GENERALISED GEOMETRY IN STRING THEORY

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INTRODUCTION

- Renewed interest for extended symmetries in string theory and M-theory
  - the $10-d$ effective actions of type II and M-theory compactified on tori have large symmetry groups

dimension	7	6	5	4	3
glob. symm	SL(5)	SO(5,5)	$E_{6(6)}$	$E_{7(7)}$	$E_{8(8)}$

  - these are the U-duality groups
    - U-duality contains $O(d,d)$ (T-duality) as a subgroup
    - U-duality cannot be understood from symmetries of the conventional formulation of type II or M-theory
  - new formulations of type II and M-theory where
    - extended symmetries have a geometrical interpretation
    - role of the higher rank gauge fields

- Two different approaches
  - DFT/EFT: Double and/or Exceptional Field Theories

    [hull, zwiebach 09; hohm, hull, zwiebach 10; hohm, samtleben 13; ...]
    - U-duality covariant formulation
    - enlarge the space-time

glob. symm	SL(5)	SO(5,5)	$E_{6(6)}$	$E_{7(7)}$
dimension	7 + 10	6 + 16	5 + 27	4 + 56

    - diffeo and p-form gauge transformation correspond to coordinate transformations in the larger space
    - consistency of the generalised algebra reduces the coordinates to a physical subset $\leftrightarrow$ section condition

- GCG/EGG: Generalised Complex and/or Exceptional Generalised Geometry
  - enlarge the tangent space

[hitchin 02; gualtieri 04; hull 07; pacheco, waldram 08, ...]

	Riemannian	G C G	E G G
tangent b.	TM	$TM \oplus T^*M$	$T \oplus T^* \oplus \Lambda^{\pm} \oplus \Lambda^5 T^* \oplus (T^* \otimes \Lambda^6 T^*)$
structure group	$SO(6)$	$O(6,6)$ T-duality	$E_{7(7)}$ U-duality

  - charges of extended objects
  - the transition functions involve RR and NS potentials as generalised diffeomorphims
  - the structure group is the duality group on the internal manifold

- Unified framework to study
  - patterns of supergravities in different dimensions
  - compactifications, effective actions
  - gauge/gravity duality, consistent truncations

- In this talk
  - introduction to the main features of generalised geometry
  - applications to string theory and supergravity
    - supersymmetric compactifications with fluxes
    - geometry of the gauge/gravity duality
    - consistent truncations

GENERALISED GEOMETRIES

GENERALISED COMPLEX GEOMETRY IN d=6

[hitchin 02; gualtieri 04]

• We are interested in compactifications of the NS sector of 10d SUGRA on backgrounds of the type

$$M_{10} = M_4 \times M_6$$

- the symmetries of the bosonic sector are diffeomorphisms and B-field gauge transformations

$$\begin{aligned} \text{metric} \quad &\longrightarrow \quad \delta g = \mathcal{L}_v g \\ \text{dilaton} \quad &\longrightarrow \quad \delta\phi = \mathcal{L}_v \phi \\ \text{two-form} \quad &\longrightarrow \quad \delta B_{(\alpha)} = \mathcal{L}_v B_{(\alpha)} - \mathrm{d}\lambda_{(\alpha)} \end{aligned}$$

where v : diffeomorphisms

$\mathrm{d}\lambda$: B-field transformations

$$B_{(\alpha)} = B_{(\alpha)} - \mathrm{d}\Lambda_{(\alpha\beta)} \qquad U_\alpha \cap U_\beta$$

$$\mathrm{d}\lambda_{(\alpha)} = \mathrm{d}\lambda_{(\beta)} - \mathcal{L}_v \mathrm{d}\Lambda_{(\alpha\beta)}$$

$$\Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = g_{(\alpha\beta\gamma)} \mathrm{d} g_{(\alpha\beta\gamma)} \qquad U_\alpha \cap U_\beta \cap U_\gamma$$

Generalised tangent bundle

- The idea of Generalised Geometry is to treat diffeomorphisms and B-field gauge transformations on the same footing

  - define the sum of tangent and cotangent bundle of M_6

$$\hat{E} = TM \oplus T^*M$$

$$\hat{V} = \hat{v} + \hat{\xi} \qquad \leftarrow \qquad \text{untwisted generalised vectors}$$

  - natural pairing of vectors and forms

$$\eta(V_1, V_2) = (v_1 + \xi_1, v_2 + \xi_2) = \frac{1}{2}(\xi_1(v_2) + \xi_2(v_1))$$

$\rightarrow$ reduce the structure group to O(6,6)

  - the O(6,6) adjoint $R \in \mathrm{ad}F = (T \otimes T^*) \oplus \Lambda^2 T^* \oplus \Lambda^2 T$ acts on V as

$$V' = R \cdot V \qquad \longleftrightarrow \qquad \begin{cases} v' = r \cdot v - \beta \lrcorner \xi \\ \xi' = r \cdot \lambda - \iota_v b \end{cases}$$

- the generalised tangent bundle E is a non-trivial fibration of T^*M over TM (extension of T by T^*)

$$0 \longrightarrow T^*M \longrightarrow E \stackrel{\pi}{\longrightarrow} TM \longrightarrow 0$$

whose sections are generalised vectors $V = v + \xi$. On $U_{(\alpha)}$

$$V_{(\alpha)} = e^{B_{(\alpha)}} \hat{V} = [e^B(\hat{v} + \hat{\xi})]_{(\alpha)} = [\hat{v} + (\hat{\xi} - \iota_{\hat{v}} B)]_{(\alpha)}$$

and have non trivial patching on $U_\alpha \cap U_\beta$

$$\begin{aligned} v_{(\alpha)} + \xi_{(\alpha)} &= r_{(\alpha\beta)} v_{(\beta)} + \left[r^{-T}_{(\alpha\beta)} \xi_{(\beta)} - i_{r_{(\alpha\beta)} v_{(\beta)}} \mathrm{d}\lambda_{(\alpha\beta)} \right] \\ V_{(\alpha)} &= e^{-\mathrm{d}\lambda_{(\alpha\beta)}} R_{(\alpha\beta)} V_{(\beta)} \end{aligned} \qquad \left\{ \begin{array}{l} r \in GL(6, \mathbb{R}) \\ \mathrm{d}\lambda_{(\alpha\beta)} \in \Lambda^2(M_6) \end{array} \right.$$

- Observations
  - the split $V = e^B \hat{V}$ give non-canonical isomorphism between E and $T \oplus T^*$
  - the split and the patching of V imply the gauge transformations of the B-field
  - the patching further reduces it to $GL(R,6) \ltimes \Lambda^2(M) \to$ generalised diffemorphisms
- To include the dilaton we need a further generalisation of E

$$\tilde{E} = (\det T^* M) E$$

with structure group $GL(R,6) \ltimes \Lambda^2(M) \times \mathbb{R}^+$

Differential structure

- The ordinary Lie derivative generates diffeomorphisms

$$\mathcal{L}_v v'^m = v^n \partial_n v'^m - v'^n \partial_n v^m = v^n \partial_n v'^m - (\partial \times_{\mathrm{ad}} v)^m{}_n v'^n$$

GL(6) adjoint action $\hookleftarrow$

- The Dorfmann derivative generates generalised diffeos

$$L_V V'^M = V^N \partial_N V'^M - (\partial \times_{\mathrm{ad}} V)^M{}_N V'^N \qquad \partial_N = (\partial_n, 0)$$

O(6,6) adjoint action $\hookleftarrow$

  - in components ($V = v + \xi$ and $V' = v' + \xi'$)

$$L_V V' = \mathcal{L}_v v' + (\mathcal{L}_v \xi' - \iota_{v'} \mathrm{d}\xi)$$

- Define the twisted Dorfmann derivative acting on the untwisted gen. vectors $\hat{V}$

$$\hat{L}_{\hat{V}} \hat{V}' = e^B L_V V' = \mathcal{L}_{\hat{v}} \hat{v}' + (\mathcal{L}_{\hat{v}} \hat{\xi}' - \iota_{\hat{v}'} \mathrm{d}\hat{\xi} + \iota_{\hat{v}'} \iota_{\hat{v}} H)$$

twist by the two-form flux $H = \mathrm{d}B \hookleftarrow$

Generalised Metric and Vielbeine

- The metric g and the B-field combine into a single object $\to$ generalised metric

$$\mathcal{G} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

  - defines an $O(6) \times O(6)$ structure on E
  - parametrises the coset

$$\frac{O(6,6)}{O(6) \times O(6)} \quad \longleftrightarrow \quad \text{NS moduli of toroidal compactifications}$$

- Define generalised vielbeine E_A on E such that

$$\mathcal{G}^{-1} = \delta^{AB} E_A \otimes E_B$$

  - take a frame in $\{\hat{e}^a\} \in TM$ and the dual frame $\{e_a\} \in T^*M$ ($a = 1, \ldots, 6$), then

$$E_A = (\hat{e}^a + \iota_{\hat{e}^a} B) + e_a$$

EXCEPTIONAL GEOMETRY: TYPE IIA

[grana, louis, sim, waldram 09; cassani, de felice, m.p., strickland constable, waldram 16]

- The construction can be extended to include all potentials of type II SUGRA $\rightarrow$ IIA
  - the IIA potentials (democratic formulation [bergshoeff et al. 01])

NS two-form	RR polyform	NS six-form
B	$C = \sum_{k=0}^{4} C_{2k+1}$	$\tilde{B}$
$H = \mathrm{d}B$	$F = \mathrm{d}C - H \wedge C + m e^{B}$	$\tilde{H} = \mathrm{d}\tilde{B} - \frac{1}{2}[s(F) \wedge C + m e^{-B} \wedge C]_{(7)}$

  and $m = F_0$ is the Romans mass

  - with gauge transformations

$$\delta_V B = \mathcal{L}_v B - \mathrm{d}\lambda$$

$$\delta_V C = \mathcal{L}_v C - e^{B} \wedge (\mathrm{d}\omega - m\lambda)$$

$$\delta_V \tilde{B} = \mathcal{L}_v \tilde{B} - (\mathrm{d}\sigma + m\,\omega_6) - \frac{1}{2}\left[e^{B} \wedge (\mathrm{d}\omega - m\lambda) \wedge s(C)\right]_6$$

- The generalised tangent bundle is $E \in \mathbf{56}$ of $E_{7(7)}$ and decomposes under $GL(6)$

$$E \simeq TM \oplus T^*M \oplus \Lambda^5 T^*M \oplus (T^* \otimes \Lambda^6 T^*) \oplus \Lambda^{\text{even}} T^*M$$

charges of branes allowed in six dimensions

  - generalised vectors

$$V = (v + \lambda + \sigma + \tau + \omega) \qquad \Longleftrightarrow \qquad V = e^{\tilde{B}}\, e^{-B}\, e^{C} \hat{V}$$

with patching on $U_{(\alpha)} \cap U_{(\beta)}$

$$V_{(\alpha)} = e^{\mathrm{d}\tilde{\Lambda}_{(\alpha\beta)}}\, e^{\mathrm{d}\Omega_{(\alpha\beta)} + m\,\Omega_{6(\alpha\beta)}}\, e^{-\mathrm{d}\Lambda_{(\alpha\beta)} - m\,\Lambda_{(\alpha\beta)}} \cdot V_{(\beta)}$$

- The adjoint bundle is $\mathrm{ad} \in \mathbf{133} + \mathbf{1}$ of $E_{7(7)} \times \mathbb{R}^+$ ($GL(6)$ decomposition)

$$\mathrm{ad} = \mathbb{R}_\Delta \oplus \mathbb{R}_\phi \oplus (T \otimes T^*) \oplus \Lambda^2 T \oplus \Lambda^2 T^* \oplus \Lambda^6 T \oplus \Lambda^6 T^* \oplus \Lambda^{\text{odd}} T \oplus \Lambda^{\text{odd}} T^*$$

$$R = l + \varphi + r + \beta + B + \tilde{\beta} + \tilde{B} + \Gamma + C$$

includes $GL(6)$ action, shifts of warp factor and dilaton, NSNS and RR potentials

- Define a generalised frame and metric
  - Given the generalised tangent bundle

$$E \simeq TM \oplus T^*M \oplus \Lambda^5 T^*M \oplus (T^* \otimes \Lambda^6 T^*) \oplus \Lambda^{\text{even}} T^*M$$

    define the conformal split frame as a twist

$$E_A = e^{\tilde{B}} e^{-B} e^{C} e^{\Delta} e^{\phi} \cdot \hat{E}_A$$
$$\{\hat{E}_A\} = \{\hat{e}_a\} \cup \{e^a\} \cup \{e^{a_1 \dots a_5}\} \cup \{e^{a, a_1 \dots a_6}\} \cup \{1\} \cup \{e^{a_1 a_2}\} \cup \{e^{a_1 \dots a_4}\} \cup \{e^{a_1 \dots a_6}\}$$

  - Define the inverse generalised metric

$$\mathcal{G}^{-1} = \delta^{AB} E_A \otimes E_B$$

- The Dorfman derivative is

$$L_V V' = L_V^{(m=0)} V' + \underline{m}(V) \cdot V'$$

with
$$\begin{aligned} L_V^{(m=0)} V' &= \mathcal{L}_v v' + (\mathcal{L}_v \lambda' - \iota_{v'} \mathrm{d}\lambda) + (\mathcal{L}_v \sigma' - \iota_{v'} \mathrm{d}\sigma + [s(\omega') \wedge \mathrm{d}\omega]_5) \\ &\quad + (\mathcal{L}_v \tau' + j\sigma' \wedge \mathrm{d}\lambda + \lambda' \otimes \mathrm{d}\sigma + js(\omega') \wedge \mathrm{d}\omega) \\ &\quad + (\mathcal{L}_v \omega' + \mathrm{d}\lambda \wedge \omega' - (\iota_{v'} + \lambda' \wedge)\mathrm{d}\omega) \\ \underline{m}(V) \cdot V' &= m\,(-\iota_{v'}\omega_6 - \lambda \wedge \omega'_4 + \lambda' \otimes \omega_6 - \lambda \otimes \omega'_6 + \iota_{v'}\lambda + \lambda' \wedge \lambda) \end{aligned}$$

- it generates generalised diffeos : diffeos plus NSNS and RR gauge transformations

$$L_V \mathcal{G}, \quad V = (v, \lambda, \sigma, \omega) \quad \Longrightarrow \quad \begin{cases} \delta_V g = \mathcal{L}_v g \\ \delta_V B = \mathcal{L}_v B - \mathrm{d}\lambda \\ \delta_V C = \mathcal{L}_v C - (\mathrm{d}\omega - m\lambda) + \dots \\ \delta_V \tilde{B} = \mathcal{L}_v \tilde{B} - (\mathrm{d}\sigma + m\omega_6) - \frac{1}{2}(\mathrm{d}\omega - m\lambda) \wedge s(C) + \dots \end{cases}$$

APPLICATION 1:
SUPERSYMMETRIC COMPACTIFICATIONS

CALABI-YAU vs FLUX GEOMETRIES

- Generalised geometries are useful tools to study flux compactifications

EX: compactifications to

$$M_{10} = M_4 \times M_6$$

with 10d type II SUSY parameters

$$\begin{aligned} \epsilon_1 &= \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_- \\ \epsilon_2 &= \zeta_+ \otimes \eta^2_\mp + \zeta_- \otimes \eta^2_\pm \end{aligned} \quad \rightarrow \quad \begin{aligned} (\eta^i_+)^* &= \eta^i_- \quad && \text{6}d \text{ Weyl spinors} \\ \zeta^*_+ &= \zeta_- && \text{4}d \text{ Weyl spinors} \end{aligned}$$

- with no fluxes, minimal SUSY $\Rightarrow$ the internal manifold is a Calabi-Yau

$$\nabla_m \eta^{(i)}_+ = 0 \qquad \Rightarrow \qquad \begin{cases} \text{SU(3) holonomy} \\ \text{Ricci flatness} \qquad R_{mn} = 0 \end{cases}$$

- for flux compactifications SUSY $\Rightarrow$ the internal manifold is no longer Calabi-Yau

$$(\nabla_m \pm \frac{1}{4} H_m)\eta^i_+ + \frac{1}{16} e^\phi \epsilon_{ij} \not{F} \gamma_m \eta^j_- = 0$$

- We still say something about the geometry of the internal manifold
  - go from spinors to structures
  - go to Generalised Geometry

- Calabi Yau compactifications revisited
  - $\exists$ globally defined, invariant forms (bilinears in the susy parameters)

$$\begin{array}{ll} J_{\rm CY} & (\text{ Kähler form }) \\ \Omega_{\rm CY} & (\text{holomorphic 3-form}) \end{array} \quad \Longrightarrow \quad GL(3,\mathbb{C}) \cap Sp(6,\mathbb{R}) = SU(3) \text{ structure on } TM_6$$

  - SUSY is equivalent to the closure of SU(3) structure

$$\mathrm{d}J = 0 \qquad \mathrm{d}\Omega = 0$$

  - SUSY deformations

$$\begin{array}{rl} \text{Kähler def.} & (\mathrm{d}\delta J = 0) \\ \text{complex structure def.} & (\mathrm{d}\delta\Omega = 0) \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} \text{vector and hypermultiplets} \\ \text{of } \mathcal{N} = 2 \text{ SUGRA in 4 d} \end{array}$$

$\mathcal{N}=1$ VACUA AND $O(d,d)$ PURE SPINORS

- Use GCG to give a geometrical interpretation of $\mathcal{N}=1$ flux vacua
  - build polyforms on M_6 as bispinors of the susy parameters

$$\Phi_\pm = \eta^1_+ \otimes \bar{\eta}^2_\pm = \frac{1}{8}\sum_p \frac{1}{p!}\left(\bar{\eta}^2_\pm \gamma_{m_1\ldots m_p}\eta^1_+\right) \mathrm{d}x^{m_p\ldots m_1}$$

  - $\Phi_\pm$ are spinors on $TM_6 \oplus T^*M_6$

positive chirality $\leftrightarrow$ $\Phi_+ \in \Lambda^{even}T^*(M)$ even forms

negative chirality $\leftrightarrow$ $\Phi_- \in \Lambda^{odd}T^*(M)$ odd forms

  - pure spinors $\to$ vacuum of Cliff(6,6)
  - define a SU(3) $\times$ SU(3) structure on $T \oplus T^*$

SU(2) on T^*	$\eta^1_+ = a\eta_+$ $\eta^2_+ = c_1\eta_+ + c_2 z\cdot\eta_-$	$\Phi_+ = \frac{a}{8}(\bar{c}_1 e^{-ij} - i\bar{c_2}\omega)\wedge e^{z\wedge\bar{z}/2}$ $\Phi_- = -i\frac{a}{8}(\bar{c}_2 e^{-ij} + i\bar{c_1}\omega)\wedge z$
SU(3) on T^*	$\eta^1_+ = a\eta_+$ $\eta^2_+ = b\eta_+$	$\Phi_+ = \frac{a\bar{b}}{8}e^{-iJ}$ $\Phi_- = -i\frac{ab}{8}\Omega$

- 10d SUSY variations are equivalent to the differential conditions
  - one spinor is closed

$$\mathrm{d}(e^{3A}\Phi_1) = 0 \to \text{generalised Calabi Yau}$$

  - the RR fields act as torsion

$$\mathrm{d}(e^{3A}\Phi_2) = e^{3A}\mathrm{d}A \wedge \bar{\Phi}_2 + \frac{i}{8}e^{3A} * \lambda(F)$$

	zero fluxes	fluxes
	T	$T \oplus T^*$
pure spinor	η_0	Φ
integrability	$\nabla_m \eta_0 = 0$	$\mathrm{d}\Phi = 0$
	Calabi Yau	Generalised Calabi Yau

- unified description of type IIA and IIB
- extend to other dimensions

$\mathcal{N} = 2$ VACUA AND EXCEPTIONAL STRUCTURES

- General description of string and M-theory backgrounds with 8 supercharges

$$\mathrm{d}s^2 = e^{2\Delta}\mathrm{d}s_D^2 + \mathrm{d}s_d^2 \qquad \begin{array}{l} D = 4 (d = 6 \text{ or } d = 7) \\ D = 5 (d = 5 \text{ or } d = 6) \end{array}$$

- apply to Minkowski and AdS compactifications of M-theory and type II
  - CY compactifications to 4 and 5 dimensions
  - M2 and wrapped M5 branes
  - D3 branes

- define exceptional structures on M_d
  - bilinears in the background susy parameters
  - parametrise the scalars of the D-dim SUGRA
    - VM structure $\rightarrow$ scalar fields of vector multiplets

$$\mathcal{M}_v = \frac{E_{7(7)} \times \mathbb{R}^+}{E_{6(2)}} \qquad \longleftrightarrow \qquad K \in \mathbf{56}$$

    - H structure $\rightarrow$ scalar fields of hyper multiplets

$$\mathcal{M}_{\text{HK}} = \frac{E_{7(7)} \times \mathbb{R}^+}{Spin^\star(12)} \qquad \longleftrightarrow \qquad J_a \in \mathbf{133}_{1/2} \qquad \left(\mathcal{M}_{\text{QK}} = \frac{\mathcal{M}_{\text{HK}}}{SU(2) \times \mathbb{R}^+} \right)$$

SU(2) triplets

  - together J_a and K define a $Usp(6)$ structure $\Longleftrightarrow$ $\mathcal{N} = 2$ SUSY in five dimensions

- Supersymmetry: write the SUSY variations as condtions on the generalised structures [ashmore, waldram 15, ashmore, m.p., waldram 16, grana, ntokos 16]

$$
\begin{aligned}
&L_K J_a = \epsilon_{abc} \lambda_b J_c \\
&L_K K = 0 \\
&\mu_a(V) = -\frac{1}{2} \epsilon_{abc} \int \mathrm{tr}(J_b, L_V J_c) = \lambda_a \int c(K, K, V)
\end{aligned}
$$

with L_K Dorfmann derivative (generates diffeomorphims and gauge transf.)

$\mu_a(V)$ moment maps for the action of gen. diffeomorphisms on J_a

$\lambda_a = 0$ Minkowski $\lambda_a \neq 0$ AdS

- nice dictionary with gauged supergravity

$$
\begin{aligned}
L_K J_a = \epsilon_{abc} \lambda_b J_c \quad (L_{\tilde{K}} J_a = 0) \quad &\leftrightarrow \quad \text{hyperino variations} \\
L_K K = 0 \quad &\leftrightarrow \quad \text{gaugino variations} \\
\mu_a(V) = \lambda_a \int c(K, K, V) \quad &\leftrightarrow \quad \text{gravitino variations}
\end{aligned}
$$

- generalised torsion maps to the embedding tensor

RESULTS

- Compactifications
  - examples of compactifications on GCY manifolds [grana, minasian, m.p. tomasiello 06]
  - effective actions on flux backgrounds [grana, louis, waldram 05,06; martucci, koerber 07, 08, ...]
    - expansion forms could be not closed and of mixed degree
    - difficult to identify the light modes
  - some progress in moduli counting [tomasiello 07, martucci 09,...]

- Gauge/gravity duality
  - baryonic branch of Klebanov-Strassler [grana, minasian, m.p. zaffaroni 06]
  - massive deformations of type IIA AdS_4 duals of CFT_3 [m.p, zaffaroni 09, lüst, tsimpis 09]
  - geometry of superconformal $\mathcal{N}=1$ theories [minasian, m.p, zaffaroni 06; gabella, gauntlett, palti, sparks, waldram 09, ...]
  - exactly marginal deformations of $\mathcal{N}=1$ SCFT [ashmore, gabella, grana, m.p., waldram 16]
  - AdS_7 solutions dual to $6d$ SCFT [apruzzi, fazzi, rosa, tomasiello 13, ...]

APPLICATION 2:
CONSISTENT TRUNCATIONS

LOW ENERGY EFFECTIVE ACTIONS

- A standard technique to derive lower dimensional actions is Kaluza-Klein reduction
  - take a background of the type

$$M_{10} = M_{10-d} \times M_d$$

  - expand the 10d field in harmonics on $M_d \to$ infinite towers of lower-dimensional fields
  - truncate the KK modes to a finite set in a consistent way
    - no dependence on the internal manifold in the eom and susy variations
    - full non-linear interactions and symmetries for the lower dimensional fields

- Consistent reductions establish a map between theories in different dimensions
  - all solutions of the lower dimensional theory lift to higher dimensional ones
  - insight on the higher dimensional origin of the lower dimensional gauge symmetries
  - powerful tool in AdS/CFT

ORDINARY SCHERK-SCHWARZ REDUCTiON

- Consistent truncations are rare and non-trivial
  - often the truncation ansatz is ensured by symmetry $\to$ Scherck-Schwarz reductions on group manifolds

- Consider a compactification on a group manifold M_d

$$M_{10} = M_{10-d} \times M_d$$

  - the left-invariant vector fields $\{\hat{e}_a\}$ satisfy the algebra

$$\mathcal{L}_{\hat{e}_a} \hat{e}_b = f_{ab}{}^c \hat{e}_c \qquad\qquad f_{ab}{}^c \text{ constant}$$

  - $\{\hat{e}_a\}$ generate the (right) isometries of the metric
  - $\{\hat{e}_a\}$ give a globally defined frame $\to$ M_d is parallelisable

- Truncation ansatz
  - define the twisted frame and the internal metric

$$\hat{e}'_a{}^m(x,z) = U_a{}^b(x)\,\hat{e}_b{}^m(z) \qquad U_a{}^b \in GL(d)$$

$$g^{mn}(x,z) = \mathcal{M}^{ab}(x)\,\hat{e}_a{}^m(z)\,\hat{e}_b{}^n(z) \qquad \mathcal{M}^{ab} = \delta^{cd}\,U_c{}^a U_d{}^b \in GL(d)/\,SO(d)$$

  - write the 10d SUGRA fields as

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathcal{M}_{ab}(x)(e^a - \mathcal{A}^a_\mu(x)\mathrm{d}x^\mu)(e^b - \mathcal{A}^b_\nu(x)\mathrm{d}x^\nu)$$

$$C_1(x,z) = C_\mu(x)\mathrm{d}x^\mu + C_a(x)(e^a - \mathcal{A}^a_\mu(x)\mathrm{d}x^\mu) + c_1$$

$$C_3(x,z) = \dots$$

  where $g_{\mu\nu}$, $\mathcal{M}_{ab}$, $\mathcal{A}^a_\mu$, etc are $D-d$-dimensional fields

- Features of the truncated theory
  - the reduction is consistent by symmetry
  - being parallelisable M_d has globally defined spinors $\Rightarrow$ maximal SUSY
  - the gauge group comes from Killing symmetries and gauge transformations of the potentials.

GENERALISED SCHERCK-SCHWARZ REDUCTION

- Extend to EGG the notion of parallelisability $\rightarrow$ Generalised Leibniz parallelisation
  - $\exists$ a globally defined frame $\{E_A\}$ for the $E_{d+1(d+1)} \times \mathbb{R}^+$ generalised tangent bundle on M_d.
  - the frame must satisfy the algebra

$$L_{E_A} E_B = X_{AB}{}^C E_C \qquad\qquad X_{AB}{}^C \text{ constant}$$

  - $X_{AB}{}^C$ generate the gauge algebra

$$[X_A, X_B] \;=\; -X_{AB}{}^C X_C$$

and are related to the embedding tensor of the $D-d$ gauged supergravity

$$X_{AB}{}^C = \Theta_A{}^\alpha (t_\alpha)_B{}^C$$

embedding tensor $\hookleftarrow$ $\hookrightarrow$ U-duality generators

  - GLP implies the manifold is a coset $M_d \sim G/H$ and maximal SUSY

- Truncation ansatz
  - decompose all 10d fields according to $SO(1,9) \to SO(1,9-d) \times SO(d)$

$$
\begin{aligned}
\mathrm{d}s^2 &= e^{2\Delta} g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn}(x,z) Dz^m Dz^n \\
B &= \tfrac{1}{2} B_{m_1 m_2}(x,z) Dz^{m_1} \wedge Dz^{m_2} + \overline{B}_{\mu m}(x,z)\mathrm{d}x^\mu \wedge Dz^m + \tfrac{1}{2}\overline{B}_{\mu\nu}(x,z)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
C_1 &= C_m Dz^m + \overline{C}_\mu \, \mathrm{d}x^\mu \\
&\ldots
\end{aligned}
$$

$$Dz^m = \mathrm{d}z^m - h_\mu^m(x,z)\mathrm{d}x^\mu$$

  - field redefinition $\to$ covariance under gen diffeos

$$
\begin{aligned}
B_\mu &= \overline{B}_\mu & B_{\mu\nu} &= \overline{B}_{\mu\nu} + \iota_{h_{[\mu}} B_{\nu]} \\
C_\mu &= e^{-B} \wedge \overline{C}_\mu & \tilde{B}_{\mu\nu} &= \overline{\tilde{B}}_{\mu\nu} - \tfrac{1}{2}\big[\,\overline{C}_{\mu\nu} \wedge s(C)\,\big]_4 + \iota_{h_{[\mu}} \tilde{B}_{\nu]} \\
\tilde{B}_\mu &= \overline{\tilde{B}}_\mu - \tfrac{1}{2}[\,\overline{C}_\mu \wedge s(C)\,]_5 & C_{\mu\nu} &= e^{-B} \wedge \overline{C}_{\mu\nu} + \iota_{h_{[\mu}} C_{\nu]} + B_{[\mu} \wedge C_{\nu]}
\end{aligned}
$$

    - B, C are field on the internal manifold
    - reproduce the gauge transformation of the lower-dimensional supergravity

- arrange fields with the same external indices in $E_{d+1(d+1)} \times \mathbb{R}^+$ tensors
  - scalars $\rightarrow$ gen metric

$$\{g_{mn},\, B_{m_1 m_2},\, \tilde{B}_{m_1\dots m_6},\, C_m,\, C_{m_1 m_2 m_3},\, C_{m_1\dots m_5}\}$$

  - vectors $\rightarrow$ gen vector ${\mathcal{A}_\mu}^M \in E$

$${\mathcal{A}_\mu}^M = \{{h_\mu}^m,\, B_{\mu m},\, \tilde{B}_{\mu m_1\dots m_5},\, C_\mu,\, C_{\mu m_1 m_2},\, C_{\mu m_1\dots m_4},\, C_{\mu m_1\dots m_6}\}$$

  - two-forms $\rightarrow$ ${\mathcal{B}_{\mu\nu}}^{MN} \in N' \subset (E \otimes E)_{\text{sym}}$

$${\mathcal{B}_{\mu\nu}}^{MN} = \{B_{\mu\nu},\, \tilde{B}_{\mu\nu m_1\dots m_4},\, C_{\mu\nu m},\, C_{\mu\nu m_1 m_2 m_3},\, C_{\mu\nu m_1\dots m_5}\}$$

- twist the generalised frame and metric

$$E'_A{}^M(x,z) = U_A{}^B(x) E_B{}^M(z) \qquad \mathcal{G}^{MN}(x,z) = \mathcal{M}^{AB}(x) E_A{}^M(z) E_B{}^N(z)$$

$$E_{d+1(d+1)} \hookleftarrow \qquad\qquad \hookrightarrow \frac{E_{d+1(d+1)}}{K}$$

with K maximally compact subgroup of $E_{d+1(d+1)}$

- write the 10d fields as

$$\mathcal{A}_\mu{}^M(x,z) = \mathcal{A}_\mu{}^A(x) E_A{}^M(z)$$

$$\mathcal{B}_{\mu\nu}{}^{MN}(x,z) = \frac{1}{2} \mathcal{B}_{\mu\nu}{}^{AB}(x) (E_A \otimes_{N'} E_B)^{MN}(z)$$

a similar construction should work for higher rank forms in $D-d$-dimensions

- Partial check of the consistency of the reduction $\rightarrow$ recover the gauge transformations of the lower dimensional gauged SUGRA

GENERALISED SPHERE REDUCTIONS

Sphere reductions are an interesting application of generalised Scherk-Schwarz reduction

- only S^1, S^3, anf S^7 are parallelisable
- all spheres are generalised parallelisable [lee, strickland-constable, waldram 14]
  - a d-sphere is a coset

$$S^d = \frac{SO(d+1)}{SO(d)} \quad \text{with} \quad F_d = \mathrm{d}A_{d-1}$$

  - $GL(d+1,\mathbb{R})$ generalised geometry: define the gen. tangent bundle (twisting given by A_{d-1})

$$E_{GL(d+1)} = T \oplus \Lambda^{d-2} T^* \quad \to \frac{d(d+1)}{2} \text{ dim. bivector of} GL(d+1,\mathbb{R})$$

  - $E_{GL(d+1)}$ always admits a globally defined frame

$$E_{ij} = v_{ij} + \tfrac{R^{d?2}}{(d-2)!} \epsilon_{ijk_1 \dots k_{d-1}} y^{k_1} \mathrm{d}y^{k_2} \wedge \dots y^{k_{d-1}} + \iota_{v_{ij}} A$$

$SO(d+1)$ killing vectors $\hookleftarrow$ $\hookrightarrow$ constrained coordinates on $\mathbb{R}^d$ $\delta_{ij} y^i y^j = 1$

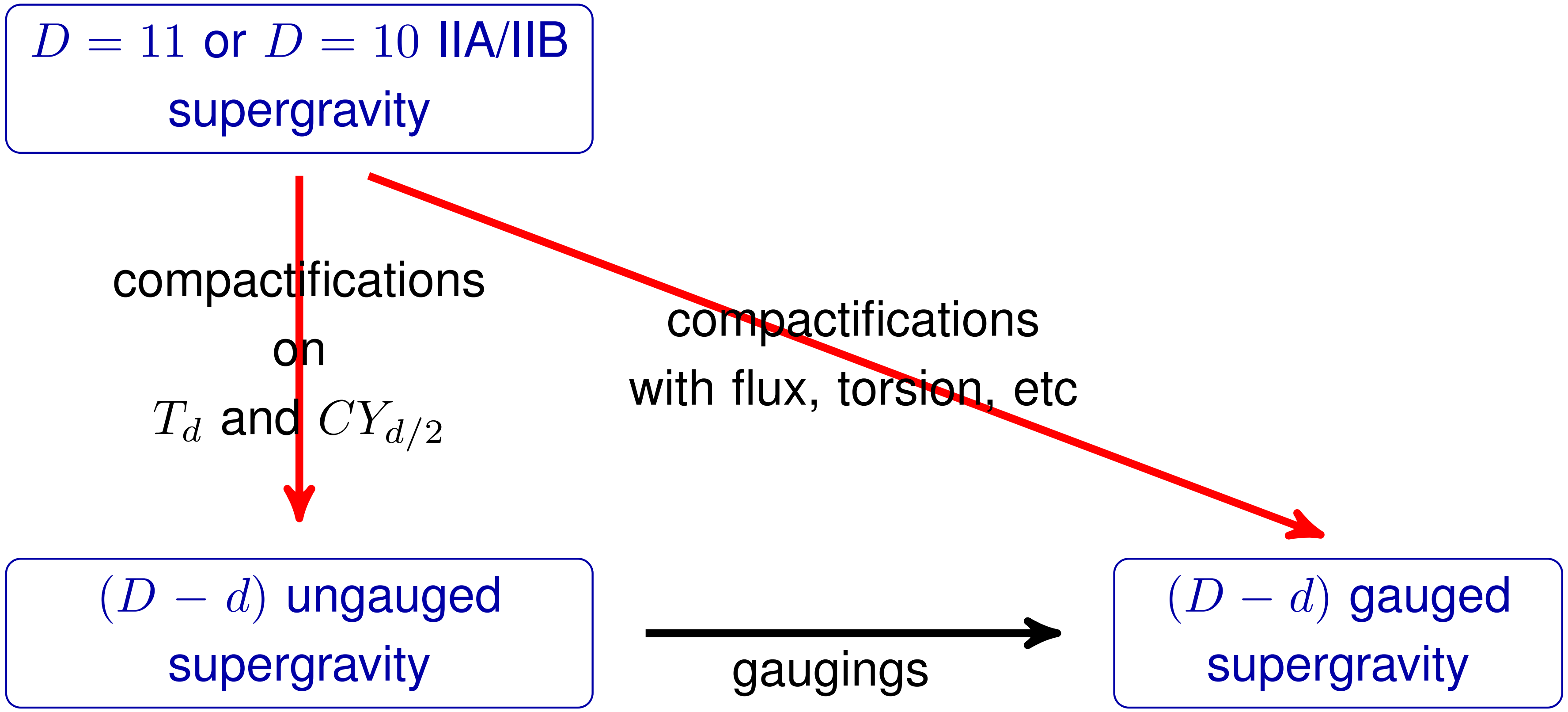
RESULTS

- New results about consistent truncations to maximal supergravity from EGG and EFT
  - consistency of the $AdS_5 \times S^5$ truncation [hohm, samtleben 14]
  - extension to hyperbolic spaces
  - no-go theorems for consistent truncations for massive IIA [ciceri, gaurino, inverso 16, cassani, de felice, m.p. strickland-constable, waldram 16]
- Extension to less than maximal supergravity [work in progress]

CONCLUSIONS

- Generalised geometry is a powerful tool to study generic supergravity backgrounds
- GCG geometrise NS fluxes
  - largely exploited in several dimensions
  - interesting new results for instance in AdS/CFT
- EGC geometrise both NS and RR
  - nicely reproduces and unify knwon results
  - new consistent truncations

- More generally $\rightarrow$ deeper understanding of flux backgrounds



- moduli problem and deformations
- trunctations with less SUSY
- phenomenological effective actions
- topological models
- more insight into the AdS/CFT correspondence

GENERALISED SPHERE REDUCTIONS

- Apply generalised Scherk-Schwarz reduction to sphere compactifications in massless and massive type IIA

	$m = 0$		$m \neq 0$	
S^6	$ISO(7)$	[guarino,varela 15]	$ISO(7)_m$	[guarino,varela 15]
S^4	$SO(5)$	[cowdall 88; cvetic et al 00]	–	
S^3	$ISO(4)$	[malek, samtleben 15]	–	
S^2	$SO(3)$	[salam, sezgin 85]	–	

$\rightarrow$ only S^6 admits a parallelisation for massive IIA

- Sphere backgrounds and generalised parallelisable [lee, strickland-constable, waldram 14]
  - a $S^d = SO(d+1)/SO(d)$ backgrounds has a d-form $F_d = \mathrm{d}A_{d-1}$
  - define the gen. tangent bundle where the twisting is given by A_{d-1}

$$E_{GL(d+1)} = T \oplus \Lambda^{d-2}T^*$$

    - $E_{GL(d+1)}$ is in $\frac{1}{2}d(d+1)$ dim. bivector representation of $GL(d+1,\mathbb{R})$
      $\Rightarrow GL(d+1,\mathbb{R})$ generalised geometry
    - $E_{GL(d+1)}$ can be see as

$$E_{GL(d+1)} = W \wedge W \qquad\qquad W \sim (\det T^*)^{1/2}(T + \Lambda^d T)$$

      W is the fundamental of $GL(d+1)$.

  - $E_{GL(d+1)}$ always admits a globally defined frame

$$E_{ij} = v_{ij} + \frac{R^{d?2}}{(d-2)!}\epsilon_{ijk_1\ldots k_{d-1}} y^{k_1}\mathrm{d}y^{k_2}\wedge\ldots y^{k_{d-1}} + \iota_{v_{ij}}A$$

$SO(d+1)$ killing vectors $\hookleftarrow$ $\hookrightarrow$ constrained coordinates on $\mathbb{R}^d$ $\delta_{ij}y^iy^j = 1$

    - $E_{ij} = E_i \wedge E_j$ with E_j frame on W

- For massless IIA the full $E_{d+1(d+1)}$ tangent bundle is parallelisable
  - define a frame

$$E_A = E_{ij} + \sum E_B \qquad E_k \in \text{ other } GL(d+1) \text{ reprs}$$

with

$$L_{E_A} E_B \;=\; X_{AB}{}^C E_C \;, \qquad [X_A, X_B] \;=\; -X_{AB}{}^C X_C$$

- For massive IIA the frame the algebra of the massless frame does not close

$$X^C_{AB} \to X^C_{AB} + E_A^{\ M} E_B^{\ N} E^C_{\ P} m_{MN}^{\ \ P}$$

  - to have a GLP we would need

$$E_A^{\ M} E_B^{\ N} E^C_{\ P} m_{MN}^{\ \ P} = const$$

  - this is possible if $E_A \in G_m$, the stabiliser of the Romans mass $m \to$ true only for S^6

EXAMPLE: S^3 REDUCTIONS

- Describe $S^3 = SO(4)/SO(3)$ in constrained coordinates
  - introduce on $\mathbb{R}^4$ the coordinates

$$x^i = r y^i \qquad \text{such that} \qquad \delta_{ij} y^i y^j = 1 \qquad i = 1, \ldots, 4$$

  - S^3 of radius R is obtained fixing $r = R$. The metric and volume form are

$$g_{S^3} = R^2 \, \delta_{ij} \mathrm{d}y^i \mathrm{d}y^j \qquad \mathrm{vol}_{S^3} = \frac{R^3}{3!} \, \epsilon_{i_1 \ldots i_4} y^{i_1} \mathrm{d}y^{i_2} \wedge \cdots \wedge \mathrm{d}y^{i_4}$$

  - the $SO(4)$ Killing vectors are

$$\begin{aligned} v_{ij} &= R^{-1} \, (y_i k_j - y_j k_i) \\ \mathcal{L}_{v_{ij}} v_{kl} &= R^{-1} \, (\delta_{ik} v_{lj} - \delta_{il} v_{kj} - \delta_{jk} v_{li} + \delta_{jl} v_{ki}) \end{aligned}$$

  with k_i conformal Killing vectors

$$\mathcal{L}_{k_i} g = -2 y^i g \qquad k_i(y_j) \equiv \iota_{k_i} \mathrm{d}y_j = \delta_{ij} - y_i y_j$$

- Generalised geometry for S^3
  - the U-duality group on M_3 is $E_{4(4)} \sim SL(5,\mathbb{R})$
  - the generalised tangent bundle

$$E \sim T \oplus T^* \oplus \mathbb{R} \oplus \Lambda^2 T^*$$
$$V = v + \lambda + \omega_0 + \omega_2$$

is in $\mathbf{10}$ of $SL(5,\mathbb{R})$ and is the product $E = W \wedge W$

$$W \sim (\det T)^{-1/2}(T + \det T) \in \mathbf{5} \text{ of } SL(5,\mathbb{R})$$

$\Rightarrow$ a parallelisation of E must also be a parallelisation of W

  - a generalised frame $\{E_{IJ}\}$ has natural decomposition under $SL(4,\mathbb{R})$ as

$$\begin{aligned} SL(5,\mathbb{R}) &\supset SL(4,\mathbb{R}) \\ \mathbf{10} &\to \mathbf{6} + \mathbf{4} \\ \{E_{IJ}\} &= \{E_{ij},\, E_{i5}\} \qquad i,j = 1,\dots,4 \end{aligned}$$

Massless IIA on S^3 is Leibniz parallelisable

- there is globally defined and orthonormal frame

$$E_{IJ} = \begin{cases} E_{ij} = v_{ij} + R\,\epsilon_{ijkl}\,y^k \mathrm{d}y^l + \iota_{v_{ij}} B \\ E_{i5} = y_i + \frac{R^2}{2}\,\epsilon_{ijkl}\,y^j \mathrm{d}y^k \wedge \mathrm{d}y^l - y_i B \end{cases} \qquad H = \mathrm{d}B = \frac{2}{R}\mathrm{vol}_3$$

- the Dorfman derivative gives the $ISO(4)$ algebra

$$L_{E_{ij}} E_{kl} = 2R^{-1}\delta_{i[k}E_{l]j} - \delta_{j[k}E_{l]i})$$
$$L_{E_{ij}} E_{k5} = -2R^{-1}\delta_{k[i}E_{j]5}$$
$$L_{E_{i5}} E_{kl} = 2R^{-1}\delta_{i[k}E_{l]5}$$
$$L_{E_{i5}} E_{k5} = 0$$

- it reproduces the truncation to maximal $D = 7$ supergravity with $ISO(4)$ gaugings

[cvetic et al.00; samtleben, weidner 05]

Towards a no-go theorem for massive IIA parallelisations on S^3

- Basic assumptions
  - since $H = \frac{1}{m}\mathrm{d}F_2$ is exact, the only non-zero flux is m, so that no twisting of E
  - it exists a parallelisation E_{IJ} that gives an $SO(4)$

$$L_{E_{ij}} E_{kl} = 2R^{-1}\left(\delta_{i[k} E_{l]j} - \delta_{j[k} E_{l]i}\right) \qquad L_{E_{i5}} E_{kl} = 0$$
$$L_{E_{ij}} E_{k5} = -2R^{-1}\delta_{k[i} E_{j]5} \qquad L_{E_{i5}} E_{k5} = 0$$

- consequences
  - E_{IJ} are generalised Killing vectors $L_{E_A}\mathcal{G}^{-1} = 0$
  - the frame is uniquely fixed

$$E_A \equiv E_{IJ} = \begin{cases} E_{ij} &= v_{ij} + R^2\,\mathrm{d}y_i \wedge \mathrm{d}y_j \\ E_{i5} &= R\,(m\,y_i + \mathrm{d}y_i)\ , \end{cases}$$

  but is not a $SL(5,\mathbb{R})$ frame since $E_{IJ} \neq E_I \wedge E_J$ with $E_I \in W$

- the same results hold for $SO(5)$ and $ISO(4)$ algrebrae

- There is no parallelisations E_{IJ} on S^3