\rho^\dagger_M\left(\partial_s x^* - m\frac{x}{2}^*\right) \end{pmatrix}$$

$$A = \frac{1}{\sqrt{2}z^2}\phi_M \rho^{MN} z_N - \frac{1}{\sqrt{2}z}\phi + \mathrm{i}\,\frac{z_N}{z^2}\rho^{MN}\,\partial_t z^M$$

Green-Schwarz string in the null cusp background

After linearization the Lagrangian reads ($m \sim P_+$)

$$
\begin{aligned}
\mathcal{L}_{\rm cusp} \;&= |\partial_t x + \frac{m}{2} x|^2 + \frac{1}{z^4}|\partial_s x - \frac{m}{2} x|^2 + (\partial_t z^M + \frac{m}{2} z^M)^2 + \frac{1}{z^4}(\partial_s z^M - \frac{m}{2} z^M)^2 \\
&+ \frac{1}{2}\phi^2 + \frac{1}{2}(\phi_M)^2 + \psi^T O_F \psi \,,
\end{aligned}
$$

A naive discretization $p_\mu \to \mathring{p}_\mu \equiv \frac{1}{a}\sin(a\, p_\mu)$ leads to fermion doublers.

Add to the action a “Wilson term” W

$$
\hat{O}_F = \begin{pmatrix}
W_+ & -\mathring{p}_0 \mathbb{1} & (\mathring{p}_1 - i\frac{m}{2})\rho^M \frac{z^M}{z^3} & 0 \\
-\mathring{p}_0 \mathbb{1} & -W_+^\dagger & 0 & \rho_M^\dagger(\mathring{p}_1 - i\frac{m}{2})\frac{z^M}{z^3} \\
-(\mathring{p}_1 + i\frac{m}{2})\rho^M \frac{z^M}{z^3} & 0 & 2\frac{z^M}{z^4}\rho^M\left(\partial_s x - m\frac{x}{2}\right) + W_- & -\mathring{p}_0 \mathbb{1} - A^T \\
0 & -\rho_M^\dagger(\mathring{p}_1 + i\,\frac{m}{2})\frac{z^M}{z^3} & -\mathring{p}_0 \mathbb{1} + A & -2\frac{z^M}{z^4}\rho_M^\dagger\left(\partial_s x^* - m\frac{x}{2}^*\right) - W_-^\dagger
\end{pmatrix}
$$

where $W_\pm = \frac{r}{2}\left(\hat{p}_0^2 \pm i\,\hat{p}_1^2\right)\rho^M u_M$, $|r| = 1$, and $\hat{p}_\mu \equiv \frac{2}{a}\sin\frac{p_\mu a}{2}$. It is such that

- Lattice perturbation theory reproduces its continuum counterpart for $a \to 0$
- It **preserves** the **SO(6)** global symmetry, breaks the SO(2).

The simulation: parameter space

- In the continuum model there are two parameters, $g = \frac{\sqrt{\lambda}}{4\pi}$ and $m \sim P_+$. In perturbation theory divergences cancel, dimensionless quantities are pure functions of the (bare) coupling

$$F = F(g)$$

- Our discretization cancels (1-loop) divergences, and reproduces the 1-loop cusp anomaly. Assume it is true nonperturbatively, for lattice regularization.

  Only additional scale: lattice spacing a.

  Three dimensionless (input) parameters:

$$g\,, \qquad N \equiv \frac{L}{a}\,, \qquad M \equiv m\,a$$

  Therefore

$$F_{\mathrm{LAT}} = F_{\mathrm{LAT}}(g, N, M)$$

Line of constant physics

The continuum limit must be taken through a series of simulations in a controlled way: lattice spacing $a \to 0$ while physical (renormalized) quantities should be kept constant.

Line of constant physics: curves in the bare parameter space, where dimensionless physical quantities are kept fixed as a changes.

In the continuum, “effective” masses undergo a *finite* renormalization

[Basso 2010]
[Giombi Ricci Roiban Tseytlin 2010]

$$m_x^2(g) = \frac{m^2}{2}\Big(1 - \frac{1}{8\,g} + \mathcal{O}(g^{-2})\Big) \qquad (\star)$$

The dimensionless physical quantity to keep constant when $a \to 0$ is

$$L^2\, m_x^2 = \text{const}\,, \qquad \text{leading to} \qquad (L\,m)^2 \equiv (N M)^2 = \text{const}\,,$$

if $(\star)$ is still true on the lattice and g is not (infinitely) renormalized.

Continuum limit $a \to 0$

We assume that, on the lattice, no further scale but a is present.

A generic observable

finite lattice spacing (~a) effects

finite volume (~ m L) effects

$$F_{\rm LAT} = F_{\rm LAT}(g, N, M) = F(g) + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(e^{-MN}\right)$$

where

$$g = \frac{\sqrt{\lambda}}{4\pi}\,, \qquad N = \frac{L}{a}\,, \qquad M = a\,m\,.$$

Recipe:

- fix g
- fix MN, large enough so to to keep small finite volume effects
- evaluate $F_{\rm LAT}$ for $N = 6, 8, 10, 12, 16, \cdots$
- obtain $F(g)$ extrapolating to $N \to \infty$.

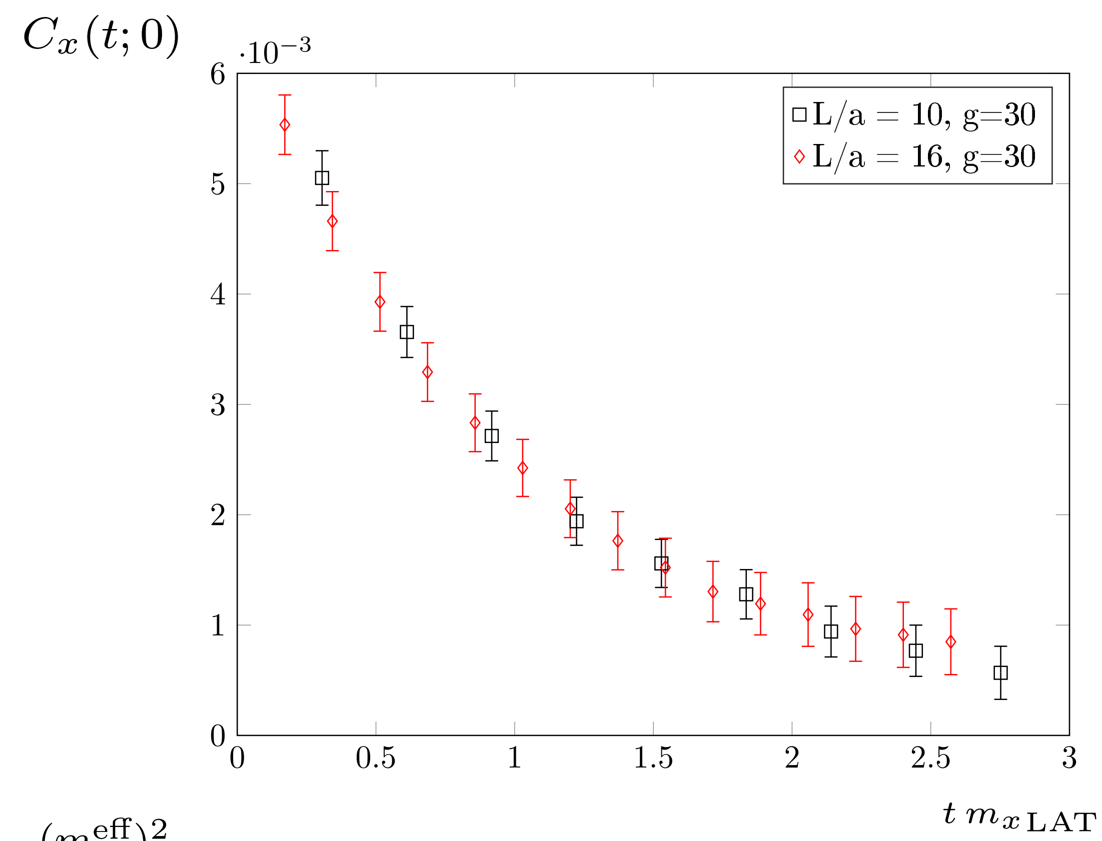
Measurement I: $\langle x, x^* \rangle$ correlator

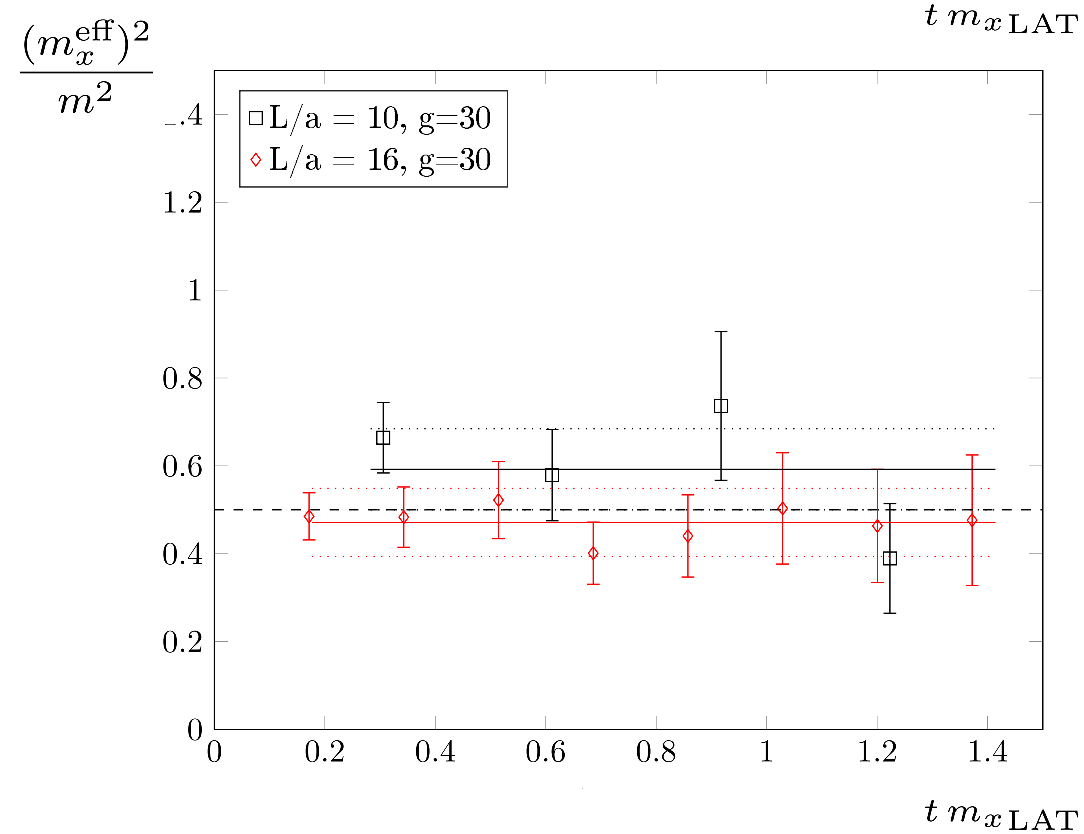
From the correlator of the x fields

$$
\begin{aligned}
C_x(t;0) &= \sum_{s_1,\, s_2} \langle x(t,\, s_1) x^*(0,\, s_2) \rangle \\
&= \sum_n |c_n|^2 e^{-tE_x(0;\,n)} \\
&\overset{t \gg 1}{\approx} e^{-t\, m_{x\,\mathrm{LAT}}}
\end{aligned}
$$

extract the x-mass

$$
\begin{aligned}
m_{x\,\mathrm{LAT}} &= \lim_{t\to\infty} m_x^{\mathrm{eff}} \\
&\equiv \lim_{t\to\infty} \frac{1}{a} \log \frac{C_x(t;0)}{C_x(t+a;\,0)}
\end{aligned}
$$

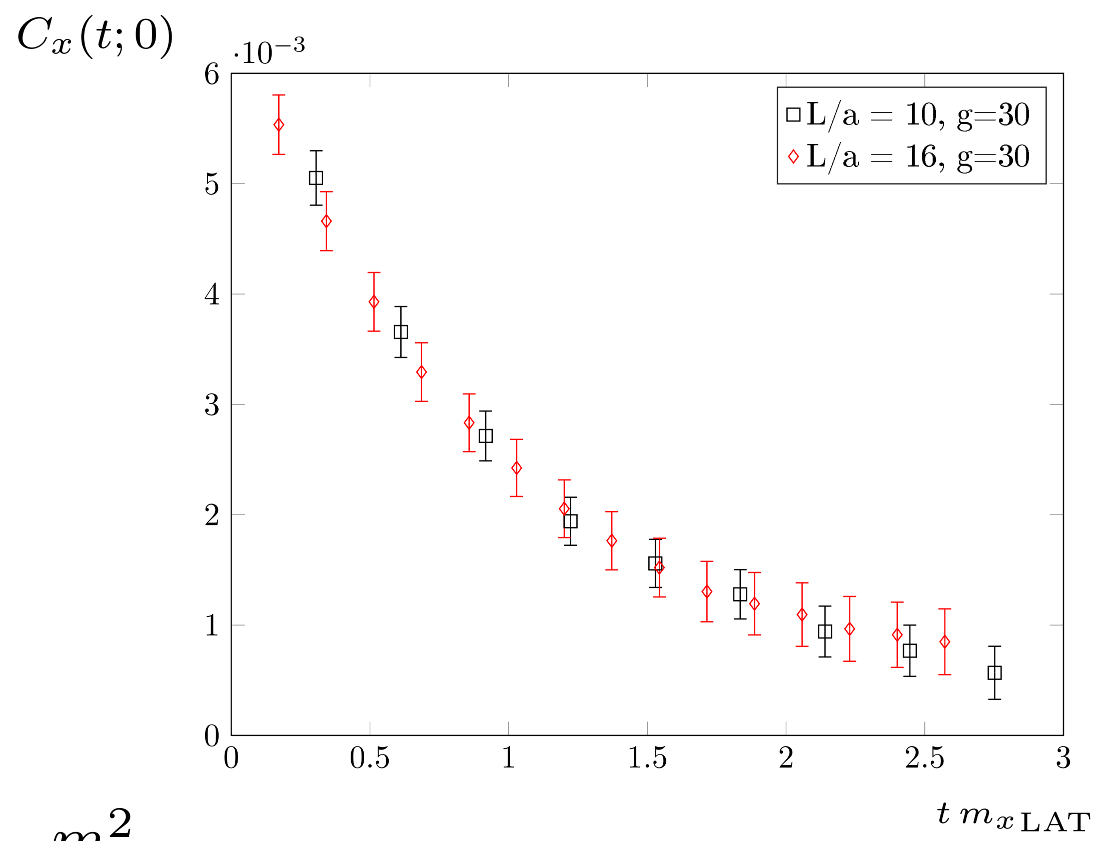


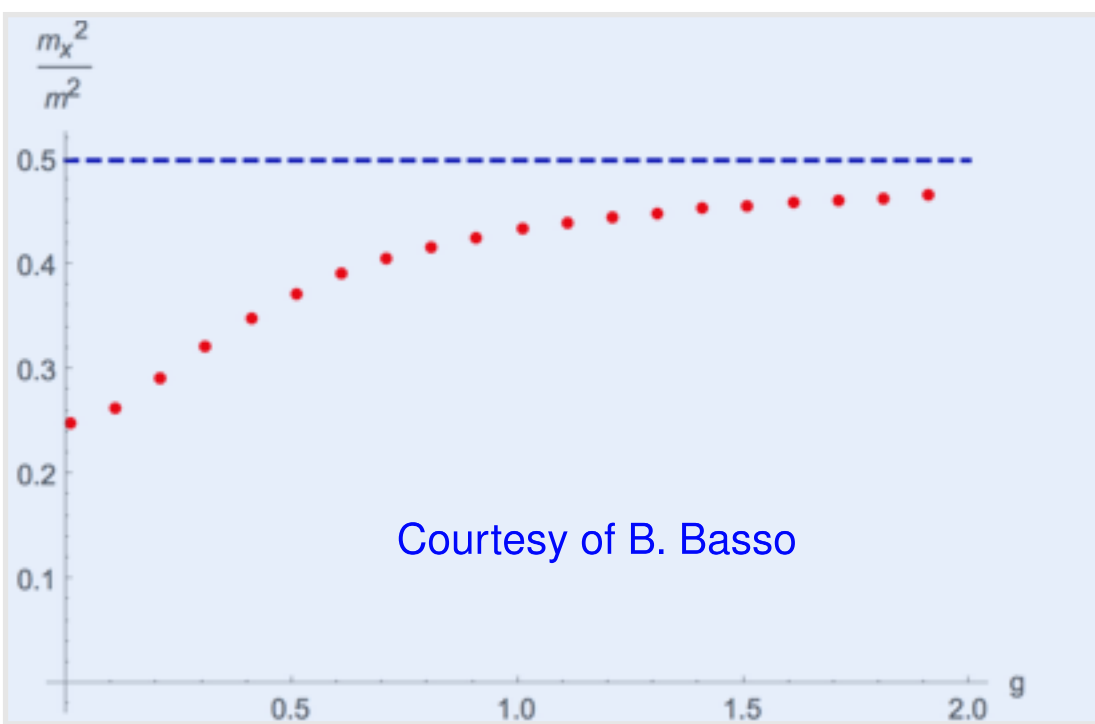


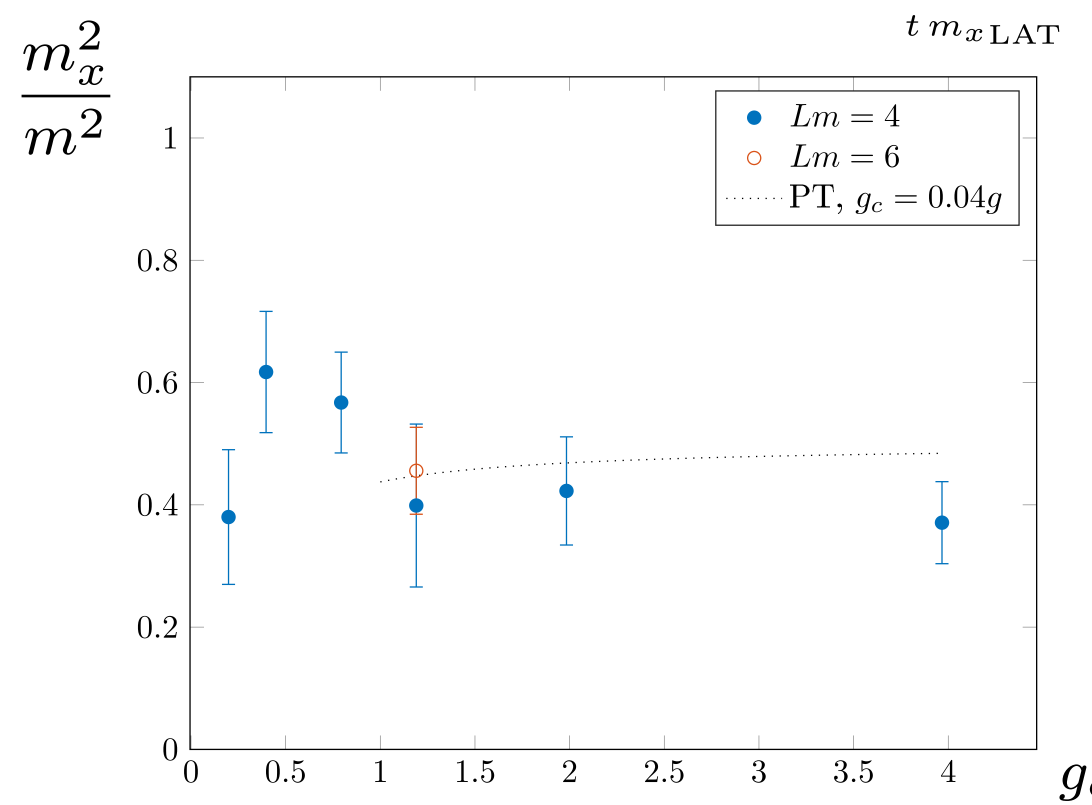
Measurement I: $\langle x, x^* \rangle$ correlator

From the correlator of the x fields

$$C_x(t;0) = \sum_{s_1,\, s_2} \langle x(t,\, s_1) x^*(0,\, s_2) \rangle$$
$$= \sum_n |c_n|^2 e^{-tE_x(0;\,n)}$$
$$\overset{t \gg 1}{\approx} e^{-t\, m_{x\,\mathrm{LAT}}}$$







Consistent with large g prediction, no clear signal of bending down.

No infinite renormalization occurring.

This corroborates our choice of line of constant physics.

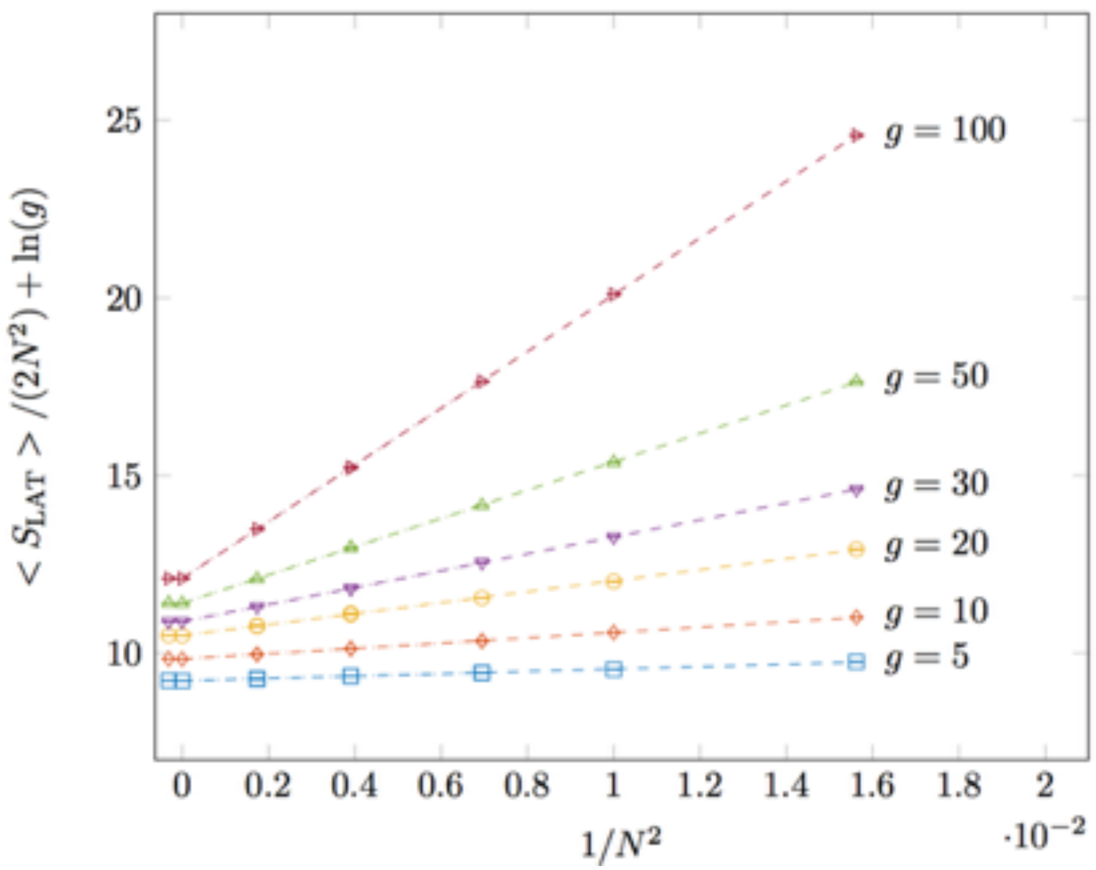
Measurement II: (derivative of the) cusp anomaly

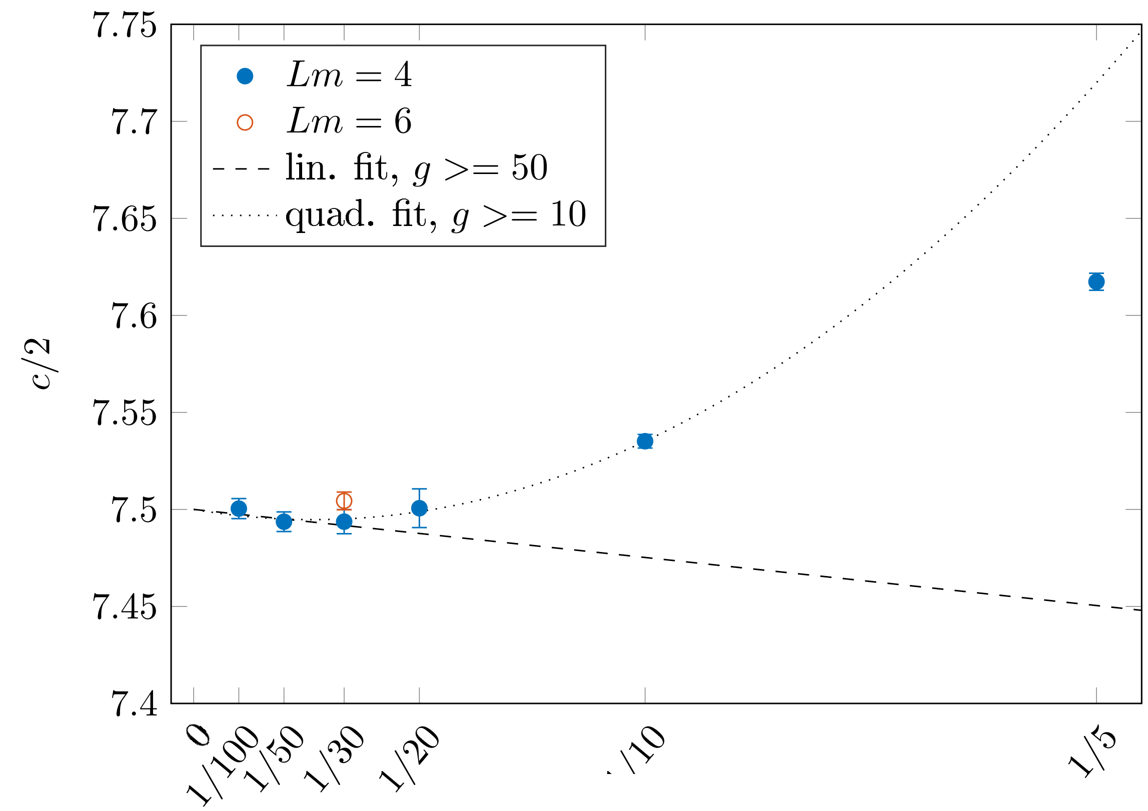
In measuring $\langle S_{\rm cusp} \rangle \equiv g\, \frac{V_2\, m^2}{8}\, f'(g)$ quadratic divergences appear.

At large g,

$$\langle S_{\rm LAT} \rangle \equiv g\, \frac{N^2\, M^2}{4}\, 4 + \frac{c}{2}(2N^2)$$

where $c = \mathsf{n}_{bos} = 8 + 7 = 15.$





This is because $\langle S \rangle = -\frac{\partial \ln Z}{\partial \ln g}$ and $Z \sim \Pi_{\mathsf{n}_{\rm bos}} (\det g\, \mathcal{O})^{-\frac{1}{2}}$.

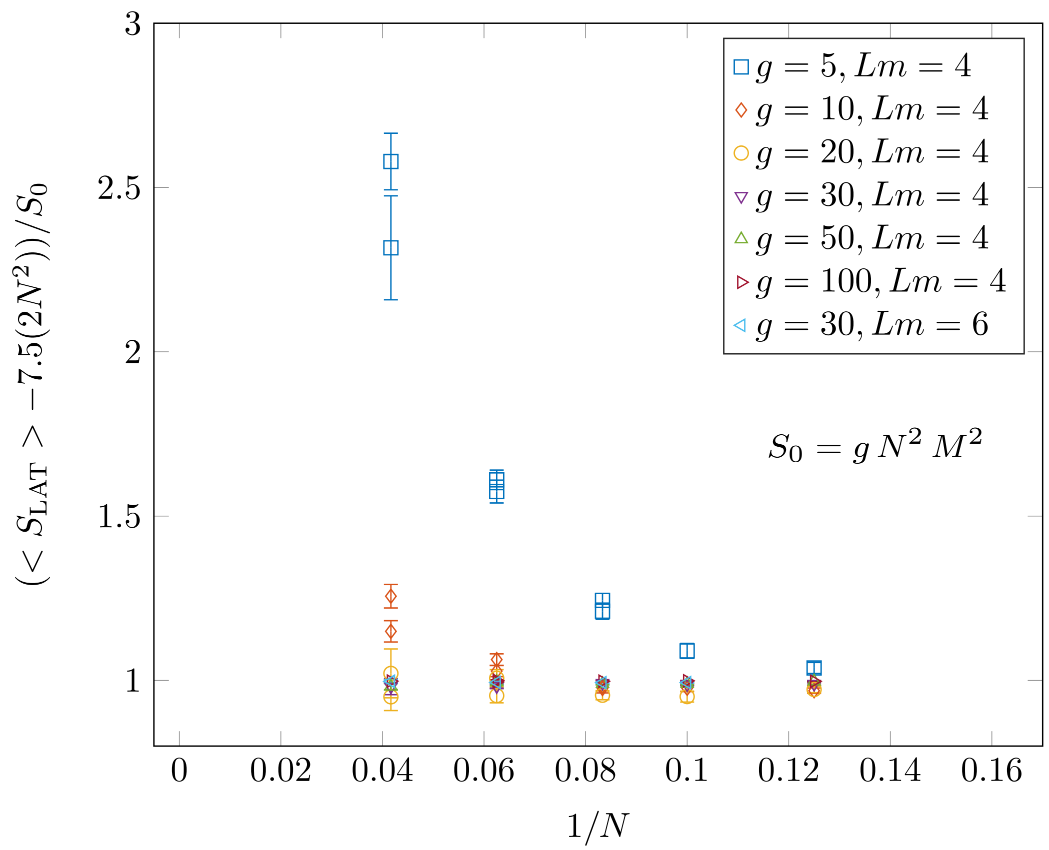
Therefore a factor proportional to $g^{-\frac{(2N^2)}{2}}$ for each bosonic field species.

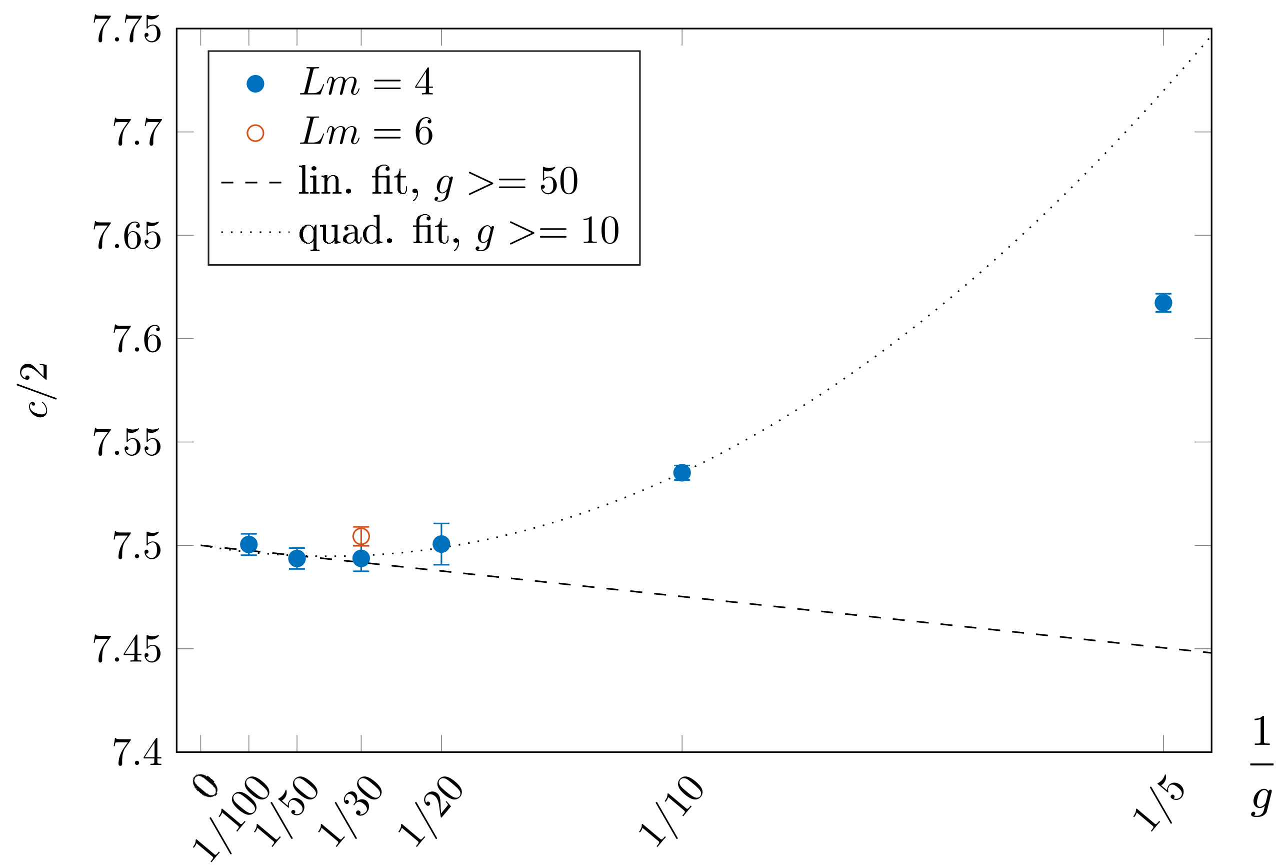
In lattice codes, coupling omitted from the (pseudo)fermionic part of the action.

Measurement II: (derivative of the) cusp anomaly

Divergences appear also at finite g,

$$\langle S_{\rm LAT} \rangle \equiv g \, \frac{N^2 \, M^2}{4} \, f'(g)_{\rm LAT} + \frac{c(g)}{2} (2N^2)$$





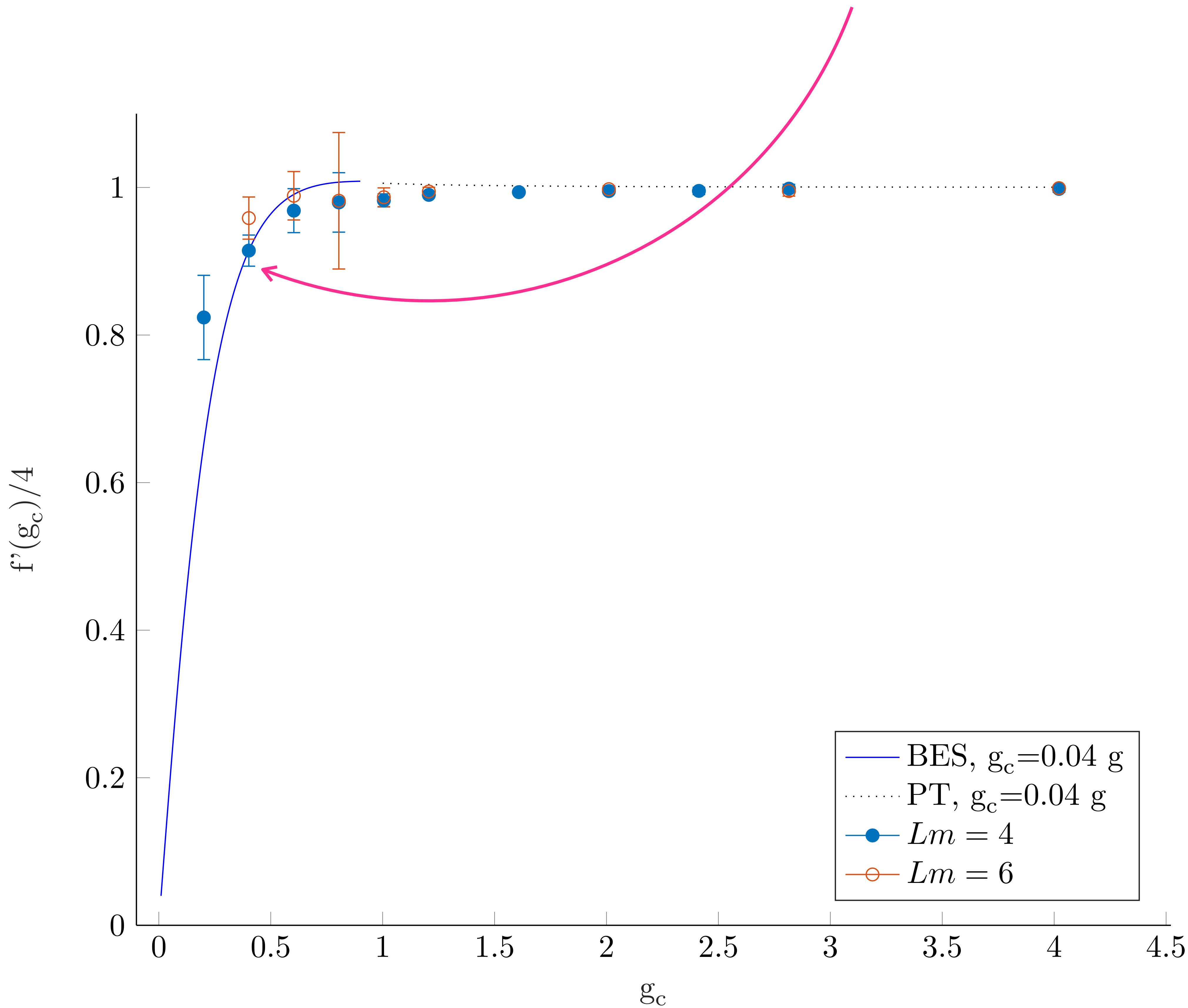
In continuum perturbation theory dim. reg. set them to zero.

Here, expected mixing of the Lagrangian with lower dimension operator

$$\mathcal{O}(\phi)_r = \sum_{\alpha:[O_\alpha]\leq D} Z_\alpha \, \mathcal{O}_\alpha(\phi) \,, \qquad Z_\alpha \sim \Lambda^{(D-[\mathcal{O}_\alpha])} \sim a^{-(D-[\mathcal{O}_\alpha])}$$

Measurement II: (derivative of the) cusp anomaly

To compare, assume $g = \alpha\, g_c$: then from $f'(g) = f'(g_c)_c$ is $g_c = 0.04g$.



The phase

After linearization $\mathcal{L}_F = \psi^T \, \mathcal{O}_F \, \psi$, integrating fermions leads to a complex Pfaffian $\mathrm{Pf}\, O_F = |(\det O_F)^{\frac{1}{2}}|\, e^{i\theta}$.

The phase is encoded in the linearization: we deal with a fermionic hermitian bilinear $b \sim \eta^2$ whose corresponding quartic interaction

$$e^{-\mathcal{L}_4^{\mathrm{ferm}}} = e^{-\frac{b^2}{4\,a}} = \int dx \, e^{-a\,x^2 + i\,b\,x}$$

comes in the exponential as with the "wrong" sign.

The phase can be treated via reweighting: incorporate the non positive part of the Boltzmann weight into the observable

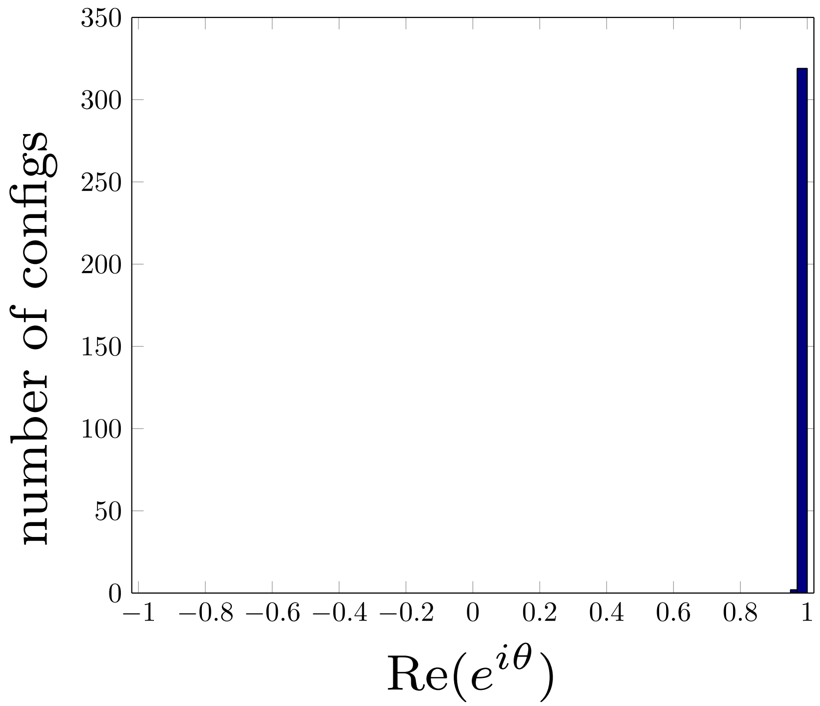
$$\langle \mathcal{O} \rangle_{\mathrm{reweight}} = \frac{\langle \mathcal{O}\, e^{i\theta} \rangle_{\theta=0}}{\langle e^{i\theta} \rangle_{\theta=0}}$$

It gives meaningful results as long as the phase does not average to zero.

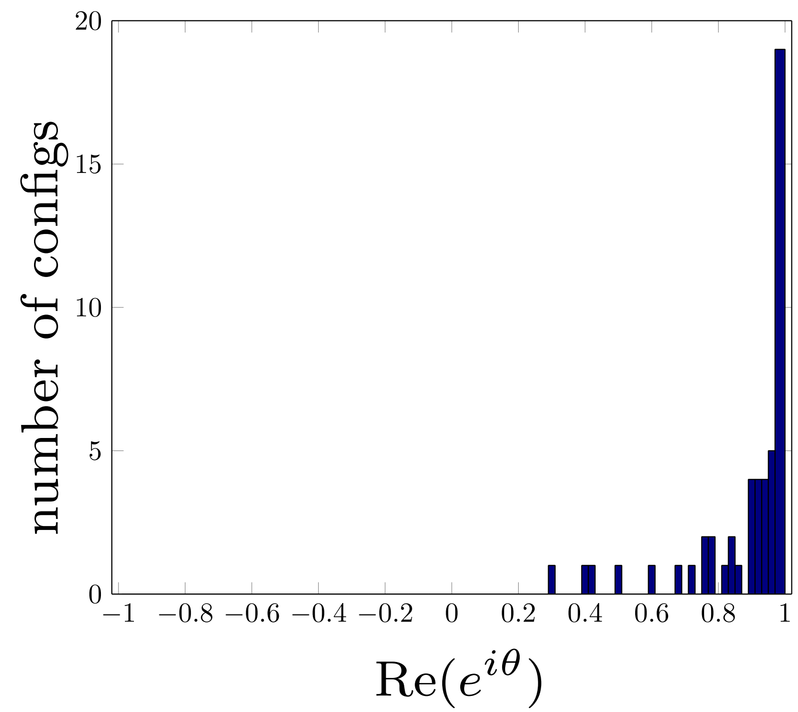
The phase

In the interesting ($g = 1$) region the phase has a flat distribution.

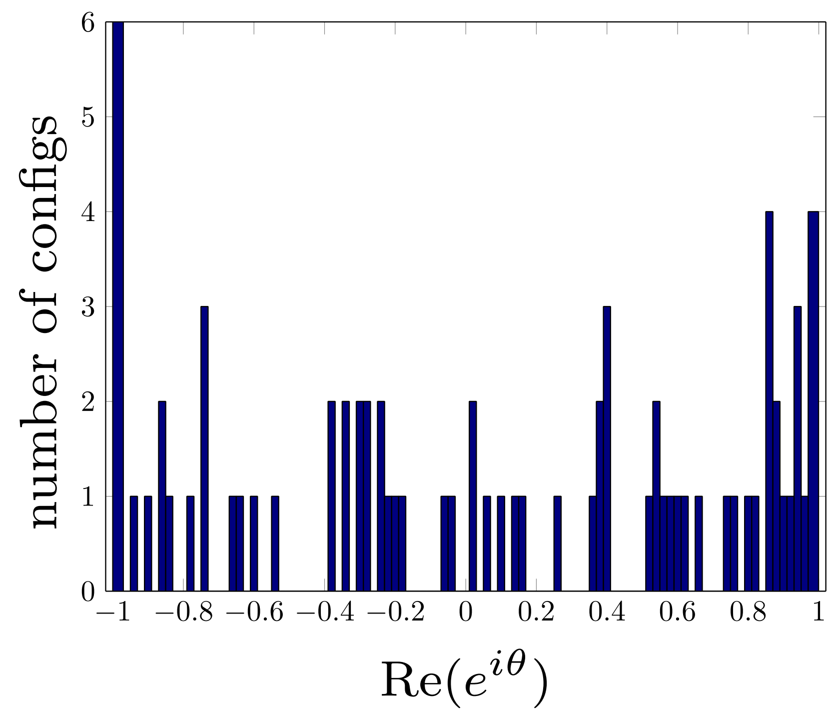
g=30



g=5



g=1



Alternative algorithms: active field of study, no general proof of convergence.

Alternative linearization

We identified a problem in the ”wrong sign” of the quartic fermionic interaction.

Consider a simple SO(4) invariant four-fermion interaction **[Catterall 2015]**

$$\mathcal{L}_{4F} = \frac{1}{2}\epsilon_{abcd}\,\psi^a(x)\,\psi^b(x)\,\psi^c(x)\,\psi^d(x) \equiv \Sigma^{ab}\,\widetilde{\Sigma}^{ab}$$

where $\Sigma^{ab} = \psi^a\psi^b\,,\;\widetilde{\Sigma}^{ab} = \frac{1}{2}\epsilon_{abcd}\,\psi^c\,\psi^d$.

Introducing the (anti)self-dual fermion bilinears

$$\Sigma^{ab}_{\pm} = \frac{1}{2}\left(\Sigma^{ab} \pm \frac{1}{2}\,\epsilon_{abcd}\,\Sigma^{cd}\right)$$

one can rewrite

$$\mathcal{L}_{4F} = \pm\, 2\left(\Sigma^{ab}_{\pm}\right)^2$$

just exploiting the Graßmann character of the underlying fermions.

$$\pm\Sigma^{ab}_{\pm}\Sigma^{ab}_{\pm} = \pm\frac{1}{4}\left[\Sigma^{ab} \pm \frac{1}{2}\epsilon_{abcd}\Sigma^{cd}\right]\left[\Sigma^{ab} \pm \frac{1}{2}\epsilon_{abef}\Sigma^{ef}\right]$$

$$= \pm\frac{1}{4}\left[\Sigma^{ab}\Sigma^{ab} \pm \frac{1}{2}\epsilon_{abcd}\left(\Sigma^{ab}\Sigma^{cd} + \Sigma^{cd}\Sigma^{ab}\right) + \frac{1}{4}\epsilon_{abcd}\,\epsilon_{abef}\,\Sigma^{cd}\Sigma^{ef}\right]$$

since $\psi^a\psi^b\psi^a\psi^b = 0$

$$= \frac{1}{4}\left(\Sigma^{ab}\widetilde{\Sigma}^{ab} + \widetilde{\Sigma}^{ab}\Sigma^{ab}\right) = \frac{1}{2}\Sigma^{ab}\widetilde{\Sigma}^{ab}$$

Alternative linearization

In our case

$$\mathcal{L}_{F4} = -\frac{1}{z^2}(\eta^2)^2 + \frac{1}{z^2}(i\,\eta_i(\rho^{MN})^i{}_j n^N \eta^j)^2$$

one analogously defines - notice $(\rho^M)^{im}(\rho^M)^{kn} = 2\epsilon^{imkn}$ - the bilinears

$$\Sigma_i{}^j = \eta_i\eta^j \qquad \widetilde{\Sigma}_j{}^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l$$

and again introduces $\Sigma_\pm{}_i^j = \Sigma_i^j \pm \widetilde{\Sigma}_i^j$ to rewrite

$$\mathcal{L}_{F4} = -\frac{1}{z^2}(\eta^2)^2 \mp \frac{2}{z^2}(\eta^2)^2 \mp \frac{1}{z^2}\Sigma_\pm{}_i^j \Sigma_\pm{}_j^i$$

Now choose the good sign ($-$).

The new set of Yukawa terms (now $1+16$ real auxiliary fields)

$$\mathcal{L}_{F4} \longrightarrow \frac{12}{z}\eta^2\phi + 6\phi^2 + \frac{2}{z}\Sigma_+{}_j^i \phi_i^j + \phi_j^i\phi_i^j$$

ensures the full Lagrangian to be hermitian, and a full (including auxiliary fields) non negative $\det O_F$.

Alternative linearization

A Γ_5-hermiticity and antisymmetry

$$O_F^\dagger = \Gamma_5\, O_F\, \Gamma_5\,, \qquad\qquad O_F^T = -O_F$$

with $\Gamma_5^\dagger \Gamma_5 = \mathbb{1}\,,\ \Gamma_5^\dagger = -\Gamma_5$ ensures $\det O_F{}^W$ to be real and non-negative.

Pfaffian is real, $(\mathrm{Pf} O_F)^2 = \det O_F \geq 0$, but not positive definite, $\mathrm{Pf} O_F = \pm \det O_F$.

Gain in computational costs: for large values of N (finer lattices) the algorithm for evaluating complex determinants is very inefficient. Now just a sign flip.

$$\langle \mathcal{O} \rangle_{\mathrm{reweight}} = \frac{\langle \mathcal{O}\, e^{i\theta} \rangle_{\theta=0}}{\langle e^{i\theta} \rangle_{\theta=0}} \qquad \longrightarrow \qquad \langle \mathcal{O} \rangle_{\mathrm{reweight}} = \frac{\langle \mathcal{O}\, w \rangle_{\sqrt{\det O_F}}}{\langle w \rangle_{\sqrt{\det O_F}}}$$

where $w = \pm 1$, and $\sqrt{\det O_F} = (\det O_F^\dagger\, O_F)^{\frac{1}{4}}$.

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where $w = \pm 1$, and $\sqrt{\det O_F} = (\det O_F^\dagger \, O_F)^{\frac{1}{4}}$.

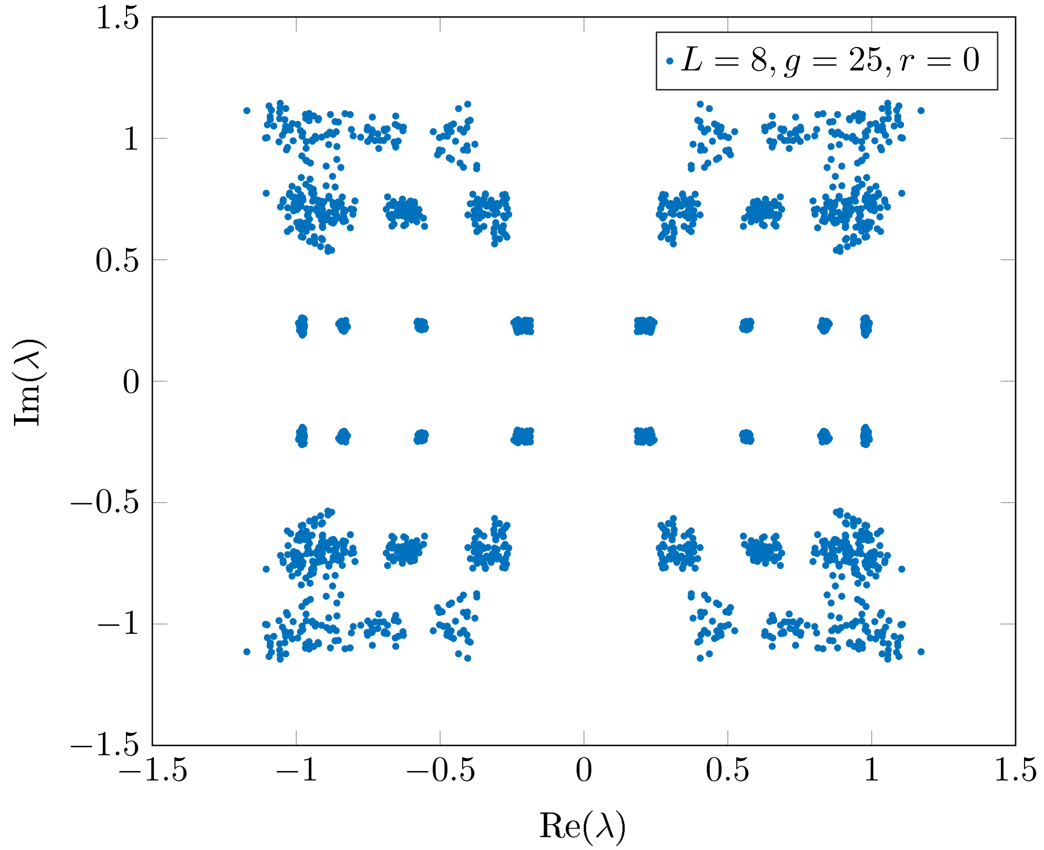
In simpler models with four-fermion interactions, similar manipulations ensure a definite positive Pfaffian. There real, antisymmetric operator with doubly degenerate eigenvalues: quartets $(ia, ia, -ia, -ia)$, $a \in \mathbb{R}$.

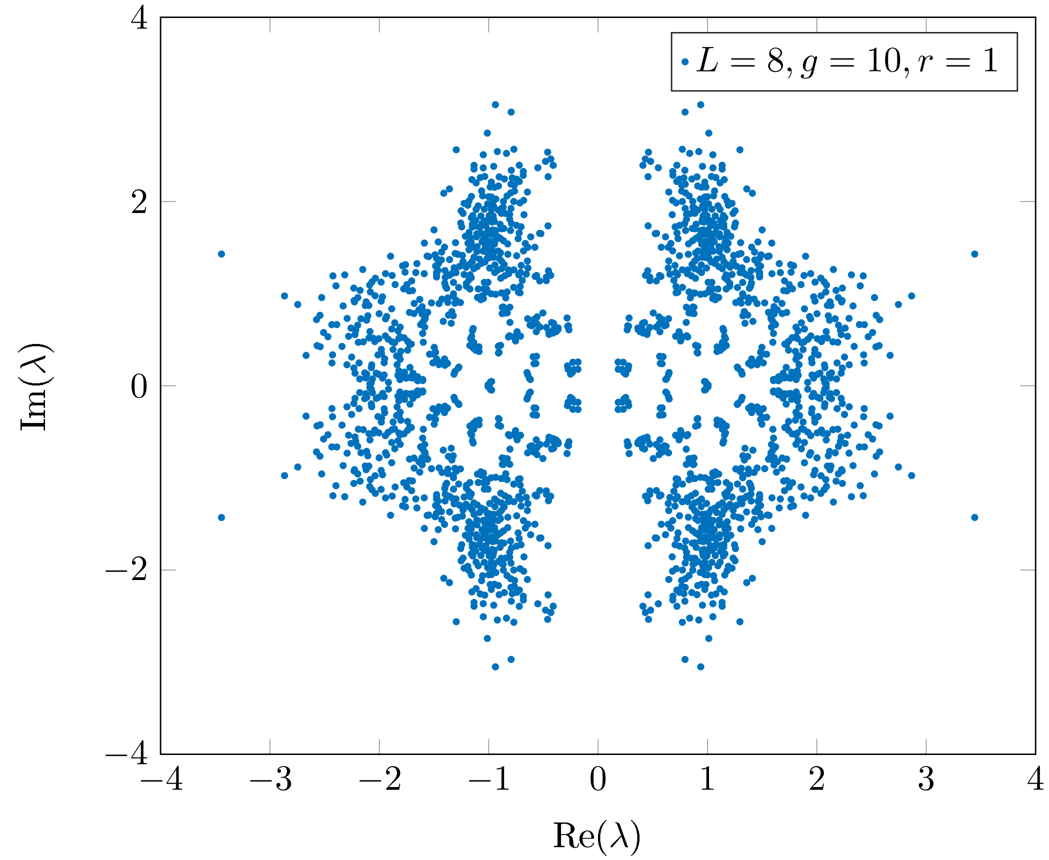
[Catterall 2016, Catterall and Schaich 2016]

Spectrum of O_F

From Γ_5-hermiticity and antisymmetry,

$$\mathcal{P}(\lambda) = \det(O_F - \lambda \mathbb{1}) = \det(\Gamma_5 \, (O_F - \lambda \mathbb{1}) \, \Gamma_5)$$
$$= \det(O_F^\dagger + \lambda \mathbb{1}) = \det(O_F + \lambda^* \mathbb{1})^* = \mathcal{P}(-\lambda^*)^*$$





Spectrum characterized by quartets $\{\lambda, -\lambda^*, -\lambda, \lambda^*\}$.

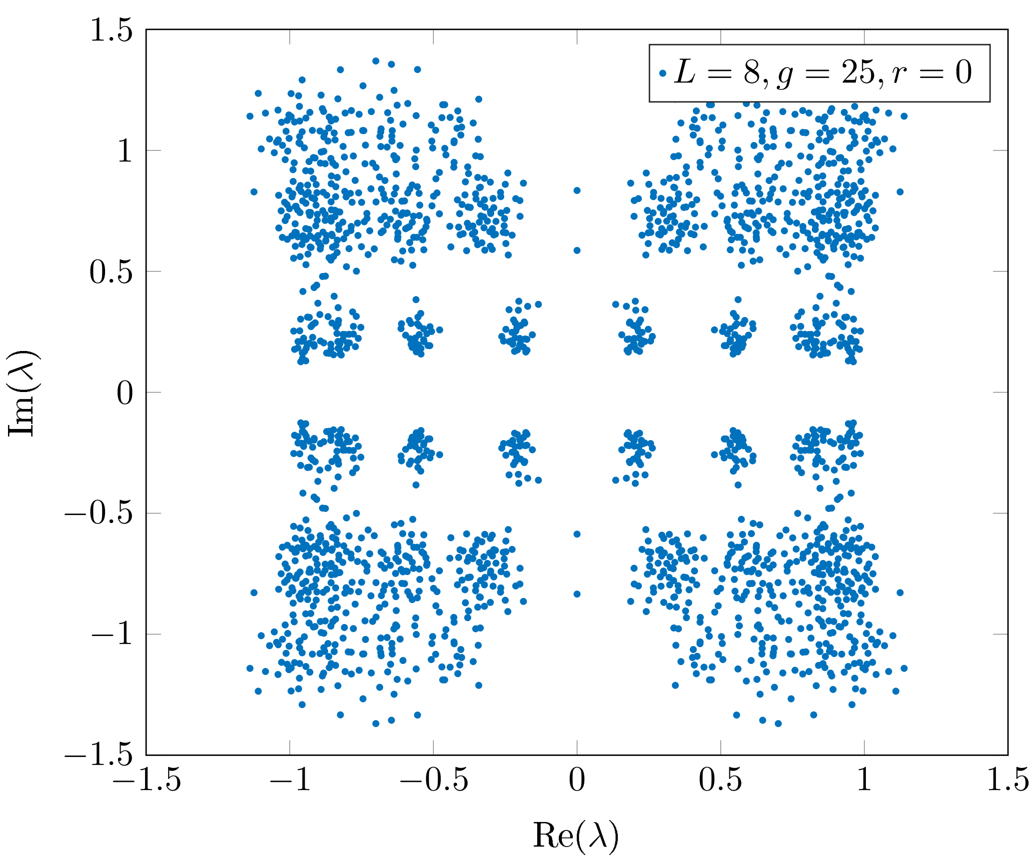
$$\det O_F = \prod_i |\lambda_i|^2 \, |\lambda_i|^2 \quad \longrightarrow \quad \mathrm{Pf}(O_F) = \pm \prod_i |\lambda_i|^2$$

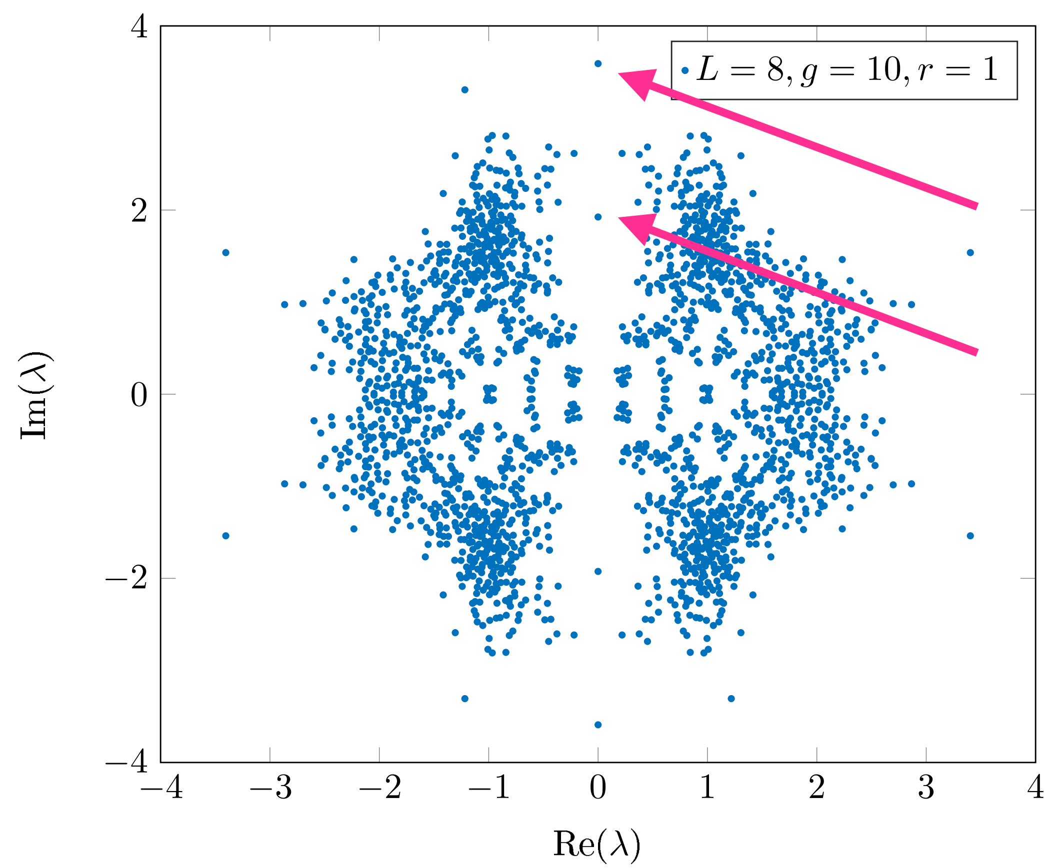
Choosing a starting configuration with positive Pfaffian, no sign change possible.

Spectrum of O_F

From Γ_5-hermiticity and antisymmetry,

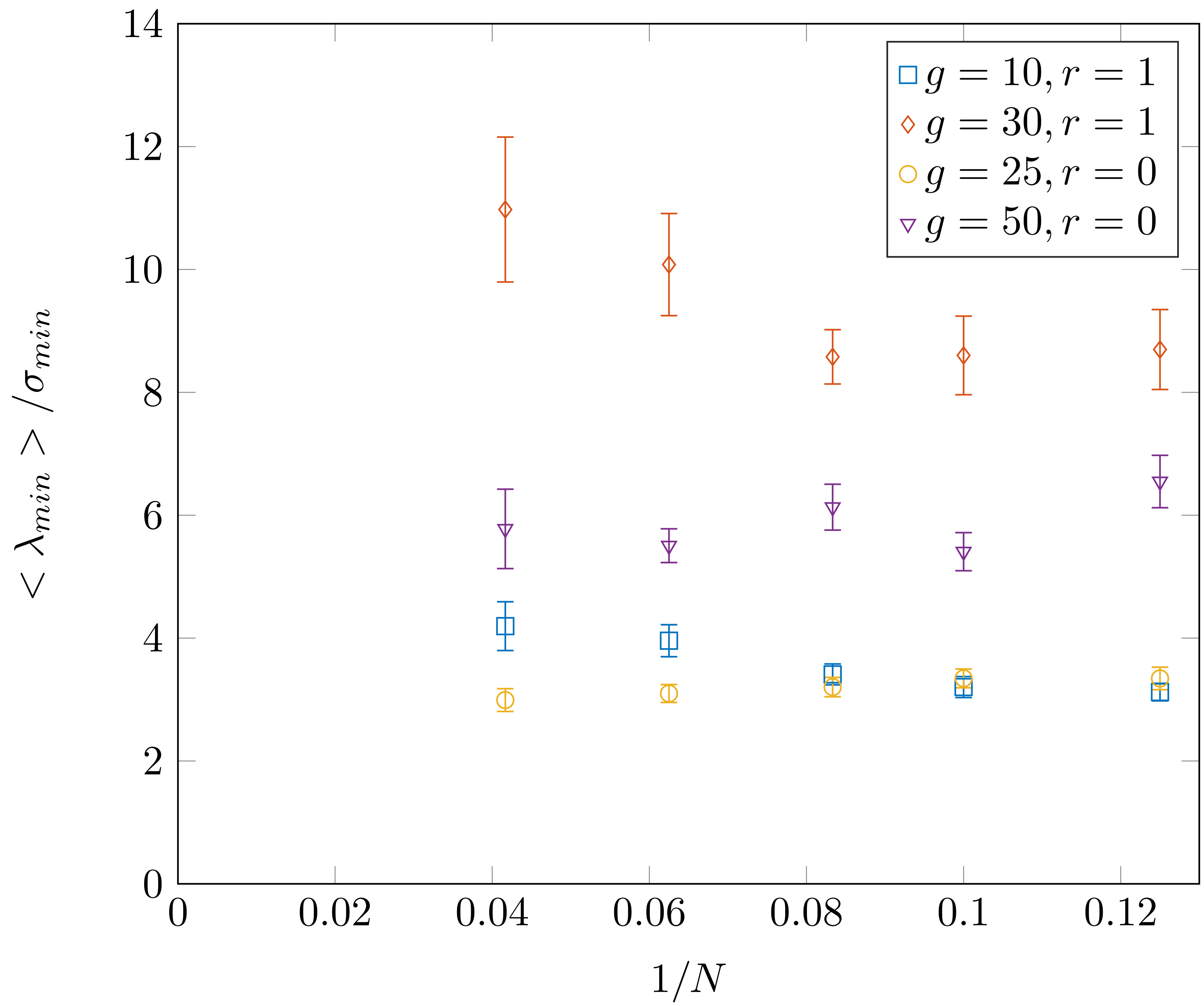
$$\mathcal{P}(\lambda) = \det(O_F - \lambda\mathbb{1}) = \det(\Gamma_5\,(O_F - \lambda\mathbb{1})\,\Gamma_5)$$
$$= \det(O_F^\dagger + \lambda\mathbb{1}) = \det(O_F + \lambda^*\mathbb{1})^* = \mathcal{P}(-\lambda^*)^*$$





For $\lambda = \pm\,\lambda^*$, no four-fold property: due to zero crossings, Pfaffian may change sign.

Alternative linearization



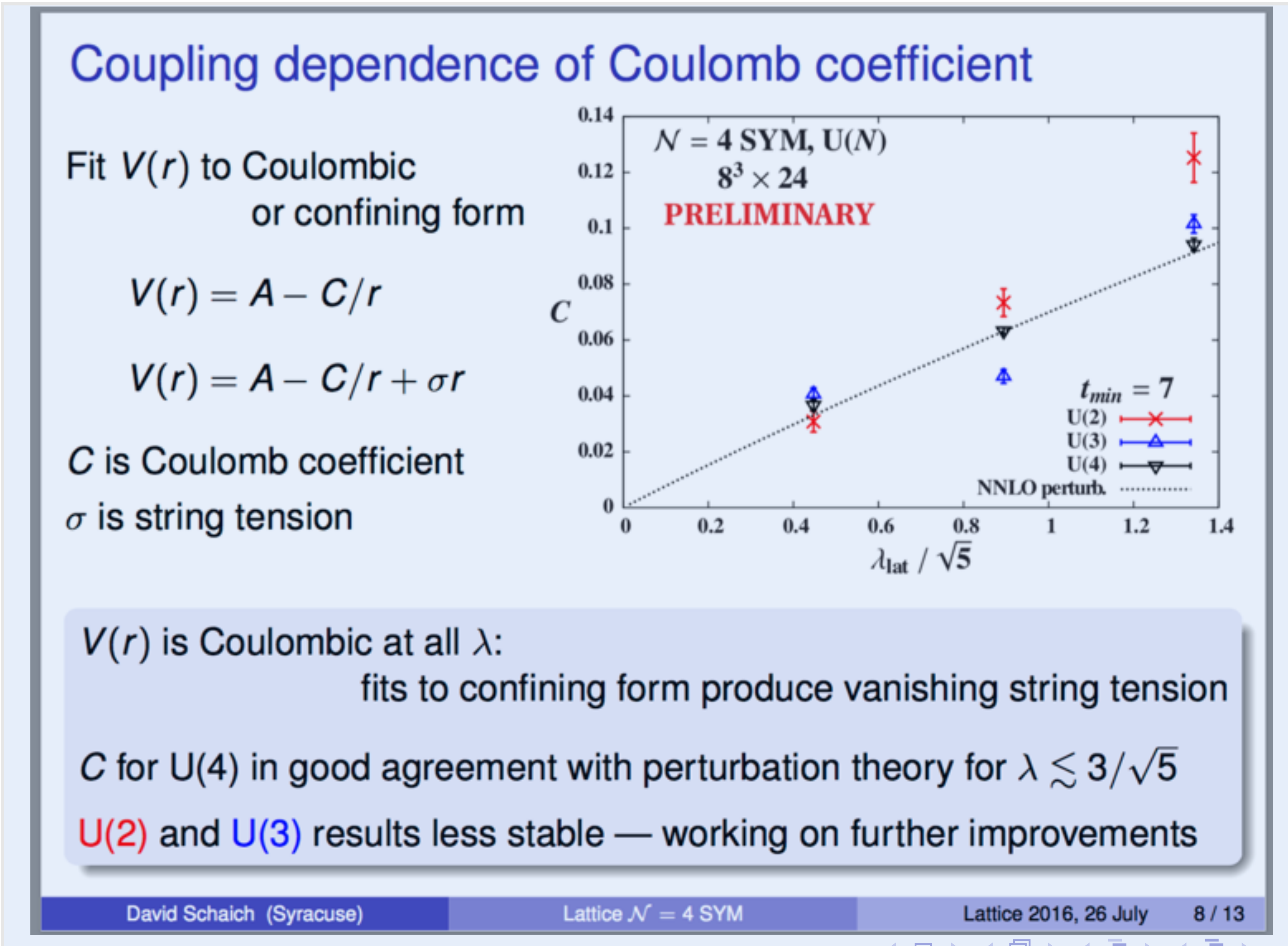
Eigenvalue distribution of fermionic operator well separated from zero,
no sign problem for $g \geq 10$, where nonperturbative physics is captured.

On the CFT side

The control is in the perturbative region (matching with NNLO).

Strong sign problem at strong coupling ($\lambda \gg 1$).

David Schaich at Lattice 2016



Concluding remarks

Solving a non-trivial 4d QFT is **hard** $\longrightarrow$ reduce the problem via AdS/CFT:

solve (finding a good regulator for) a non-trivial 2d QFT.

For Green-Schwarz string worldsheet in AdS backgrounds, it is possible to improve perturbative techniques e.g. via cross-fertilization of QFT methods.

The model is amenable to study using lattice QFT techniques (Wilson-like discretizations, standard simulation algorithms). Interesting beyond string community.

Non-perturbative definition of string theory? Not quite *yet*.
Still, suitable framework for first principle statements (proofs of AdS/CFT), and potentially efficient tool in numerical holography.

Future

- All correlators, different backgrounds (e.g. for ABJM cusp).
- Further observables? ...
- ...

Thanks for your attention.

A remark on numerics

The most difficult part of the algorithm is the inversion of the fermionic matrix

$$|\mathrm{Pf}\, O_F| \equiv (\det O_F^\dagger O_F)^{\frac{1}{4}} \equiv \int d\zeta d\bar{\zeta}\, e^{-\int d^2\xi\, \bar{\zeta}\, (O_F^\dagger O_F)^{-\frac{1}{4}}\, \zeta} .$$

The RHMC (Rational Hybrid Montecarlo) uses a rational approximation

$$\bar{\zeta}\, (O_F^\dagger O_F)^{-\frac{1}{4}}\, \zeta = \alpha_0\, \bar{\zeta}\, \zeta + \sum_{i=1}^{P} \bar{\zeta}\, \frac{\alpha_i}{O_F^\dagger O_F + \beta_i}\, \zeta$$

with α_i and β_i tuned by the range of eigenvalues of O_F.

Defining $s_i \equiv \frac{1}{O_F^\dagger O_F + \beta_i}\, \zeta$, one solves

$$(O_F^\dagger O_F + \beta_i)\, s_i = \zeta\,, \qquad\qquad i = 1, \ldots, P.$$

with a (multi-shift conjugate) solver for which

$$\text{number of iterations} \sim \lambda_{\min}^{-1}$$

In our case the spectrum of O_F has very small eigenvalues.

And: $O_F = \Big| \begin{matrix} i\partial_t \\ i\frac{z^M}{z^3}\rho^M \left(\partial_s - \frac{m}{2}\right) \end{matrix}$

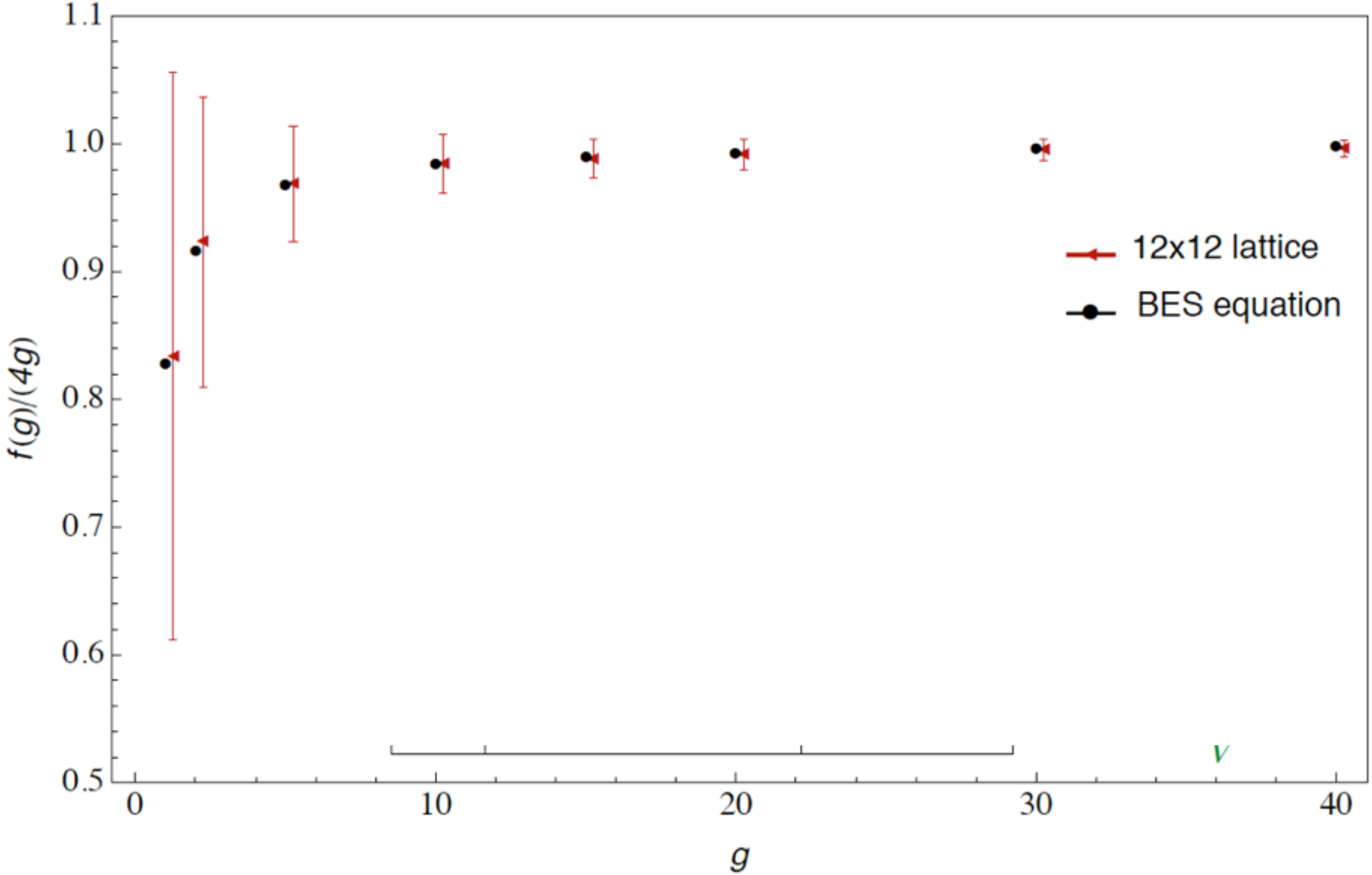
Parameters of the simulations

g	$T/a \times L/a$	Lm	am	τ_{int}^{S}	$\tau_{\text{int}}^{m_x}$	statistics [MDU]
5	16×8	4	0.50000	0.8	2.2	900
	20×10	4	0.40000	0.9	2.6	900
	24×12	4	0.33333	0.7	4.6	900,1000
	32×16	4	0.25000	0.7	4.4	850,1000
	48×24	4	0.16667	1.1	3.0	92,265
10	16×8	4	0.50000	0.9	2.1	1000
	20×10	4	0.40000	0.9	2.1	1000
	24×12	4	0.33333	1.0	2.5	1000,1000
	32×16	4	0.25000	1.0	2.7	900,1000
	48×24	4	0.16667	1.1	3.9	594,564
20	16×8	4	0.50000	5.4	1.9	1000
	20×10	4	0.40000	9.9	1.8	1000
	24×12	4	0.33333	4.4	2.0	850
	32×16	4	0.25000	7.4	2.3	850,1000
	48×24	4	0.16667	8.4	3.6	264,580
30	20×10	6	0.60000	1.3	2.9	950
	24×12	6	0.50000	1.3	2.4	950
	32×16	6	0.37500	1.7	2.3	975
	48×24	6	0.25000	1.5	2.3	533,652
	16×8	4	0.50000	1.4	1.9	1000
	20×10	4	0.40000	1.2	2.7	950
	24×12	4	0.33333	1.2	2.1	900
	32×16	4	0.25000	1.3	1.8	900,1000
	48×24	4	0.16667	1.3	4.3	150
50	16×8	4	0.50000	1.1	1.8	1000
	20×10	4	0.40000	1.2	1.8	1000
	24×12	4	0.33333	0.8	2.0	1000
	32×16	4	0.25000	1.3	2.0	900,1000
	48×24	4	0.16667	1.2	2.3	412
100	16×8	4	0.50000	1.4	2.7	1000
	20×10	4	0.40000	1.4	4.2	1000
	24×12	4	0.33333	1.3	1.8	1000
	32×16	4	0.25000	1.3	2.0	950,1000
	48×24	4	0.16667	1.4	2.4	541

Table 1: Parameters of the simulations: the coupling g, the temporal (T) and spatial (L) extent of the lattice in units of the lattice spacing a, the line of constant physics fixed by Lm and the mass parameter $M = am$. The size of the statistics after thermalization is given in the last column in terms of Molecular Dynamic Units (MDU), which equals an HMC trajectory of length one. In the case of multiple replica the statistics for each replica is given separately. The auto-correlation times τ of our main observables m_x and S are also given in the same units.

Previous study

[McKeown Roiban, arXiv: 1308.4875]



Boundary conditions

We use periodic BC for all the fields (antiperiodic temporal BC for fermions).

In the infinite volume limit BC should not play a substantial role

(unless what is studied is topological).

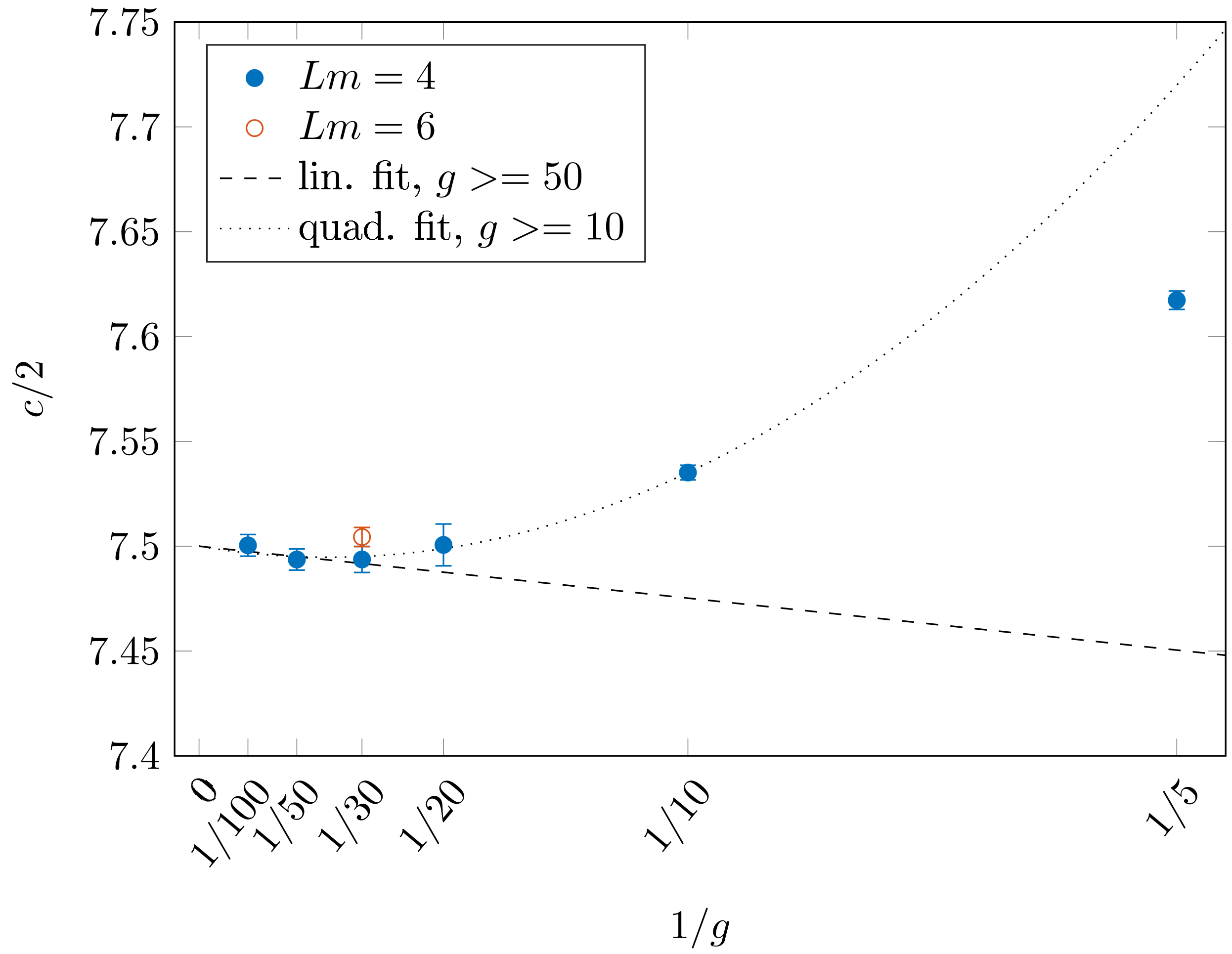
Finite volume effects $\sim e^{-m\,L} \equiv e^{-M\,N}$.

Most run are done at $M\,N = 4$ ($e^{-4} \simeq 0.02$),

some at $M\,N = 6$ ($e^{-6} \simeq 0.002$).

Appear to play a role only in evaluating

the coefficient of divergences.



Simulations with Dirichlet BC (which we are going to do) are not expected to change the outcome significantly.

Parameters of the simulations

g	$T/a \times L/a$	Lm	am	τ_{int}^{S}	$\tau_{\text{int}}^{m_x}$	statistics [MDU]
5	16×8	4	0.50000	0.8	2.2	900
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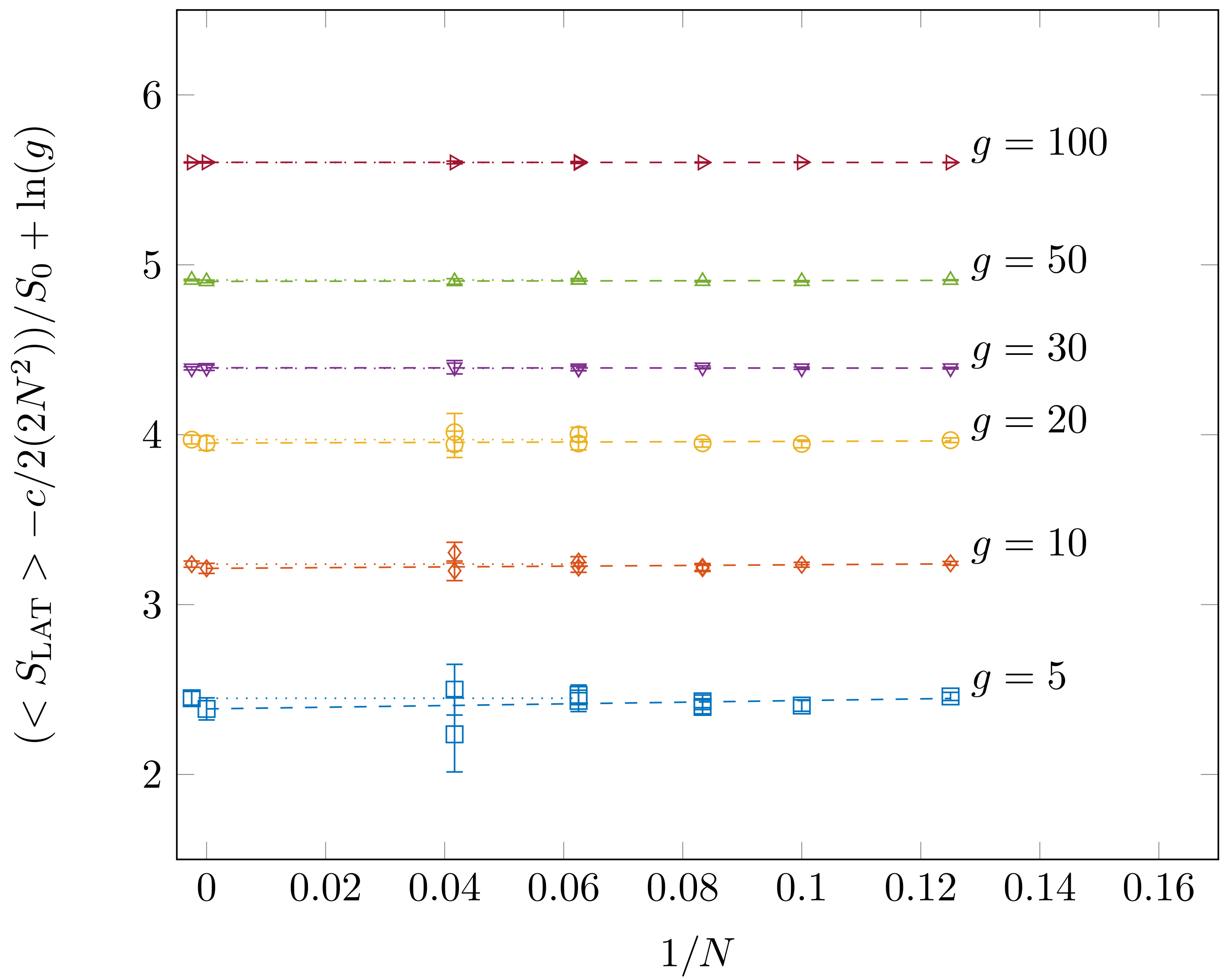
Measurement II: (derivative of the) cusp anomaly

We proceed subtracting the continuum extrapolation of $\frac{c}{2}$ multiplied by N^2: divergences appear to be completely subtracted, confirming their quadratic nature.

Errors are small, and do not diverge for $N \to \infty$.

Flatness of data points indicates very small lattice artifacts.



We can thus extrapolate at infinite N to show the continuum limit.