The Standard Model of Cosmology

Matthias Bartelmann

Universität Heidelberg, Zentrum für Astronomie
Institut für Theoretische Astrophysik

Outline

1. Geometry and Dynamics
2. Parameters, Age and Distances
3. Thermal Evolution
4. Recombination and Nucleosynthesis
5. The Growth of Perturbations
6. Statistics and Non-Linear Evolution
7. Structures in the CMB
8. Cosmological Weak Lensing
9. Type-Ia Supernovae
10. Cosmological Inflation and Dark Energy

Geometry and Dynamics

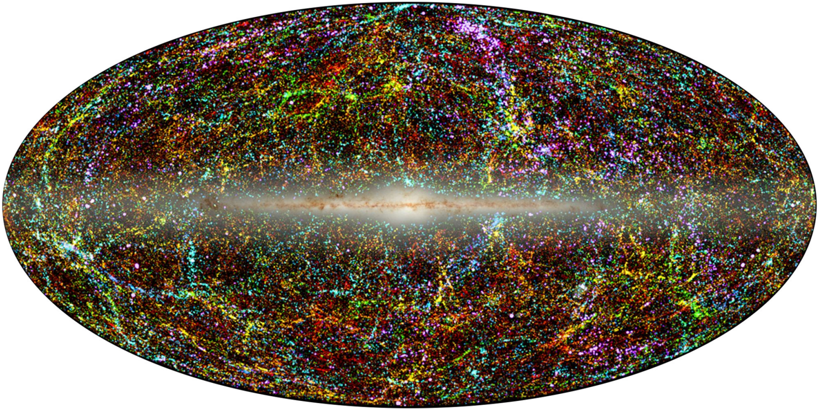
1 Geometry and Dynamics
Assumptions
Metric
Redshift
Dynamics
Remark on Newtonian Dynamics

2 Parameters, Age and Distances

Assumptions

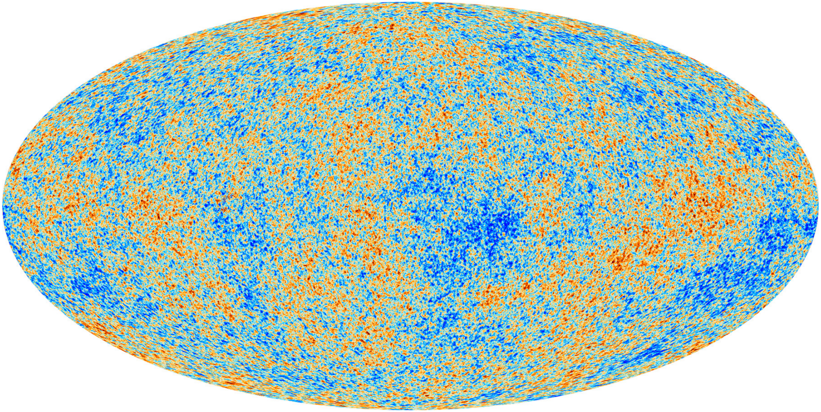
- standard cosmology makes two fundamental assumptions:
  1. observable properties of the Universe are isotropic
  2. our position in the Universe is not preferred to any other (cosmological principle);
- such a Universe is homogeneous and isotropic
- only relevant interaction is gravity: search for cosmological models in general relativity

Assumptions: Isotropy?



2-Micron All-Sky Survey, 2MASS

Assumptions: Isotropy?



CMB temperature fluctuations, measured by Planck

Metric

- metric tensor $g_{\mu\nu}$ has ten independent components: g_{00}, g_{0i}, and g_{ij}; two fundamental assumptions greatly simplify the metric

Metric

- eigentime should equal coordinate time for fundamental observers: [signature chosen: $(-,+,+,+)$]

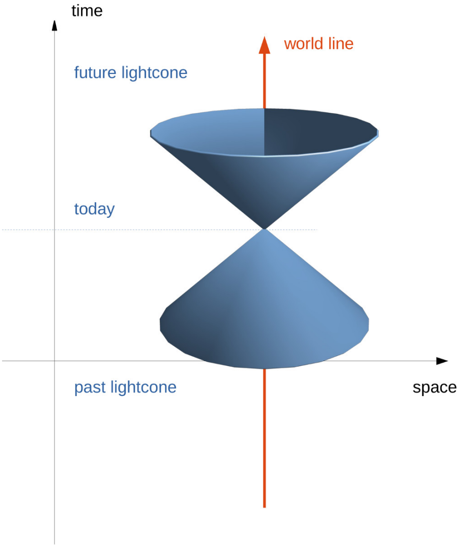
$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 = -c^2\mathrm{d}t^2 \;\Rightarrow\; g_{00} = -c^2$$

- isotropy requires $g_{0i} = 0$ and spherical symmetry for three-space, thus

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}w^2 + f_K^2(w)\mathrm{d}\Omega^2\right],$$

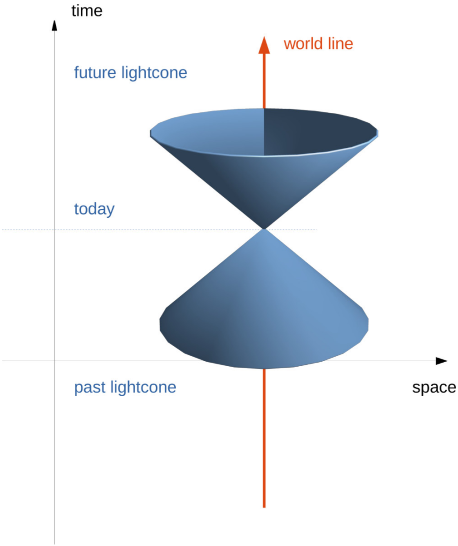
$$\text{with} \quad f_K(w) = \begin{cases} K^{-1/2}\sin(K^{1/2}w) & (K>0) \\ w & (K=0) \\ |K|^{-1/2}\sinh(|K|^{1/2}w) & (K<0) \end{cases}$$

Metric: Lightcone

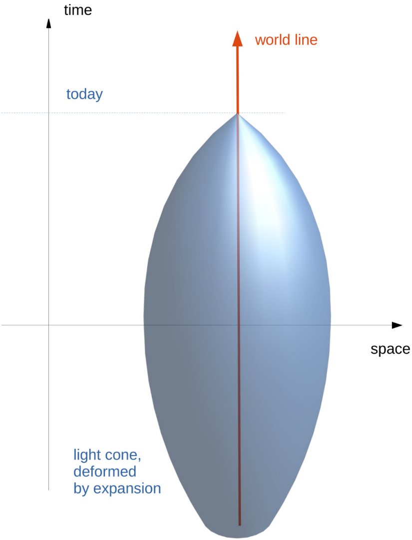


Minkowski space-time

Metric: Lightcone



Minkowski space-time



Expanding space-time

Redshift

- space can expand or shrink, leading to red- or blueshift
- propagation condition for light, $\mathrm{d}s = 0$, implies

$$\frac{\nu_\mathrm{e}}{\nu_\mathrm{o}} = \frac{\lambda_\mathrm{o}}{\lambda_\mathrm{e}} = 1 + \frac{\lambda_\mathrm{o} - \lambda_\mathrm{e}}{\lambda_\mathrm{e}} = 1 + z = \frac{a(t_\mathrm{e})}{a(t_\mathrm{o})}$$

- light is red- or blueshifted by the same amount as space expands or shrinks
- redshifts of galaxies shows that the Universe is expanding

Dynamics

- dynamics of the metric is expressed by dynamics of the scale factor $a(t)$
- Einstein's field equations reduce to Friedmann's equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}\,, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

- they can be combined to give the *adiabatic equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\rho c^2\right) + p\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\right) = 0$$

expressing energy conservation

1 Geometry and Dynamics

2 Parameters, Age and Distances
- Forms of Matter
- Parameters
- Age and Expansion of the Universe
- Distances
- Horizons

3 Thermal Evolution

Forms of Matter

- two forms of matter can broadly be distinguished, relativistic and non-relativistic; they are often called radiation and dust, respectively

- for radiation:

$$p = \frac{\rho c^2}{3}$$

which implies

$$\rho(t) = \rho_0 a^{-4} \ ,$$

($a = 1$ today)

- for dust, $p = 0$ because $p \ll \rho c^2$, and

$$\rho(t) = \rho_0 a^{-3}$$

Parameters

- Hubble parameter, relative expansion rate:

$$H(t) \equiv \frac{\dot{a}}{a}\,, \qquad H_0 \equiv H(t_0) = 100\,h\,\frac{\text{km}}{\text{s}\,\text{Mpc}} = 3.2 \times 10^{-18}\,h\,\text{s}^{-1}$$

Parameters

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- Hubble parameter, relative expansion rate:

$$H(t) \equiv \frac{\dot{a}}{a}$$

- critical density

$$\rho_{\rm cr}(t) \equiv \frac{3H^2(t)}{8\pi G}\,, \quad \rho_{\rm cr0} \equiv \rho_{\rm cr}(t_0) = \frac{3H_0^2}{8\pi G} = 1.9 \times 10^{-29}\, h^2\, {\rm g\, cm^{-3}}$$

Parameters

- Hubble parameter, relative expansion rate:

$$H(t) \equiv \frac{\dot{a}}{a}$$

- critical density

$$\rho_{\mathrm{cr}}(t) \equiv \frac{3H^2(t)}{8\pi G}$$

- dimension-less density parameters

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{\mathrm{cr}}(t)}\,, \quad \Omega_0 \equiv \frac{\rho(t_0)}{\rho_{\mathrm{cr}0}}\,, \quad \Omega_\Lambda(t) = \frac{\Lambda c^2}{3H^2(t)}\,, \quad \Omega_{\Lambda 0} \equiv \frac{\Lambda c^2}{3H_0^2}$$

Parameters

- Hubble parameter, relative expansion rate:

$$H(t) \equiv \frac{\dot{a}}{a}$$

- dimension-less density parameters

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{\mathrm{cr}}(t)}\ , \quad \Omega_0 \equiv \frac{\rho(t_0)}{\rho_{\mathrm{cr}0}}\ , \quad \Omega_\Lambda(t) = \frac{\Lambda c^2}{3H^2(t)}\ , \quad \Omega_{\Lambda 0} \equiv \frac{\Lambda c^2}{3H_0^2}$$

- Friedmann's equation becomes

$$H^2(a) = H_0^2 \left[\Omega_{\mathrm{r}0} a^{-4} + \Omega_{\mathrm{m}0} a^{-3} + \Omega_{\Lambda 0} - \frac{Kc^2}{a^2} \right]$$

Parameters

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- Friedmann's equation becomes

$$H^2(a) = H_0^2 \left[\Omega_{r0} a^{-4} + \Omega_{m0} a^{-3} + \Omega_{\Lambda 0} - \frac{Kc^2}{a^2} \right]$$

specialising to $a = 1$ allows to solve for K,

$$-Kc^2 = 1 - \Omega_{r0} - \Omega_{m0} - \Omega_{\Lambda 0} \equiv \Omega_K$$

Parameters

- final form for Friedmann's equation

$$H^2(a) = H_0^2 \left[\Omega_{r0} a^{-4} + \Omega_{m0} a^{-3} + \Omega_{\Lambda 0} + \Omega_K a^{-2}\right] \equiv H_0^2 E^2(a)$$

Parameters

- final form for Friedmann's equation

$$H^2(a) = H_0^2 \left[\Omega_{r0} a^{-4} + \Omega_{m0} a^{-3} + \Omega_{\Lambda 0} + \Omega_K a^{-2} \right] \equiv H_0^2 E^2(a)$$

- radiation density exceeded matter density before

$$a_{eq} = \frac{\Omega_{r0}}{\Omega_{m0}}$$

Parameters

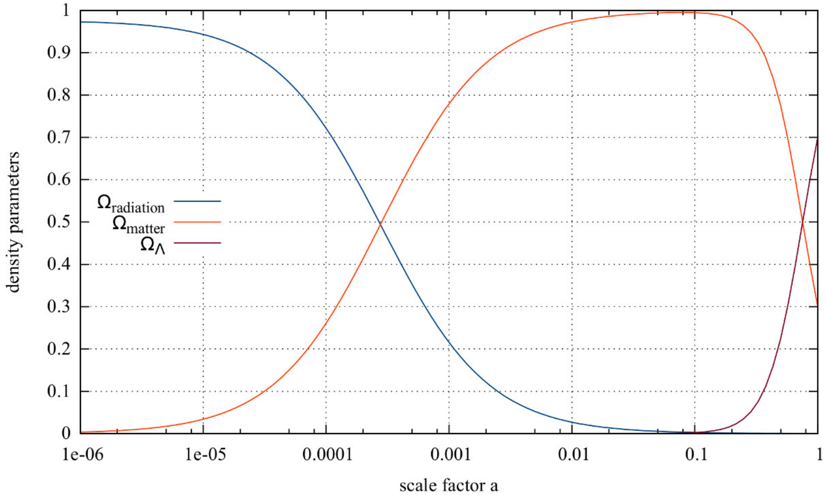
- the density parameters change with time:

$$\Omega_{\mathrm{m}}(a) = \frac{\Omega_{\mathrm{m0}}}{a + \Omega_{\mathrm{m0}}(1-a) + \Omega_{\Lambda 0}(a^3 - a)},$$

$$\Omega_{\Lambda}(a) = \frac{\Omega_{\Lambda 0} a^3}{a + \Omega_{\mathrm{m0}}(1-a) + \Omega_{\Lambda 0}(a^3 - a)}$$

- this implies: $\Omega_{\mathrm{m}}(a) \to 1$ and $\Omega_{\Lambda}(a) \to 0$ for $a \to 0$ regardless of their present values; if $\Omega_{\mathrm{m0}} + \Omega_{\Lambda 0} = 1$, remains so for $a < 1$

Parameters



Evolution of density parameters

Parameters: Planck 2015

Hubble constant	h	0.6727 ± 0.0066
dark-matter density	Ω_{c0}	0.2647 ± 0.0042
cosmological constant	$\Omega_{\Lambda 0}$	0.6844 ± 0.0091
baryon density	Ω_B	0.04917 ± 0.0006
radiation density	Ω_{r0}	$(8.51 \pm 0.050) \cdot 10^{-5}$
Hubble time age of the Universe	H_0^{-1} t_0	$14.60 \pm 0.14\,\mathrm{Gyr}$ $13.813 \pm 0.026\,\mathrm{Gyr}$
matter-radiation equality	a_{eq} z_{eq}	$(2.711 \pm 0.056) \times 10^{-4}$ 3687.2 ± 76.9
optical depth	τ	0.079 ± 0.017
fluctuation amplitude	σ_8	0.831 ± 0.013

Age and Expansion of the Universe

- since $H = \dot{a}/a$, the age of the Universe is determined by

$$\frac{\mathrm{d}a}{\mathrm{d}t} = H_0 a E(a) \;\Rightarrow\; H_0 t = \int_0^a \frac{\mathrm{d}x}{x E(x)}$$

- in a flat (late) universe with $\Omega_{\mathrm{m}0} \neq 0$ and $\Omega_\Lambda = 1 - \Omega_{\mathrm{m}0} \neq 0$:

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{\mathrm{m}0}}} \operatorname{arcsinh}\left[\sqrt{\frac{1-\Omega_{\mathrm{m}0}}{\Omega_{\mathrm{m}0}}}\, a^{3/2}\right]$$

the age of our universe is

$$t(a=1) = \frac{0.94}{H_0} = (13.813 \pm 0.026) \times 10^9\ \mathrm{yr}$$

Age and Expansion of the Universe: Constraints

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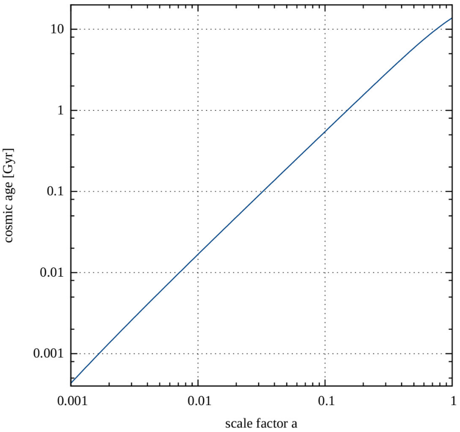
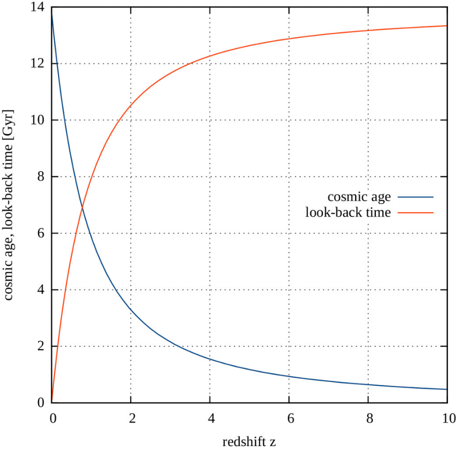
- the Universe should be older than its oldest parts
- three ways of measuring the ages:
  1. nuclear cosmo-chronology: decay of long-lived nuclei $\approx 4.6\,\mathrm{Gyr}$ for the Earth, $7\ldots13\,\mathrm{Gyr}$ for the Galaxy;
  2. ages from stellar evolution: $\gtrsim 12\,\mathrm{Gyr}$ from globular clusters;
  3. cooling of white dwarfs: $\approx 10\,\mathrm{Gyr}$
- $t(a=1) \gtrsim 11\,\mathrm{Gyr}$ needs $H_0 \lesssim 61\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ in an Einstein-de Sitter universe ($\Omega_{m0} = 1$, $\Omega_{\Lambda 0} = 0$)

Cosmic Age



Age as a function of scale factor



Age as function of redshift

Distances

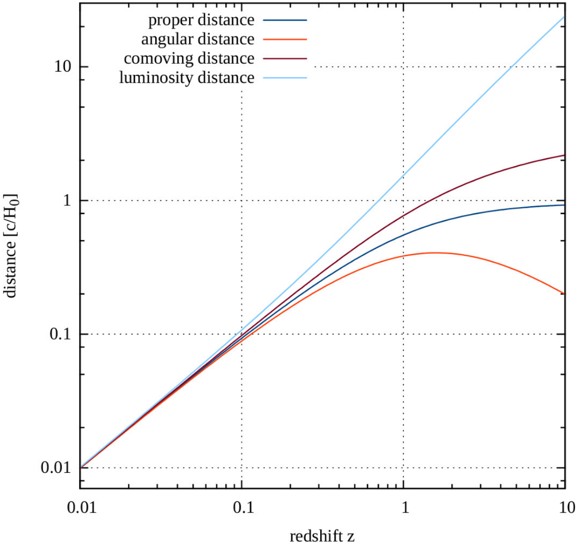
- distance measures are no longer unique in general relativity
- proper distance D_{prop}, $\mathrm{d}D_{\mathrm{prop}} = -c\mathrm{d}t = -c\mathrm{d}a/\dot{a}$
- comoving distance D_{com}, $\mathrm{d}D_{\mathrm{com}} = \mathrm{d}w$
- angular diameter distance D_{ang}

$$D_{\mathrm{ang}}(z_1, z_2) = \left(\frac{\delta A}{\delta \omega}\right)^{1/2} = a(z_2) f_K[w(z_1, z_2)]$$

- luminosity distance D_{lum},

$$D_{\mathrm{lum}}(z_1, z_2) = \left[\frac{a(z_1)}{a(z_2)}\right]^2 D_{\mathrm{ang}}(z_1, z_2)$$

Distances



Distances as functions of redshift

Horizons

- between t_1 and $t_2 > t_1$, light can travel across comoving distance

$$\Delta w(t_1, t_2) = \int_{t_1}^{t_2} \frac{c\mathrm{d}t}{a(t)} = c \int_{a(t_1)}^{a(t_2)} \frac{\mathrm{d}a}{a\dot{a}} \propto a^{n/2-1} \quad \text{if} \quad \rho \propto \rho_0 a^{-n}$$

  if $n > 2$, light can only travel by a finite distance; there exists a particle horizon

- Hubble radius at a_{eq}, important for structure formation

$$r_{\mathrm{H,eq}} = \frac{c}{H(a_{\mathrm{eq}})} = \frac{c}{H_0} \frac{a_{\mathrm{eq}}^{3/2}}{\sqrt{2\Omega_{\mathrm{m0}}}}$$

Thermal Evolution

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2 Parameters, Age and Distances

3 Thermal Evolution
- Assumptions
- Properties of Ideal Quantum Gases
- Adiabatic Expansion of Ideal Gases
- Particle Freeze-Out

4 Recombination and Nucleosynthesis

Assumptions

- the universe expands adiabatically – isotropy requires the universe to expand adiathermally; entropy generation is completely negligible
- thermal equilibrium can be maintained despite the expansion
- the cosmic "fluids" can be treated as ideal gases
- those assumptions are the starting point of our considerations; they need to be verified as we go along

Adiabatic Expansion of Ideal Gases

- for relativistic boson or fermion gases in thermal equilibrium

$$P = \frac{u}{3} = \frac{E}{3V}$$

- first law of thermodynamics implies

$$\mathrm{d}E = -P\mathrm{d}V = 3\mathrm{d}(PV) \;\Rightarrow\; P \propto V^{-4/3}$$

i.e. $\gamma = 4/3$; for non-relativistic ideal gases, $\gamma = 5/3$

- temperature scaling:

$$T \propto P^{1/4} \propto V^{-1/3} \propto a^{-1} \quad \text{(relativistic)}$$
$$T \propto PV \propto V^{-5/3+1} \propto a^{-2} \quad \text{(non-relativistic)}$$

Particle Freeze-Out

- expansion time-scale during radiation-dominated era

$$t_{\rm exp} \approx (G\rho)^{-1/2} \propto a^{-2}$$

- collision rate and time-scale

$$\Gamma \equiv n\langle\sigma v\rangle \propto n \propto T^3 \propto a^{-3}\,, \quad t_{\rm coll} = \Gamma^{-1} \propto a^3$$

- ratio $t_{\rm exp}/t_{\rm coll} \propto a^{-1}$, thermal equilibrium can be maintained despite the expansion at early times;
- thermal equilibrium breaks down when $\Gamma \ll H$
- relativistic particle species retain their thermal-equilibrium density!

Recombination and Nucleosynthesis

3 Thermal Evolution

4 Recombination and Nucleosynthesis
Neutrino Background
Photons and Baryons
Recombination Process
Primordial Nucleosynthesis

5 The Growth of Perturbations

Neutrino Background

- weak interaction

$$\nu + \bar{\nu} \leftrightarrow e^+ + e^-$$

  freezes out when temperature drops to
  $T_\nu \approx 10^{10.5}\,\mathrm{K} \approx 2.7\,\mathrm{MeV}$
- electron-positron pairs annihilate when temperature drops below $T \approx 2m_\mathrm{e}c^2 \approx 1\,\mathrm{MeV} \approx 10^{10}\,\mathrm{K}$
- their decay heats the photon gas, but not the neutrinos
- photon temperature is $\approx 40\%$ higher than neutrino temperature:

$$T_\gamma = \left(\frac{11}{4}\right)^{1/3} T_\nu$$

Photons and Baryons

- number density of baryons today is

$$
\begin{aligned}
n_{\mathrm{B}} &= \frac{\rho_{\mathrm{B}}}{m_{\mathrm{p}}} = \frac{\Omega_{\mathrm{B}}}{m_{\mathrm{p}}} \frac{3H_0^2}{8\pi G} = 1.1 \times 10^{-5}\, \Omega_{\mathrm{B}}\, h^2\, \mathrm{cm}^3 \\
\Omega_{\mathrm{B}} h^2 &\approx 0.02 \qquad (1)
\end{aligned}
$$

- the photon number density today is

$$n_\gamma = 407\, \mathrm{cm}^{-3}$$

- their ratio is constant; about one billion photons per baryon!

$$\eta \equiv \frac{n_{\mathrm{B}}}{n_\gamma} = 2.7 \times 10^{-8}\, \Omega_{\mathrm{B}} h^2$$

Recombination Process

- approximation: Saha's equation; ionisation fraction x is

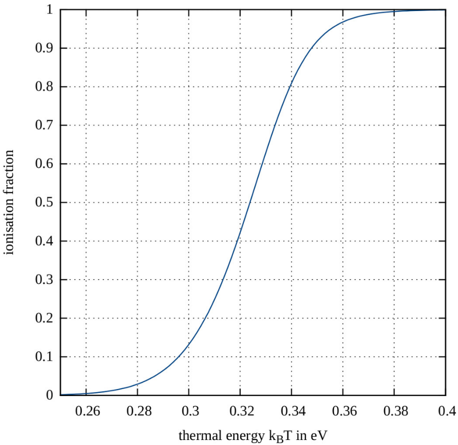
$$\frac{x^2}{1-x} = \frac{\sqrt{\pi}}{4\sqrt{2}\zeta(3)\eta}\left(\frac{m_e c^2}{kT}\right)^{3/2} e^{-\chi/kT} \approx \frac{0.26}{\eta}\left(\frac{m_e c^2}{kT}\right)^{3/2} e^{-\chi/kT}$$

- for recombination to be half-way finished, $x = 1/2 = \text{l.h.s.}$
- since $\eta^{-1} \gg 1$, $kT \ll \chi$ is required
- setting $x = 1/2$ yields $kT_{\text{rec}} = 0.32\,\text{eV}$, or

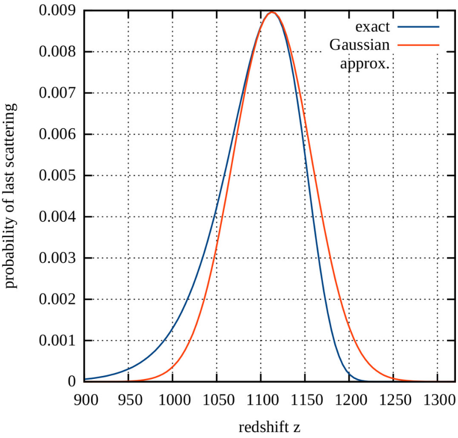
$$T_{\text{rec}} \approx 3000\,\text{K}$$

- for $\chi = 13.6\,\text{eV}$, $T_{\text{rec}} \approx 10^5\,\text{K}$: very large photon-to-baryon ratio η^{-1} delays recombination considerably

Recombination Process



Ionisation as function of temperature



Probability of last photon scattering

Recombination Process: Time and Duration

- recombination time follows from expansion history,

$$t_{\rm rec} = \int_0^{a_{\rm rec}} \frac{\mathrm{d}a}{aH(a)} \approx 374\,\mathrm{kyr}$$

- width of "recombination shell" in redshift,

$$\delta z \approx \left.\frac{\partial z}{\partial T}\right|_{z_{\rm rec}} \delta T \approx 75$$

- corresponds to time interval

$$\delta t \approx \frac{\delta a}{a_{\rm rec} H(a_{\rm rec})} = \frac{a_{\rm rec}\delta z}{H(a_{\rm rec})} \approx 40\,\mathrm{kyr}$$

- provides indirect way of constraining relativistic particle species

Recombination Process: CMB Spectrum

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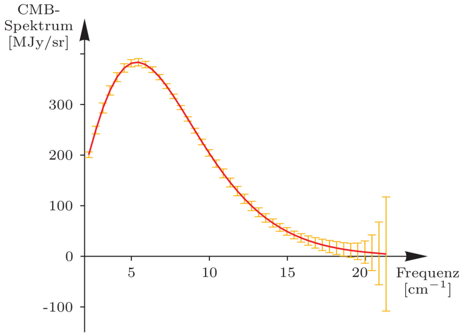
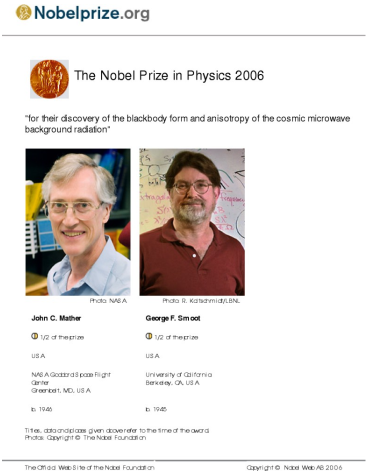


COBE-FIRAS CMB spectrum, COBE



Nobelprize.org

The Nobel Prize in Physics 2006

"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation"

Photo: NASA

Photo: R. Kaltschmidt/LBNL

John C. Mather	George F. Smoot
1/2 of the prize	1/2 of the prize
USA	USA
NASA Goddard Space Flight Center Greenbelt, MD, USA	University of California Berkeley, CA, USA
b. 1946	b. 1945

Titles, data and places given above refer to the time of the award.
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Primordial Nucleosynthesis: Concepts

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- ^{4}He abundance is $\approx 25\,\%$ by mass, far more than stars can have produced
- Universe must have acted as a fusion reactor
- MeV energies require scale factor

$$a \lesssim \frac{\mathrm{meV}}{\mathrm{MeV}} \approx 10^{-9} \ll a_{\mathrm{eq}}$$

- only radiation was important at that time
- baryon-to-photon ratio η is the only relevant parameter,

$$\eta = 10^{10}\eta_{10}\ , \quad \eta_{10} = 273\Omega_{\mathrm{B}}h^2$$

- time scale during early radiation-dominated era

$$t \approx 0.89\left(\frac{T}{\mathrm{MeV}}\right)^{-2}\ \mathrm{s}$$

Primordial Nucleosynthesis: Helium Abundance

- deuterium fusion is crucial, delayed by photon background until $T_{\mathrm{D}} \approx 78\,\mathrm{keV}$
- further fusion builds upon two-body processes, e.g.

$$n + p \to {}^{2}\mathrm{H} + \gamma\,,\;\; {}^{2}\mathrm{H} + {}^{2}\mathrm{H} \to {}^{3}\mathrm{He} + n\,,$$
$${}^{3}\mathrm{He} + {}^{2}\mathrm{H} \to {}^{4}\mathrm{He} + p\,,\;\; {}^{4}\mathrm{H} + {}^{3}\mathrm{H} \to {}^{7}\mathrm{Li} + \gamma \qquad (2)$$

- neutrons form when weak interaction freezes out at $T_n \approx 0.87\,\mathrm{MeV}$, at $t \approx 2\mathrm{s}$
- abundance controlled by Boltzmann factor,

$$\frac{n_n}{n_p} = \exp\left(-\frac{Q}{kT_n}\right)$$

- and subsequent neutron decay with half-life

$$\tau_n = (885.7 \pm 0.8)\,\mathrm{s}$$

Primordial Nucleosynthesis: Results

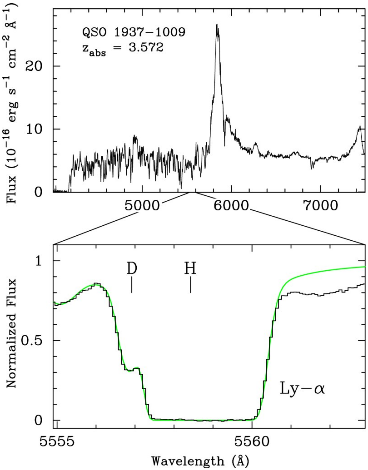
- neutron abundance $X_n \approx 0.17$ by mass at freeze-out
- neutron decay until onset of fusion at $t_\mathrm{D} \approx 150\,\mathrm{s}$ reduces this to $X_n \approx 0.14$, which implies $^4\mathrm{He}$ abundance of $Y \approx 0.28$
- D is most trustworthy baryometer; measured abundance

$$\frac{n_\mathrm{D}}{n_\mathrm{H}} = \left(2.68^{+0.27}_{-0.25}\right) \times 10^{-5}$$

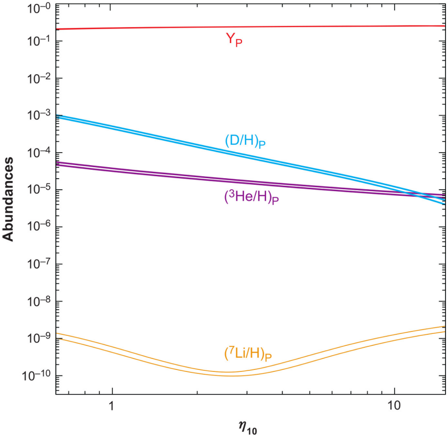
- abundances of D and $^3\mathrm{He}$ decrease with η, $^4\mathrm{He}$ increases, $^7\mathrm{Li}$ has characteristic valley
- measured element abundances imply

$$0.0207 \lesssim \Omega_\mathrm{B} h^2 \lesssim 0.0234$$

Primordial Nucleosynthesis: Results



Deuterium signal in QSO spectrum



From Steigman 2007

The Growth of Perturbations

4 Recombination and Nucleosynthesis

5 The Growth of Perturbations
- Newtonian Equations
- Perturbation Equations
- Velocity Perturbations

6 Statistics and Non-Linear Evolution

Newtonian Equations

- Newtonian hydrodynamics is a valid approximation (flatness, no retardation, short mean free path)
- continuity equation (mass conservation)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

- Euler's equation (momentum conservation)

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\vec{\nabla}\Phi - \frac{\vec{\nabla} p}{\rho}$$

- Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho$$

Density Perturbations: Equations

- split ρ and $\vec{v}$ into background and fluctuations,

$$\rho(t,\vec{x}) = \langle\rho\rangle(t) + \delta\rho(t,\vec{x}) \ , \quad \vec{v}(t,\vec{x}) = \langle\vec{v}\,\rangle(t) + \delta\vec{v}(t,\vec{x})$$

  split velocity into Hubble flow and peculiar velocity

$$\vec{v} = \dot{\vec{r}} = \dot{a}\vec{x} + a\dot{\vec{x}} = H\vec{r} + a\dot{\vec{x}} = \langle\vec{v}\rangle + \delta\vec{v}$$

- comoving coordinates $\vec{x} = \vec{r}/a$, comoving peculiar velocities $\vec{u} \equiv \delta\vec{v}/a$, density contrast $\delta = \delta\rho/\langle\rho\rangle$
- we are now left with the three equations

$$\dot{\delta} + \vec{\nabla}\cdot\vec{u} = 0 \ , \quad \dot{\vec{u}} + 2H\vec{u} = -\frac{\vec{\nabla}\delta\Phi}{a^2} - \frac{\vec{\nabla}\delta p}{a^2\langle\rho\rangle} \ , \quad \nabla^2\delta\Phi = 4\pi G\langle\rho\rangle a^2\delta$$

Density Perturbations: Equations

- combining these, decomposing δ into plane waves

$$\ddot{\delta} + 2H\dot{\delta} = \left(4\pi G\langle\rho\rangle - \frac{c_s^2 k^2}{a^2}\right)\delta$$

sound speed $c_s^2 = \delta p/\delta\rho$

Density Perturbations: Equations

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- combining these, decomposing δ into plane waves

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\langle\rho\rangle\delta$$

sound speed $c_s^2 = \delta p/\delta\rho$

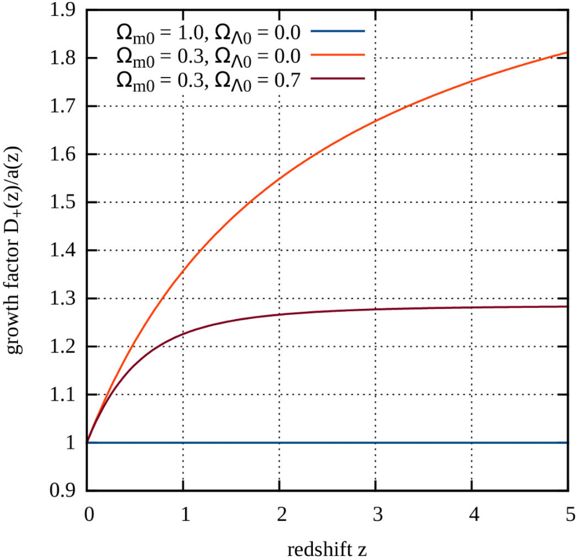
- assume large perturbations,

$$\frac{c_s^2 k^2}{a^2} \ll 4\pi G\langle\rho\rangle \quad \Rightarrow \quad k \ll a\frac{\sqrt{4\pi G\langle\rho\rangle}}{c_s}$$

- linear growth factor

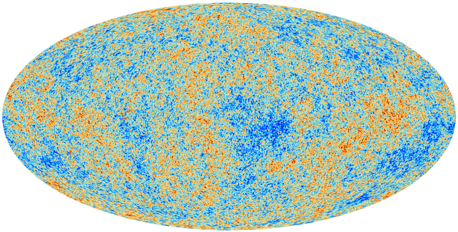
$$\delta(a) = \delta_0 D_+(a)$$

Density Perturbations: Growth

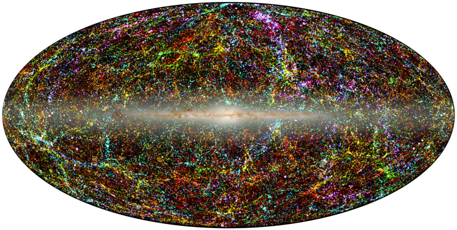


Growth factor divided by scale factor

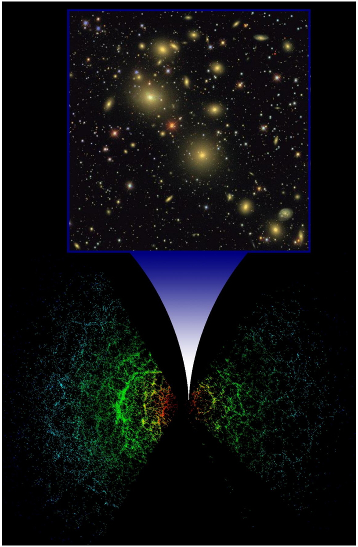
Density Perturbations: Growth



Planck CMB map



2MASS sky map



SDSS map

Statistics and Non-Linear Evolution

5 The Growth of Perturbations

6 Statistics and Non-Linear Evolution
- Power Spectra
- Evolution of the Power Spectrum
- The Zel'dovich Approximation
- Nonlinear Evolution

7 Structures in the CMB

Power Spectra

- variance of δ in Fourier space defines the power spectrum $P(k)$,

$$\left\langle \hat{\delta}(\vec{k})\hat{\delta}^*(\vec{k}')\right\rangle \equiv (2\pi)^3 P(k)\delta_{\rm D}(\vec{k}-\vec{k}')$$

- variance of δ on spatial scale R:

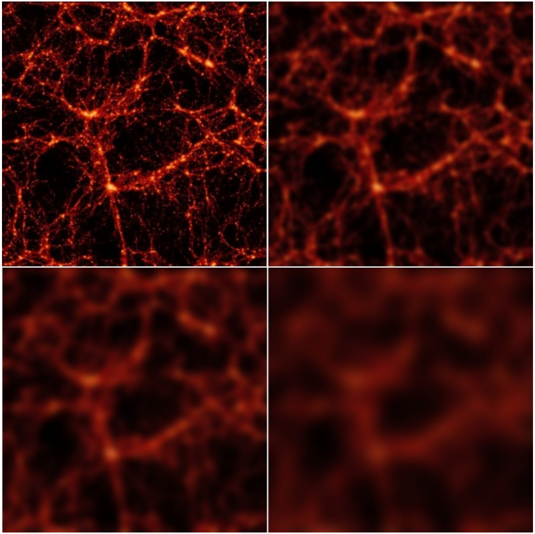
$$\bar{\delta}_R(\vec{x}) \equiv \int \mathrm{d}^3y\, \delta(\vec{x})\, W_R(|\vec{x}-\vec{y}|)$$

- the variance of the filtered density-contrast field is

$$\sigma_R^2 = 4\pi \int \frac{k^2 \mathrm{d}k}{(2\pi)^3} P(k)\hat{W}_R^2(k)$$

σ_8 is often used for normalizing the power spectrum

Power Spectra: Smoothing



Simulated density field, progressively smoothed

Evolution of the Power Spectrum

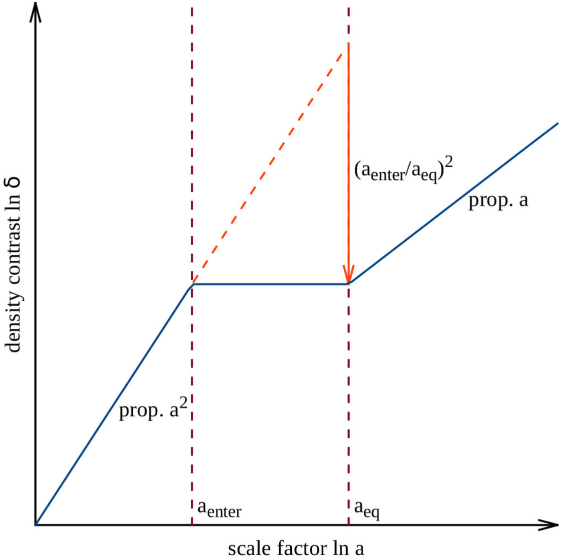
- modes entering the horizon (Hubble radius) while radiation dominates are relatively suppressed compared to larger modes
- assumed time-independence of fluctuation power entering the horizon, combined with suppression for $k > k_{\text{eq}}$ gives

$$P(k) \propto \begin{cases} k^n & (k < k_{\text{eq}}) \\ k^{n-4} & (k \gg k_{\text{eq}}) \end{cases}$$

with $n \approx 1$

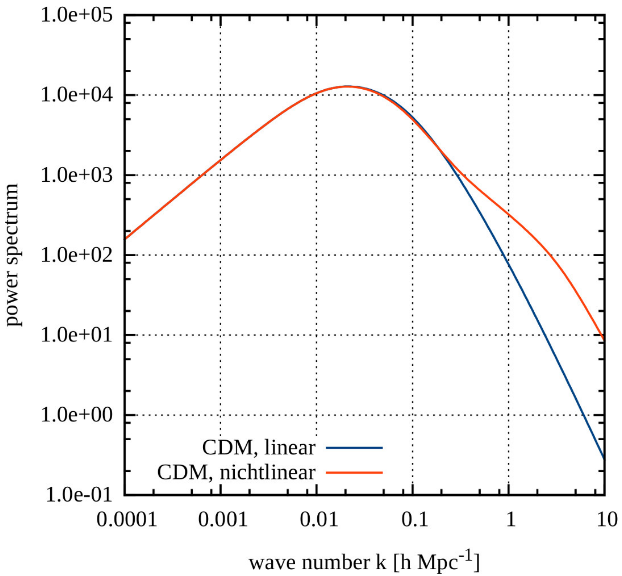
- this is the shape of the spectrum for cold dark matter (CDM)
- hot dark matter (HDM) cuts off the spectrum exponentially

Evolution of the Power Spectrum: Suppression



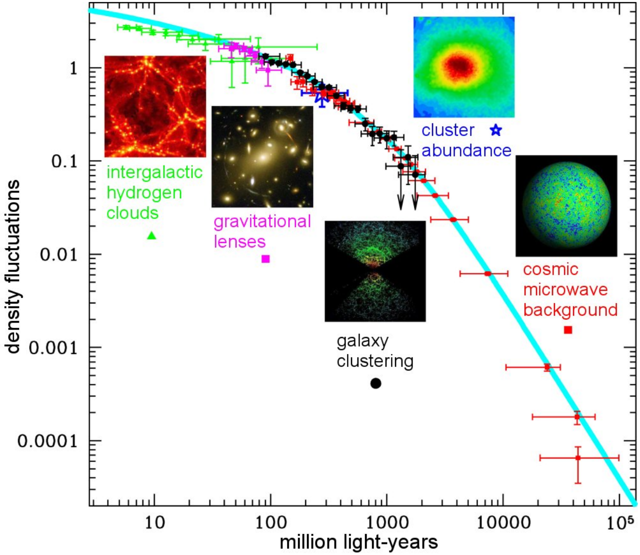
Suppression of small perturbations

Evolution of the Power Spectrum: Nonlinear



Linear and nonlinear density-fluctuation spectra

Power Spectrum: Measurements



Measured density-fluctuation power

Structures in the CMB

6 Statistics and Non-Linear Evolution

7 Structures in the CMB
- Simplified Theory of CMB Temperature Fluctuations
- CMB Power Spectra and Cosmological Parameters
- Foregrounds

8 Cosmological Weak Lensing

CMB Theory: Dipole, Fluctuation Level

- Earth's motion causes temperature dipole,

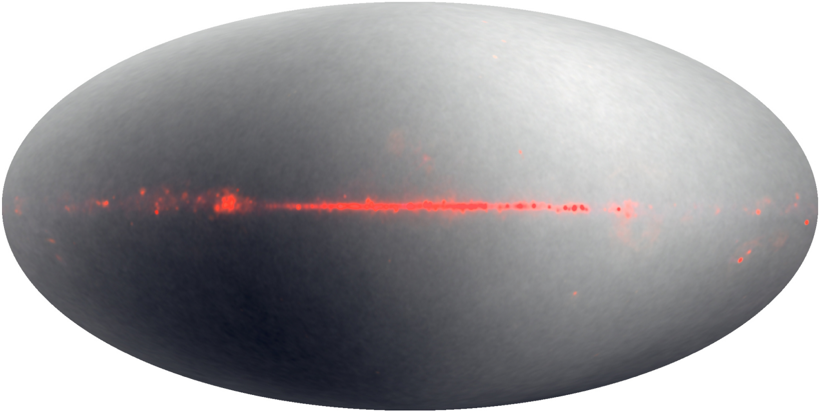
$$T(\theta) = T_0 \left(1 + \frac{v}{c} \cos\theta\right) + \mathcal{O}\left(\frac{v^2}{c^2}\right)$$

- $\delta \gtrsim 1$ today implies

$$\delta(a_{\rm CMB}) = \frac{\delta(a = 1)}{D_+(a_{\rm CMB})} \gtrsim a_{\rm CMB}^{-1} \approx 10^{-3}$$

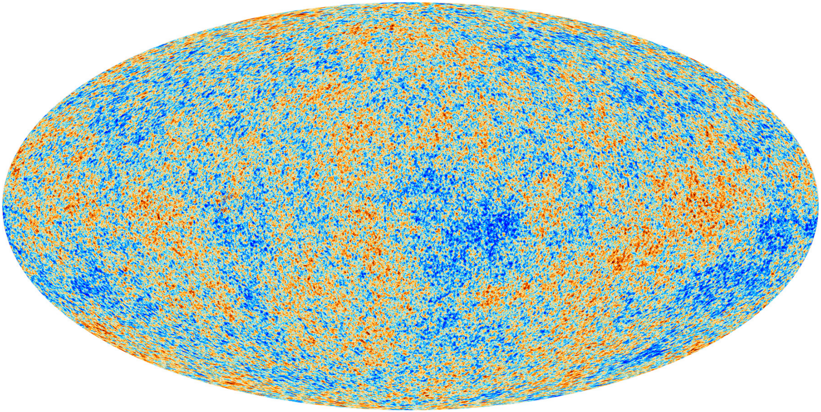
  and similar temperature fluctuations in the CMB
- such fluctuations are not found
- assuming dark matter, temperature fluctuations are expected to be $\delta T / T \approx 10^{-5}$
- detected by COBE in 1992; strongest argument for dark matter

CMB Theory: Dipole



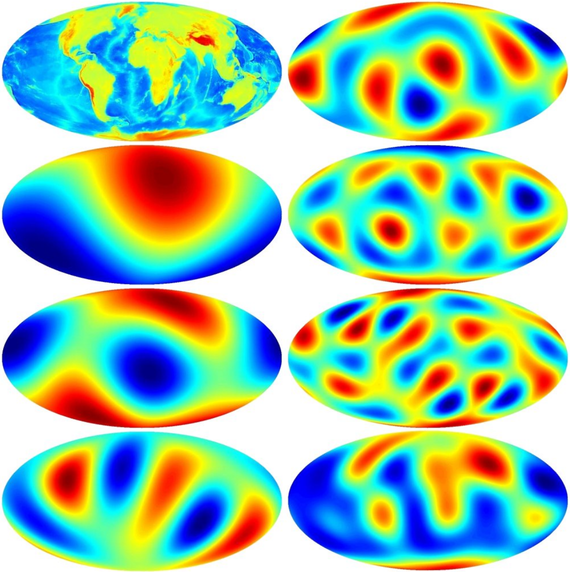
CMB dipole measured by WMAP

CMB Theory: Temperature Fluctuations



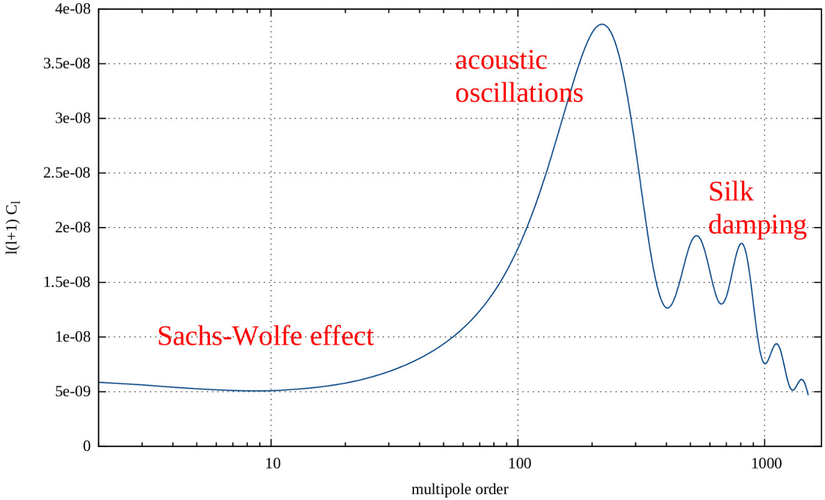
CMB temperature fluctuations measured by Planck

CMB Theory: Multipoles



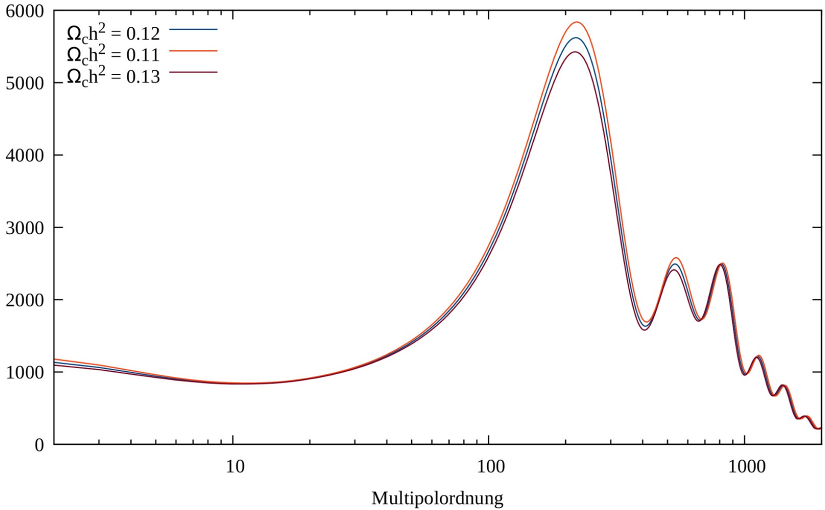
Low-order multipoles of Earth map (ETOPO-5)

CMB Spectra: Principal Effects



Principal physical effects on the CMB spectrum

CMB Spectra: Cosmological Parameters



Effect of dark-matter density

Polarization

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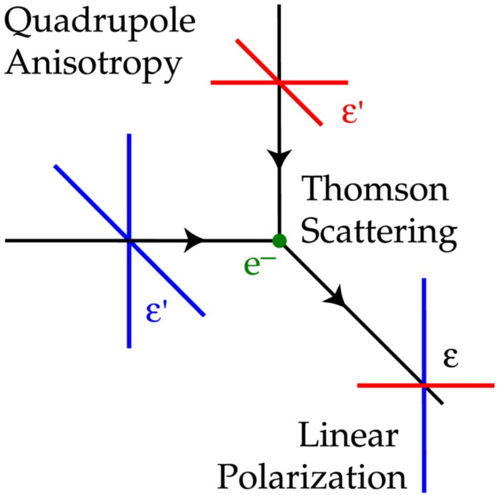
- Thomson scattering (of CMB photons) is anisotropic:

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi} \left|\vec{e}' \cdot \vec{e}\right|^2$$

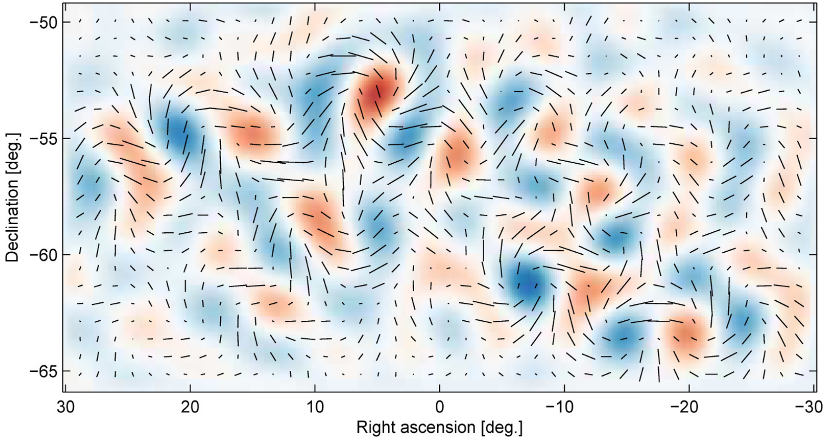
  $\vec{e}'$ and $\vec{e}$: polarization directions of incoming and scattered light
- quadrupolar intensity anisotropy of infalling radiation causes scattered radiation to be polarized
- CMB is expected to be linearly polarized
- intensity of polarized light should be $\approx 10\%$ that of the unpolarized light, amplitude of order 10^{-6} K

Polarization



Linear polarization by Thomson scattering

Polarization



BICEP-2 E-mode polarization map

Instruments and Measurements

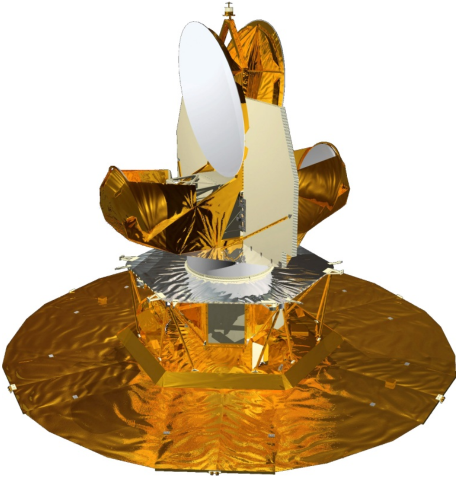
COBE

Instruments and Measurements

COBE

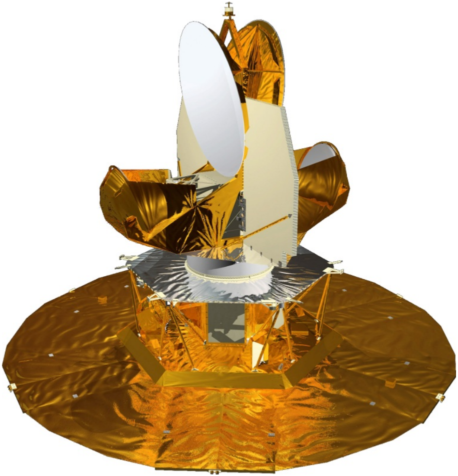
Boomerang, Maxima

Instruments and Measurements

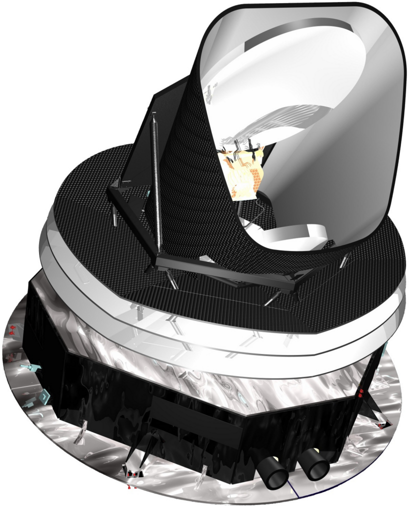


WMAP

Boomerang, Maxima

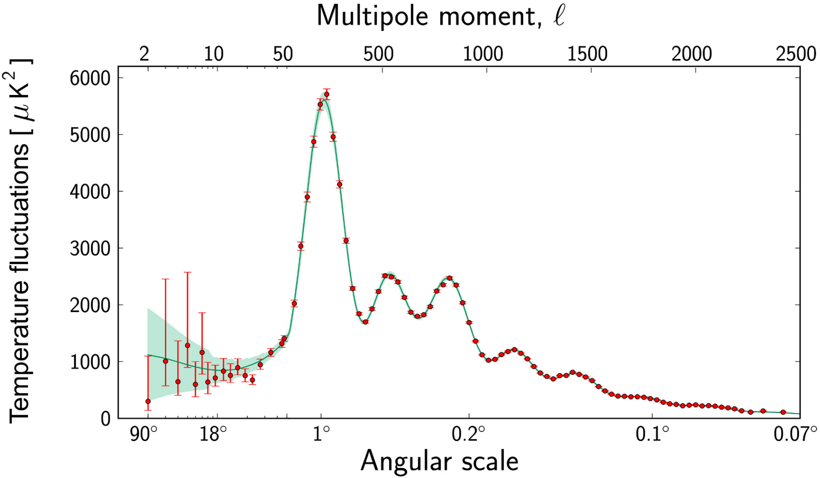


WMAP



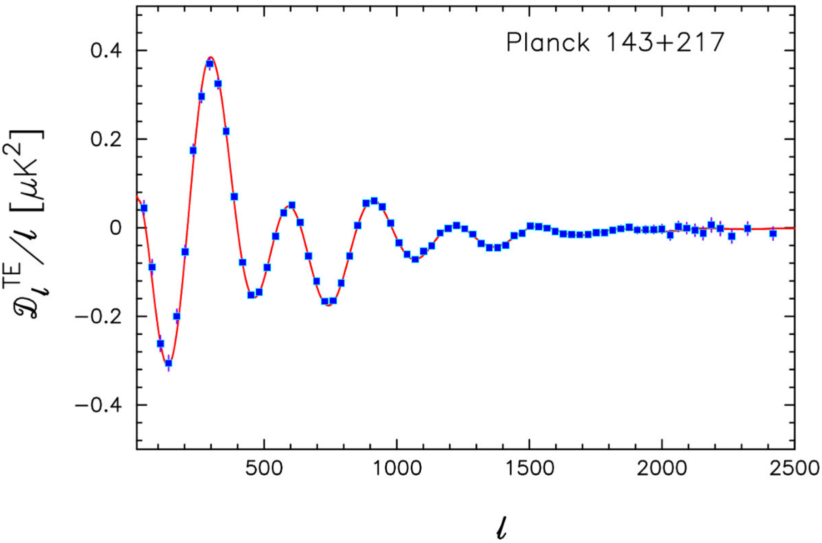
Planck

Instruments and Measurements



T power spectrum, measured by Planck

Instruments and Measurements



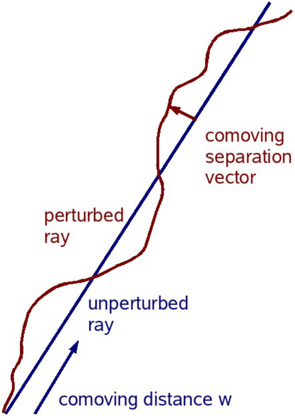
T-E power spectrum, measured by Planck

7 Structures in the CMB

8 Cosmological Weak Lensing
Light Deflection
Measurements

9 Type-Ia Supernovae

Light Deflection



- density inhomogeneities deflect light: gravitational lensing
- astigmatism causes coherent distortions with power spectrum

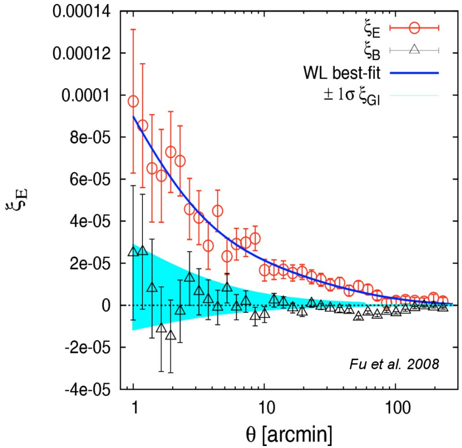
$$P_\gamma(l) = \Omega_{\mathrm{m}0}^2 \int_0^w \mathrm{d}w' \, W^2 P_\delta\left(\frac{l}{f_K(w')}\right)$$

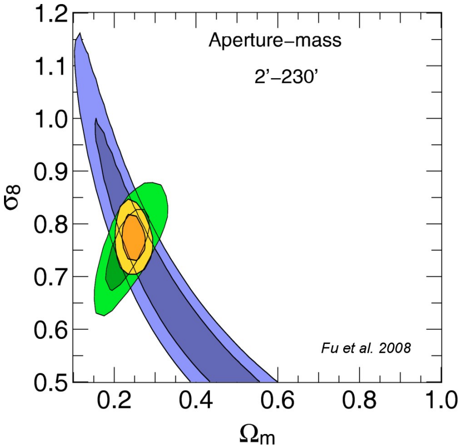
- correlation functions

$$\xi_\gamma(\phi) = \int_0^\infty \frac{l\mathrm{d}l}{2\pi} P_\gamma(l) \mathrm{J}_0(l\phi)$$

are measurable through coherent distortions of distant galaxy images

Measurements





constraints from CFHTLenS, Fu et al. 2008

8 Cosmological Weak Lensing

9 Type-Ia Supernovae
   Classification and Principle
   Measurements

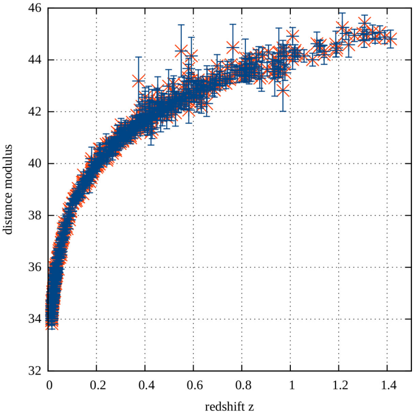
10 Cosmological Inflation and Dark Energy

Classification and Principle

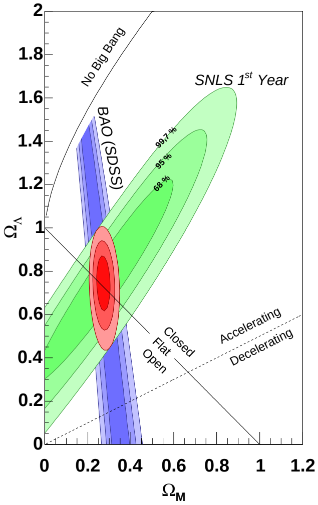
Supernova 1994d

- SN-Ia: white dwarfs in binaries, driven over Chandrasekhar limit
- "standardisable" candles; measured flux gives distance; spectrum gives redshift
- spectra required for classification: no hydrogen, but silicon lines
- expansion history of the Universe can be recovered

Measurements



Union data set



Astier et al.

type-Ia supernovae require

$$\Omega_{m0} = 0.263 \pm 0.037$$

9 Type-Ia Supernovae

10 Cosmological Inflation and Dark Energy
- Problems
- Inflation
- Accelerated Expansion
- Modified Equation of State
- Effects on Cosmology

Problems

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- angular size of the particle horizon at recombination is

$$\theta_{\rm rec} = \frac{a_{\rm rec}\Delta w(0, a_{\rm rec})}{D_{\rm ang}(0, z_{\rm rec})} \approx \sqrt{\Omega_0 a_{\rm rec}} \approx 1.7^\circ \sqrt{\Omega_0}$$

  causal connection? horizon problem

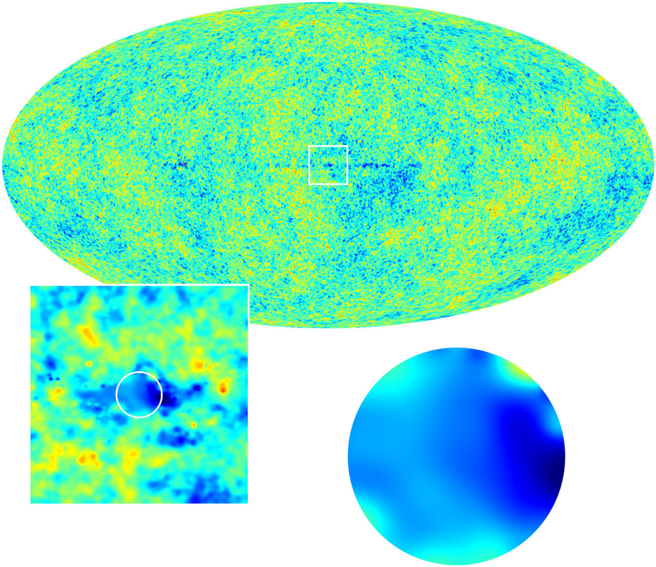
- evolution of flatness:

$$|\Omega_{\rm total} - 1| \propto \begin{cases} t & \text{radiation-dominated era} \\ t^{2/3} & \text{early matter-dominated era} \end{cases}$$

  tiny deviations of $\Omega_{\rm total}$ from unity grow rapidly! flatness problem

- where do structures originate from in the first place?

Problems: Causality Problem



Size of the causal horizon

Inflation: Idea

- accelerated expansion, $\ddot{a} > 0$, can drive universe towards flatness; this seems incompatible with gravity
- Friedmann's equation: accelerated expansion if

$$\rho c^2 + 3p < 0 \,, \quad p < -\frac{\rho c^2}{3}$$

- simple scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$$

has negative pressure if

$$\dot{\phi}^2 < V(\phi)$$

Inflation: Slow Roll, Measurements

- slow-roll conditions:

$$\epsilon \equiv \frac{1}{24\pi G}\left(\frac{V'}{V}\right)^2 \ll 1 \ ,$$

$$\eta \equiv \frac{1}{8\pi G}\left(\frac{V''}{V}\right) \ll 1$$

Inflation: Slow Roll, Measurements

- slow-roll conditions: $\epsilon, \eta \ll 1$
- flatness requires increase in scale factor by $\approx \mathrm{e}^{60}$
- this would also solve the horizon (or causality) problem
- inflaton field must decay through some coupling to “ordinary” matter: reheating

Inflation: Slow Roll, Measurements

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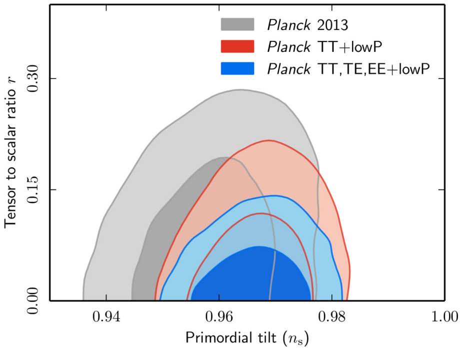

Planck 2015

$$n_s = 0.9677 \pm 0.0060$$

$$\epsilon < 0.011$$

$$\eta = -0.0092^{+0.0074}_{-0.0127}$$

Expansion of the Universe

- CMB: Universe is spatially flat, i.e. its total energy density equals the critical density
- dark matter contributes $\lesssim 30\%$ to the total energy density; light-element abundances requires the baryon density to be much lower
- type-Ia supernovae reveal need for cosmological constant or accelerated expansion
- high-z supernovae show transition from decelaration to acceleration near $z \sim 1$

Modified Equation of State

- cosmological constant may be dissatisfactory
- as for inflation, assume scalar field (“cosmon”, “quintessence”) with negative pressure,

$$p = w\rho c^2 \,, \quad w < -\frac{1}{3}$$

- for constant w,

$$\rho_{\mathrm{Q}} = \rho_{\mathrm{Q0}} a^{-3(1+w)}$$

- Friedmann equation becomes

$$H^2(a) = H_0^2 \left[\Omega_{\mathrm{m0}} a^{-3} + (1 - \Omega_{\mathrm{m0}} - \Omega_{\mathrm{Q0}}) a^{-2} + \Omega_{\mathrm{Q0}} a^{-3(1+w)} \right]$$

Effects on Cosmology

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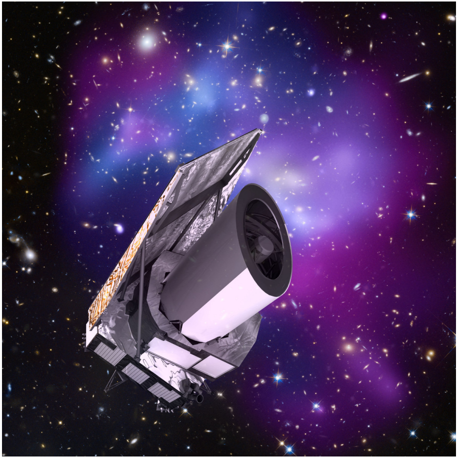
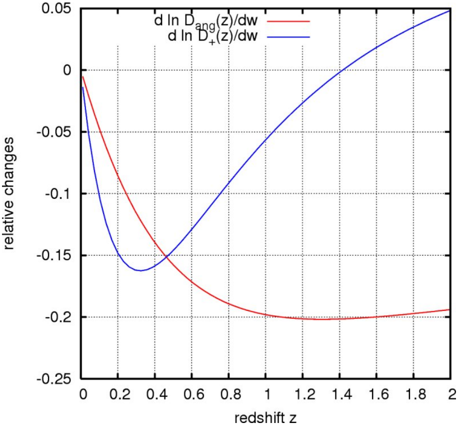
- early expansion is tightly constrained by light-element abundances
- effects on the CMB: width of the recombination shell, amount of Silk damping
- modified angular-diameter and luminosity distances affect supernovae of type Ia, apparent size of CMB fluctuations, cosmic volume, overall geometry of the universe, gravitational lensing
- growth factor is modified; structures form earlier in quintessence models
- dark-matter haloes tend to be denser, which may have strong effects on their appearance

Effects on Cosmology: Examples



Euclid satellite



Sensitivity of observables