Chiral Kinetic Theory from Wigner functions

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Relativistic hydrodynamics: theory and modern applications
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A game of collective rotation

Magnetic field

Rotation of many charged particles

Hydro-dynamics,

Kinetic Theory

Orbital angular momentum

Outline

- **Question:**

How to describe quantum phenomena like anomaly in a semi-classical way for charged massless (chiral) fermion in gauge field?

- **Answer: Quantum Kinetic Theory**

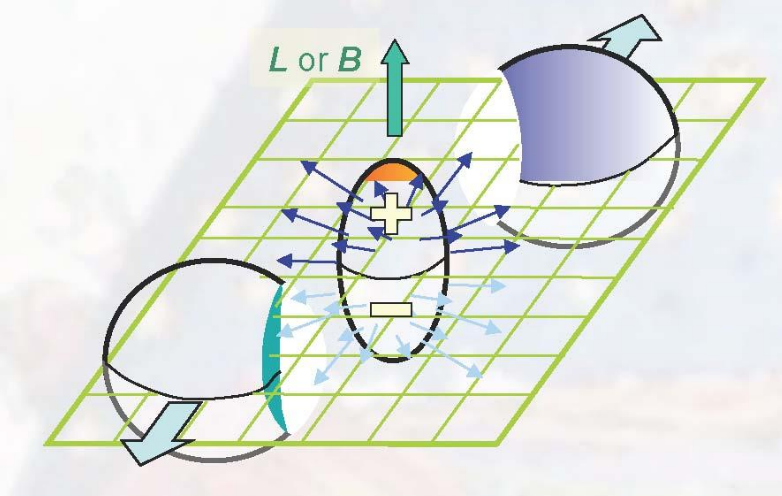
(1) Classical description of charged chiral fermion with anomaly in phase space

(2) Chiral kinetic theory from Wigner function

EM field in HIC

- **High energy HIC**

$$v = \sqrt{(s-m_n^2)/s} \sim 1 - \frac{m_n^2}{2s}$$
$$\gamma = 1/\sqrt{1-v^2/c^2} \sim \frac{\sqrt{s}}{m_n}$$



- **Electric field in cms frame of nucleus,**

$$\mathbf{E} = \frac{Ze}{R^2}\hat{\mathbf{r}}$$

- **Boost to Lab frame (v_z= 0.99995 c for 200GeV),**

Scale of strong interaction

$$\mathbf{B} = -\gamma \mathbf{v}_z \times \mathbf{E} \rightarrow eB \rightarrow 2\gamma v_z \frac{Ze^2}{R^2} \sim \boxed{1.3 m_\pi^2} \sim 2.6 \times 10^{18}\ \mathrm{Gs}$$

Kharzeev, McLerran, Warringa (2008), Skokov (2009), Deng & Huang (2012), Bloczynski,Huang, Zhang, Liao (2012); many others ……

Chirality and helicity of spin-½ fermion

Chiraltiy $\psi_R = \frac{1}{2}(1+\gamma^5)\psi \qquad \psi_L = \frac{1}{2}(1-\gamma^5)\psi,$

Helicity $h = \boldsymbol{\sigma} \cdot \frac{\mathbf{p}}{|\mathbf{p}|}$

RH (h=+1) $\uparrow\uparrow$

LH (h=-1) $\uparrow\downarrow$

In the chiral limit (massless quark) with $m_f = 0$

Helicity	RH chirality	LH chirality
Particle	+1	-1
Anti-particle	-1	+1

Phase space description of charged chiral fermion

- A charged massless fermion (unit charge) in EM field, treat (x,p) in equal footing,

$$S(\boldsymbol{x},\boldsymbol{p}) = \int dt[(\boldsymbol{p}+\boldsymbol{A}(\boldsymbol{x}))\cdot\dot{\boldsymbol{x}} - \phi(\boldsymbol{x}) - \epsilon(\boldsymbol{p})]$$

- EOM can be derived from Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{p}}} - \frac{\partial L}{\partial \boldsymbol{p}} = 0 \rightarrow \boxed{\dot{\boldsymbol{x}} = \frac{\partial \epsilon(\boldsymbol{p})}{\partial \boldsymbol{p}}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} - \frac{\partial L}{\partial \boldsymbol{x}} = 0$$

$$\rightarrow \dot{\boldsymbol{p}} + \boxed{\frac{d}{dt}\boldsymbol{A}(\boldsymbol{x})} = -\frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} + \dot{\boldsymbol{x}}_i \frac{\partial \boldsymbol{A}_i(\boldsymbol{x})}{\partial \boldsymbol{x}}$$

$$\frac{d}{dt}\boldsymbol{A}(\boldsymbol{x}) = \frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{x}) + \dot{\boldsymbol{x}}_j \frac{\partial \boldsymbol{A}_i(\boldsymbol{x})}{\partial \boldsymbol{x}_j}$$

$$\rightarrow \boxed{\dot{\boldsymbol{p}} = \boldsymbol{E} + \dot{\boldsymbol{x}} \times \boldsymbol{B}}$$

Phase space description of charged chiral fermion

- **Re-defining variables $\xi^i = x_i, \xi^{i+3} = p_i$ with $(i = 1,2,3)$**

$$S(\xi) = \int dt[\gamma_a(\xi)\dot{\xi}^a - H(\xi)]$$

No conjugate variables for $\dot{p}$

- **where** $H(\xi) = \phi(\boldsymbol{x}) + \epsilon(\boldsymbol{p})$ **and** $\gamma_a(\xi) = (\boldsymbol{p} + \boldsymbol{A}(\boldsymbol{x}), \boldsymbol{0})$
- **EOM is from Euler-Lagrange Equation**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}^a} = \frac{\partial L}{\partial \xi^a} \rightarrow \frac{\partial \gamma_a(\xi)}{\partial \xi^b}\dot{\xi}^b = \frac{\partial \gamma_b(\xi)}{\partial \xi^a}\dot{\xi}^b - \frac{\partial H(\xi)}{\partial \xi^a}$$

$$\det(\gamma) = 1$$

$$\boxed{\gamma_{ab}\dot{\xi}^b = -\frac{\partial H(\xi)}{\partial \xi^a}}$$

EOM

$$\gamma_{ab} = \frac{\partial \gamma_a(\xi)}{\partial \xi^b} - \frac{\partial \gamma_b(\xi)}{\partial \xi^a}$$

$$[\gamma_{ab}] = \begin{pmatrix} 0 & -B_3 & B_2 & 1 & 0 & 0 \\ B_3 & 0 & -B_1 & 0 & 1 & 0 \\ -B_2 & B_1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Phase space description of charged chiral fermion

- **EOM can be cast to a normal Poisson bracket in Hamiltonian dynamics**

$$\gamma_{ab}\dot{\xi}^b = -\frac{\partial H(\xi)}{\partial \xi^a} \Longrightarrow \dot{\xi}^b = -[\gamma^{-1}]_{ba}\frac{\partial H(\xi)}{\partial \xi^a}$$

from Lagrangian

$$\dot{\xi}^a = -\{\xi^a, H\}$$

from Hamiltonian

- **Now we add the Berry potential term $-\vec{a}(\vec{p})\cdot\dot{\vec{p}}$ to Lagrangian**

$$S(\boldsymbol{x},\boldsymbol{p}) = \int dt[(\boldsymbol{p}+\boldsymbol{A}(\boldsymbol{x}))\cdot\dot{\boldsymbol{x}} - \boldsymbol{a}(\boldsymbol{p})\cdot\dot{\boldsymbol{p}} - \phi(\boldsymbol{x}) - \epsilon(\boldsymbol{p})]$$

$$S(\xi) = \int dt[\gamma_a(\xi)\dot{\xi}^a - H(\xi)]$$

$$\gamma_a(\xi) = (\boldsymbol{p}+\boldsymbol{A}(\boldsymbol{x}), -\boldsymbol{a}(\boldsymbol{p}))$$

From singular property of H=**σ•p** at **p**=**0**

Phase space description of charged chiral fermion

- **EOM can be rewritten into a form which can be compared to Hamiltonian representation**

$$\gamma_{ab}\dot{\xi}^b = -\frac{\partial H(\xi)}{\partial \xi^a} \Longrightarrow \dot{\xi}^b = -[\gamma^{-1}]_{ba}\frac{\partial H(\xi)}{\partial \xi^a}$$

$$[\gamma_{ab}] = \begin{pmatrix} 0 & -B_3 & B_2 & 1 & 0 & 0 \\ B_3 & 0 & -B_1 & 0 & 1 & 0 \\ -B_2 & B_1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & \Omega_3 & -\Omega_2 \\ 0 & -1 & 0 & -\Omega_3 & 0 & \Omega_1 \\ 0 & 0 & -1 & \Omega_2 & -\Omega_1 & 0 \end{pmatrix}$$

$$\det(\gamma) = (1 + \boldsymbol{B}\cdot\boldsymbol{\Omega})^2$$

$$\boldsymbol{\Omega} = \nabla_{\boldsymbol{p}} \times \boldsymbol{a}(\boldsymbol{p})$$

Berry curvature

Phase space description of charged chiral fermion

- **The inverse matrix $[\gamma^{-1}]$ and EOM**

$$\gamma^{-1} = \frac{1}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}} \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 & -1-B_1\Omega_1 & -B_1\Omega_2 & -B_1\Omega_3 \\ -\Omega_3 & 0 & \Omega_1 & -B_2\Omega_1 & -1-B_2\Omega_2 & -B_2\Omega_3 \\ \Omega_2 & -\Omega_1 & 0 & -B_3\Omega_1 & -B_3\Omega_2 & -1-B_3\Omega_3 \\ 1+B_1\Omega_1 & B_2\Omega_1 & B_3\Omega_1 & 0 & -B_3 & B_2 \\ B_1\Omega_2 & 1+B_2\Omega_2 & B_3\Omega_2 & B_3 & 0 & -B_1 \\ B_1\Omega_3 & B_2\Omega_3 & 1+B_3\Omega_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$\boxed{\dot{\xi}^a = -[\gamma^{-1}]^{ab}\frac{\partial H(\xi)}{\partial \xi^b}} \longrightarrow \text{from Euler-Lagrange equation}$$

Hamiltonian dynamics

- **Comparing EOM from Lagranian and Hamiltonian approach**

$$\{\xi^a, \xi^b\} = [\gamma^{-1}]^{ab}$$

$$\boxed{\begin{aligned} \{x_i, x_j\} &= \frac{\epsilon_{ijk}\Omega_k}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}} \\ \{p_i, p_j\} &= \frac{-\epsilon_{ijk}B_k}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}} \\ \{x_i, p_j\} &= -\frac{\delta_{ij}+B_i\Omega_j}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}} \end{aligned}}$$

Chang & Niu, PRL 75, 1348 (1995);
Xiao,Chang,Niu, RMP 82,1959(2010);
Duval,et al, MPLB 20,373(2006)

- **Explicit EOM**

$$\begin{aligned} \dot{x}_i &= -\{x_i, x_j\}\frac{\partial H}{\partial x_j} - \{x_i, p_j\}\frac{\partial H}{\partial p_j} \\ &= \frac{1}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}}\left[\boldsymbol{E}\times\boldsymbol{\Omega}+\boldsymbol{v}+\boldsymbol{B}(\boldsymbol{v}\cdot\boldsymbol{\Omega})\right]_i \end{aligned}$$

$$\begin{aligned} \dot{p}_i &= -\{p_i, x_j\}\frac{\partial H}{\partial x_j} - \{p_i, p_j\}\frac{\partial H}{\partial p_j} \\ &= \frac{1}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}}\left[\boldsymbol{E}+(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}+\boldsymbol{v}\times\boldsymbol{B}\right]_i \end{aligned}$$

Chiral Kinetic equation in 3D

- **The above can be re-written as** ($\sqrt{\det(\gamma)} = 1 + \boldsymbol{B}\cdot\boldsymbol{\Omega}$)

$$\sqrt{\det(\gamma)}\dot{\boldsymbol{x}} = \boldsymbol{v} + \boldsymbol{E}\times\boldsymbol{\Omega} + \underline{\boldsymbol{B}(\boldsymbol{v}\cdot\boldsymbol{\Omega})}$$
$$\sqrt{\det(\gamma)}\dot{\boldsymbol{p}} = \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} + \underline{(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}}$$

symmetry

$$\boldsymbol{x} \leftrightarrow \boldsymbol{p}$$
$$\boldsymbol{v} \leftrightarrow \boldsymbol{E}$$
$$\boldsymbol{B} \leftrightarrow \boldsymbol{\Omega}$$

- **Then we can evaluate**

$$\frac{d\sqrt{\det(\gamma)}}{dt} = \frac{\partial\sqrt{\det(\gamma)}}{\partial t} + \frac{\partial \dot{x}_i\sqrt{\det(\gamma)}}{\partial x_i} + \frac{\partial \dot{p}_i\sqrt{\det(\gamma)}}{\partial p_i}$$
$$= \boldsymbol{\Omega}\cdot\dot{\boldsymbol{B}} + (\nabla_x\times\boldsymbol{E})\cdot\boldsymbol{\Omega} + (\boldsymbol{E}\cdot\boldsymbol{B})(\nabla_p\cdot\boldsymbol{\Omega})$$
$$= \boxed{(\boldsymbol{E}\cdot\boldsymbol{B})}(\nabla_p\cdot\boldsymbol{\Omega})$$

anomaly

where we have used Maxwell equations

$$\nabla_x\cdot\boldsymbol{B} = 0$$
$$\nabla_x\times\boldsymbol{E} + \dot{\boldsymbol{B}} = 0$$

Chiral Kinetic Eq. in 3D:

D.T. Son, N. Yamamoto, PRL 109, 181602 (2012) ; M.A. Stephanov, Y. Yin, PRL 109, 162001 (2012)

Chiral Kinetic equation in 3D

- **Then we can prove the conservation of invariant phase space volume is violated by anomaly (with $\nabla_p \cdot \boldsymbol{\Omega} = 2\pi\delta^3(\vec{p})$)**

$$\frac{d^3x d^3p}{(2\pi)^3}\sqrt{\det(\gamma)} \quad \Rightarrow \quad \frac{d}{dt}\int \frac{d^3x d^3p}{(2\pi)^3}\sqrt{\det(\gamma)} = 2\pi \int \frac{d^3x}{(2\pi)^3}(\boldsymbol{E}\cdot\boldsymbol{B})$$

- **If $f(x,p)$ is conserved in normal phase space**

$$\frac{\partial f}{\partial t} + \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i} = 0$$

- **Then we have chiral kinetic equation in 3D**

$$\frac{\partial f\sqrt{\det(\gamma)}}{\partial t} + \frac{\partial f \dot{x}_i \sqrt{\det(\gamma)}}{\partial x_i} + \frac{\partial f \dot{p}_i \sqrt{\det(\gamma)}}{\partial p_i} = 2\pi\delta^3(\boldsymbol{p})\underline{(\boldsymbol{E}\cdot\boldsymbol{B})}f(t,\boldsymbol{x},\boldsymbol{p})$$

anomaly

Quantum Kinetic Approach in Wigner function

- **To describe dynamics of chiral fermions, we have to explicitly know their helicity (equivalently p), therefore we need to know information of (t,x,p), that's why we use kinetic approach**
- **Classical kinetic approach: f(t,x,p)**
- **Quantum kinetic approach: W(t,x,p)**

4D Wigner Function

Gauge invariant Wigner operator/function

Vasak, Gyulassy, Elze, Annals Phys. 173, 462 (1987)

$$W(x,p) = <: \hat{W}(x,p) :>$$

$$\hat{W}_{\alpha\beta}(x,p) = \int \frac{d^4 y}{(2\pi)^4} e^{-ip\cdot y} \bar{\psi}_\beta\left(x+\frac{1}{2}y\right) \mathcal{P}U\left(A, x+\frac{1}{2}y, x-\frac{1}{2}y\right) \psi_\alpha\left(x-\frac{1}{2}y\right)$$

Gauge link $\mathcal{P}U\left(A, x+\frac{1}{2}y, x-\frac{1}{2}y\right) \equiv \mathcal{P}\mathrm{Exp}\left(-iey^\mu \int_0^1 ds A_\mu\left(x-\frac{1}{2}y+sy\right)\right)$

Dirac equation in electromagnetic field

$$[i\gamma^\mu D_\mu(x) - m]\,\psi(x) = 0, \qquad \bar{\psi}(x)\left[i\gamma^\mu D^\dagger_\mu(x) + m\right] = 0$$

Quantum Kinetic Equation for Wigner function for massless fermion in homogeneous (constant) EM field

$$\gamma_\mu\left(p^\mu + \frac{1}{2}i\nabla^\mu\right) W(x,p) = 0$$

phase space derivative

$$\nabla^\mu \equiv \partial^\mu_x - QF^{\mu\nu}\partial^p_\nu$$

Wigner functions

- **For collisionless fermions in constant EM field**

$$\gamma_\mu \left(p^\mu + \frac{1}{2} i \nabla^\mu \right) W(x,p) = 0.$$

- **Wigner function decomposition in 16 generators of Clifford algebra**

$$W = \frac{1}{4}\left[\mathscr{F} + i\gamma^5 \mathscr{P} + \gamma^\mu \mathscr{V}_\mu + \gamma^5 \gamma^\mu \mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu} \mathscr{S}_{\mu\nu} \right]$$

scalar p-scalar vector axial-vector tensor

$$j^\mu = \int d^4 p \mathscr{V}^\mu, \qquad j_5^\mu = \int d^4 p \mathscr{A}^\mu, \qquad T^{\mu\nu} = \int d^4 p p^\mu \mathscr{V}^\nu$$

Vasak, Gyulassy and Elze, Annals Phys. 173, 462 (1987);
Elze, Gyulassy and Vasak, Nucl. Phys. B 276, 706(1986);
Zhuang and Heinz, Annals Phys. 245, 311 (1996).

Wigner functions for massless fermions

- **Decoupled equations for RH (s=+1) and LH (s=-1) components**

$$\begin{aligned} p^\mu \mathscr{J}^s_\mu(x,p) &= 0, \\ \nabla^\mu \mathscr{J}^s_\mu(x,p) &= 0, \\ 2s(p^\lambda \mathscr{J}^\rho_s - p^\rho \mathscr{J}^\lambda_s) &= -\epsilon^{\mu\nu\lambda\rho}\nabla_\mu \mathscr{J}^s_\nu, \end{aligned}$$

$$\boxed{\nabla^\mu = \partial^\mu_x - QF^{\mu\nu}\partial^p_\nu}$$

- **where RH and LH components are (s=+1, -1)**

$$\mathscr{J}^s_\mu(x,p) = \frac{1}{2}[\mathscr{V}_\mu(x,p) + s\mathscr{A}_\mu(x,p)]$$

- **Expand vector and axial vector in powers of** ∂^μ_x **and** $F^{\mu\nu}$

$$\mathscr{J}^s_\mu = \mathscr{J}^{s(0)}_\mu + \mathscr{J}^{s(1)}_\mu + \mathscr{J}^{s(2)}_\mu + \mathscr{J}^{s(3)}_\mu \ldots$$

Solution of Wigner function

- The solutions to $(F_{\mu\nu})^1$ and $(\partial_x)^1$ encodes a lot of information !!

$$\mathscr{J}_s^\rho(x,p) = \mathscr{V}^\rho + s\mathscr{A}^\rho$$

$$\begin{aligned}
\mathscr{J}^\rho_{(0)s}(x,p) &= p^\rho f_s \delta(p^2) \\
\mathscr{J}^\rho_{(1)s}(x,p) &= -\frac{s}{2}\tilde{\Omega}^{\rho\beta}p_\beta \frac{df_s}{dp_0}\delta(p^2) - \frac{s}{p^2}Q\tilde{F}^{\rho\lambda}p_\lambda f_s \delta(p^2)
\end{aligned}$$

- where

Fermi-Dirac Distr.

$$f_s(x,p) = \frac{2}{(2\pi)^3}\left[\Theta(p_0)\underline{f_F(p_0-\mu_s)} + \Theta(-p_0)\underline{f_F(-p_0+\mu_s)}\right]$$

$$\tilde{F}^{\rho\lambda} = \frac{1}{2}\epsilon^{\rho\lambda\mu\nu}F_{\mu\nu} \qquad \Omega_{\nu\sigma} = \frac{1}{2}(\partial_\nu u_\sigma - \partial_\sigma u_\nu) \qquad \mu_s = \mu + s\mu_5$$

$$\tilde{\Omega}^{\xi\eta} = \frac{1}{2}\epsilon^{\xi\eta\nu\sigma}\Omega_{\nu\sigma}$$

Gao, Liang, Pu, QW, Wang, PRL 109, 232301(2012)

Decoding the solution:
(1) CME , CVE, CSE and Chiral Anomly

- **Vector current** $j^\mu = nu^\mu + \xi\omega^\mu + \xi_B B^\mu$

Gao, Liang, S. Pu, QW, Wang, PRL 109, 232301(2012)

CME: $\xi_B = Q\dfrac{\mu_5}{2\pi^2}$ **CVE:** $\xi = \dfrac{1}{\pi^2}\mu\mu_5$

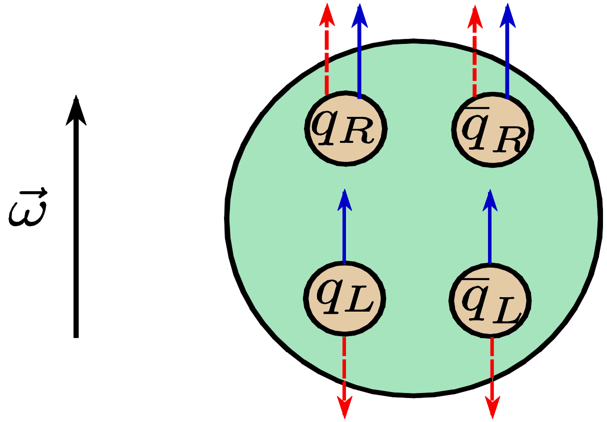
- **Axial current** $j_5^\mu = n_5 u^\mu + \xi_5\omega^\mu + \xi_{B5}B^\mu$

-Chiral separation effect:

$$\xi_{B5} = Q\frac{\mu}{2\pi^2}$$

-Local polarization effect

$$\xi_5 = \frac{1}{6}T^2 + \frac{1}{2\pi^2}\left(\mu^2 + \mu_5^2\right)$$



- **Derived conservation laws**

$$\partial_\alpha T^{\alpha\beta} = QF^{\alpha\mu}j_\mu, \qquad \partial_\alpha j^\alpha = 0, \qquad \partial_\alpha j_5^\alpha = -\frac{Q^2}{2\pi^2}E\cdot B$$

CME conductivity: discrete symmetries

$$\mathbf{j} = Q^2 \frac{\mu_5}{2\pi^2} \mathbf{B}$$

P-odd

T-odd

P-even

T-odd

P-odd effect

T-even effect
Non-dissipative current
Topologically protected

T: Time reversal
P: Parity

Ohm conductivity

$$\mathbf{j} = \sigma \mathbf{E}$$

T-odd
Dissipative

From D. Kharzeev's talk

Decoding the solution:
(2) CKE from 4D to 3D and Berry Phase

- **Covariant Chiral Kinetic Equation in 4D (CCKE)**

$$\nabla_\mu \mathscr{J}_s^\mu = 0 \quad \Rightarrow \quad \delta\left(p^2\right)\left(\frac{dx^\mu}{d\tau}\partial_\mu^x f_s + \frac{dp^\mu}{d\tau}\partial_\mu^p f_s\right) = 0$$

$$\frac{dx^\mu}{d\tau} \equiv p^\mu + s\epsilon^{\mu\nu\alpha\beta}b_\nu F_{\alpha\beta}, \qquad \frac{dp^\mu}{d\tau} \equiv F^{\mu\nu}p_\nu - s\left(E\cdot B\right)b^\mu$$

Berry Curvature In 4D $\quad b^\mu \equiv -\frac{p^\mu}{p^2}$

- **Chiral Kinetic Equation in 3D**

$$\int dp_0 \nabla_\mu \mathscr{J}_s^\mu = 0 \quad \Rightarrow \quad \partial_t f_s + \frac{d\mathbf{x}}{dt}\cdot\nabla_{\mathbf{x}} f_s + \frac{d\mathbf{p}}{dt}\cdot\nabla_{\mathbf{p}} f_s = 0$$

$$\frac{d\mathbf{x}}{dt} \equiv \frac{\hat{\mathbf{p}} + s\left[(\hat{\mathbf{p}}\cdot\mathbf{\Omega})\mathbf{B} + \mathbf{E}\times\mathbf{\Omega}\right]}{1 + s\mathbf{\Omega}\cdot\mathbf{B}}, \qquad \frac{d\mathbf{p}}{dt} \equiv \frac{\mathbf{E} + \hat{\mathbf{p}}\times\mathbf{B} + s(\mathbf{E}\cdot\mathbf{B})\mathbf{\Omega}}{1 + s\mathbf{\Omega}\cdot\mathbf{B}}$$

Chen, Pu, QW, Wang, PRL 110, 262301 (2013)
Son, Yamamoto, PRL 109, 181602 (2012)
Stephanov, Yin, PRL 109, 162001 (2012)

Berry Curvature In 3D $\quad \mathbf{\Omega} = \frac{\hat{\mathbf{p}}}{2\mathbf{p}^2}$

Decoding the solution:
(3) Energy shift and magnetic Moment

- **Particle and energy density with** $\mu_s(x,\mathbf{p})\ \ (\mu_s \equiv \mu + s\mu_5)$

$$
\boxed{\begin{aligned}
n_s &= \int d^3\mathbf{p} f_s + \int d^3\mathbf{p}\frac{s}{2E_p^2}(\mathbf{v}\cdot\mathbf{B})f_s - \int d^3\mathbf{p}\frac{s}{2E_p}(\mathbf{v}\cdot\mathbf{B})\frac{d}{dE_p}f_s \\
&\approx \int \frac{d^3\mathbf{p}}{(2\pi)^3}\sqrt{\gamma_s}\left[f_F\left(E_p' - \mu_s(x,\mathbf{p})\right) - f_F\left(E_p' + \mu_s(x,\mathbf{p})\right)\right],
\end{aligned}}
$$

$$
\boxed{\begin{aligned}
\epsilon_s &= \int d^3\mathbf{p} E_p f_s + \int d^3\mathbf{p}\frac{s}{2E_p}(\mathbf{v}\cdot\mathbf{B})f_s - \int d^3\mathbf{p}\frac{s}{2}(\mathbf{v}\cdot\mathbf{B})\frac{d}{dE_p}f_s \\
&\approx \int \frac{d^3\mathbf{p}}{(2\pi)^3}\sqrt{\gamma_s}E_p'\left[f_F\left(E_p' - \mu_s(x,\mathbf{p})\right) + f_F\left(E_p' + \mu_s(x,\mathbf{p})\right)\right],
\end{aligned}}
$$

- **Phase-space measure:** $\sqrt{\gamma_s} \equiv (1 + s\Omega\cdot\mathrm{B})$
- **Berry curvature:** $\Omega = \dfrac{\hat{\mathrm{p}}}{2\mathrm{p}^2}$

Gao, QW, PLB749, 542 (2015)

Decoding the solution: (3) Energy shift and magnetic moment

- **Effective Energy of Chiral Fermion:**

$$E'_p = |\mathbf{p}| + \Delta E_B$$

- **Energy Shift:**

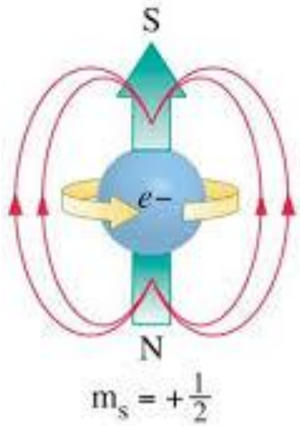
$$\Delta E_B = -\boldsymbol{\mu}_m \cdot \mathbf{B}$$

- **Magnetic moment of massless fermion:**

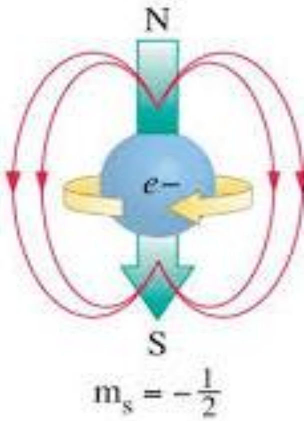
$$\boldsymbol{\mu}_m = \frac{Q}{|\mathbf{p}|}\mathbf{S}$$

- **Spin:** $\mathbf{S} = \frac{s}{2}\hat{\mathbf{p}}$ $\qquad s = \pm 1$

Son, Yamamoto (2013); Manual, Torres-Rincon (2014); Satow, Yee (2014)



$$\boldsymbol{\mu}_m = \frac{Q}{\mathrm{m}}\mathbf{S}$$



Decoding the solution:
(4) Energy Shift from Spin-Vorticity Coupling

- **Particle and energy density with** $\mu_s(x, \mathbf{p})$

$$\boxed{\begin{aligned} n_s &= \int d^3\mathbf{p} f_s - \int d^3\mathbf{p} \frac{s}{2}(\boldsymbol{\omega}\cdot\mathbf{v})\frac{d}{dE_p} f_s \\ &\approx \int \frac{d^3\mathbf{p}}{(2\pi)^3}\left[f_F\left(E'_p - \mu_s(x,\mathbf{p})\right) - f_F\left(E'_p + \mu_s(x,\mathbf{p})\right)\right], \end{aligned}}$$

$$\boxed{\begin{aligned} \epsilon_s &= \int d^3\mathbf{p} E_p f_s - \int d^3\mathbf{p} \frac{s}{2}(\mathbf{v}\cdot\boldsymbol{\omega}) E_p \frac{d}{dE_p} f_s \\ &\approx \int \frac{d^3\mathbf{p}}{(2\pi)^3} E'_p \left[f_F\left(E'_p - \mu_s(x,\mathbf{p})\right) + f_F\left(E'_p + \mu_s(x,\mathbf{p})\right)\right], \end{aligned}}$$

- **No phase-space measure, no Berry curvature**
- **Effective energy:**

$$\Delta E_\omega = -\boldsymbol{\omega}\cdot\mathbf{S}$$

$$E'_p = E_p + \Delta E_\omega$$

Gao, QW, PLB749, 542 (2015)

(5) Another solution: Linear Response Theory for Wigner Functions

- **Fields not constant, expansion in $(F_{\mu\nu})^n$ [not in $(\partial_x)^n$!] , the first order equation:**

$$p^\mu \mathscr{J}^{(1)}_{s\mu} + \delta\Pi^\mu \mathscr{J}^{(0)}_{s\mu} = 0, \quad \partial_x^\mu \mathscr{J}^{(1)}_{s\mu} + \delta G^\mu \mathscr{J}^{(0)}_{s\mu} = 0,$$

$$\epsilon_{\mu\nu\rho\sigma}\left[\partial_x^\rho \mathscr{J}_s^{(1)\sigma} + \delta G^\rho \mathscr{J}_s^{(0)\sigma}\right] = -2s\left(p_\mu \mathscr{J}^{(1)}_{s\nu} - p_\nu \mathscr{J}^{(1)}_{s\mu}\right) - 2s\left[\delta\Pi_\mu \mathscr{J}^{(0)}_{s\nu} - \delta\Pi_\nu \mathscr{J}^{(0)}_{s\mu}\right]$$

- **Formal solution:** $\Delta \equiv \partial^p \cdot \partial_x$

$$\mathscr{J}^{(1)}_{s\mu} = -\frac{s}{2p\cdot\partial_x}\epsilon_{\mu\nu\rho\sigma}\partial_x^\nu\left[j_0\left(\frac{\Delta}{2}\right)F^{\rho\lambda}\partial^p_\lambda \mathscr{J}_s^{(0)\sigma}\right] + \frac{1}{p\cdot\partial_x}p_\mu j_0\left(\frac{\Delta}{2}\right)F^{\nu\lambda}\partial^p_\lambda \mathscr{J}^{(0)}_{s\nu}$$
$$-\frac{1}{2p\cdot\partial_x}\partial_x^\nu\left[j_1\left(\frac{\Delta}{2}\right)\left(F_{\mu\lambda}\partial_p^\lambda \mathscr{J}^{(0)}_{s\nu} - F_{\nu\lambda}\partial_p^\lambda \mathscr{J}^{(0)}_{s\mu}\right)\right]$$

- **Parity-odd part of the Wigner function in momentum space:**

$$\mathscr{J}^{(1)}_{s\mu}(k,p) = -i\frac{s}{2p\cdot k}\epsilon_{\mu\nu\rho\sigma}k^\nu p^\sigma A^\rho(k) j_0(\Delta)(k\cdot\partial_p)[f_s\delta(p^2)]$$

EM field momentum

fermion momentum

Gao, QW, PLB 749, 542 (2015)

(5) Another solution: Chiral Magnetic Conductivity

- **Chiral Magnetic Conductivity:**

$$\vec{j}(\omega,\mathbf{k}) = \int d^4p \left(\vec{\mathscr{J}}_{+}^{(1)} + \vec{\mathscr{J}}_{-}^{(1)} \right) = \sigma_\chi(\omega,\mathbf{k})\vec{B}$$

$$\sigma_\chi = \frac{\mathbf{k}^2-\omega^2}{16\pi^2|\mathbf{k}|^3}\int d|\mathbf{p}| f(|\mathbf{p}|) \sum_{t=\pm 1} (2|\mathbf{p}|+t\omega) \ln\left[\frac{(\omega+i\epsilon+t|\mathbf{p}|-(\omega+|\mathbf{p}|)^2}{(\omega+i\epsilon+t|\mathbf{p}|-(\omega-|\mathbf{p}|)^2}\right]$$

- **HTL/HDL results from Wigner function:** $\omega, |\mathbf{k}| \ll |\mathbf{p}|$

$$\sigma_\chi(\omega,\mathbf{k}) = \sigma_\chi^{(0)}\left(1-\frac{\omega^2}{|\mathbf{k}|^2}\right)\left[1-\frac{\omega}{2|\mathbf{k}|}\ln\frac{\omega+|\mathbf{k}|}{\omega-|\mathbf{k}|}\right]$$

$$\sigma_\chi^{(0)} = Q\frac{\mu_5}{2\pi^2}$$

Laine, JHEP 0510, 056 (2005)
Kharzeev, Warringa, PRD80, 034028 (2009)

Decoding the solution:
(6) Polarization of massive fermions

- **Spin-vorticity coupling and magnetic-moment energy in magnetic field**

$$\delta E = \frac{1}{2}\hbar \mathbf{n}\cdot\vec{\omega} + \hbar Q \frac{\mathbf{n}\cdot\mathbf{B}}{E_p}$$

- **Polarization: heuristic argument**

$$\begin{aligned}\Pi &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left[f(E_p-\delta E) - f(E_p+\delta E)\right] \\ &\approx -\int \frac{d^3p}{(2\pi)^3}\delta E \boxed{\frac{\partial f(E_p)}{\partial E_p}} \longrightarrow \text{susceptiblility}\end{aligned}$$

Becattini, Ferroni, EJPC 52,597(2007); Betz, Gyulassy, Torrieri, PRC 76, 044901(2007); Becattini, Piccinini, Rizzo, PRC 77, 024906(2008); Beccatini, Csernai, Wang, PRC 87, 034905(2013); Xie, Glastad, Csernai, PRC 92,064901(2015).

Decoding the solution:
(6) Polarization of massive fermions

- **Wigner function for massive fermion: the axial vector component give polarization phase space density.**

$$n^{\mu}(\mathbf{p},\mathbf{n}) = \Lambda^{\mu}_{\nu}(\mathbf{v})n^{\nu}(\mathbf{0},\mathbf{n}) = \left(\frac{\mathbf{n}\cdot\mathbf{p}}{m},\mathbf{n} + \frac{(\mathbf{n}\cdot\mathbf{p})\mathbf{p}}{m(m+E_p)}\right)$$

spin in Lab frame

$$\begin{aligned} A^{\mu}_{(0)} &= \mathrm{Tr}[\gamma^{\mu}\gamma^{5}W] \\ &= m\left[\theta(p_0)\underline{n^{\mu}(\mathbf{p},\mathbf{n})} - \theta(-p_0)\underline{n^{\mu}(-\mathbf{p},-\mathbf{n})}\right]\delta(p^2-m^2)A, \end{aligned}$$

$$A^{\alpha}_{(1)}(x,p) = -\frac{1}{2}\hbar\tilde{\Omega}^{\alpha\sigma}p_{\sigma}\frac{dV}{d(\beta p_0)}\delta(p^2-m^2) - Q\hbar\tilde{F}^{\alpha\lambda}p_{\lambda}V\frac{\delta(p^2-m^2)}{p^2-m^2},$$

- **Where**

$$A \equiv \frac{2}{(2\pi)^3}\sum_{s} s\left[\theta(p^0)f_{\mathrm{FD}}(p_0-\mu_s) + \theta(-p^0)f_{\mathrm{FD}}(-p_0+\mu_s)\right]$$

$$V \equiv \frac{2}{(2\pi)^3}\sum_{s}\left[\theta(p^0)f_{\mathrm{FD}}(p_0-\mu_s) + \theta(-p^0)f_{\mathrm{FD}}(-p_0+\mu_s)\right]$$

Decoding the solution:
(6) Polarization of massive fermions

- **Wigner function for massive fermion: the axial vector component give polarization phase space density**

$$\Pi^{\alpha}_{(0)}(x) = \frac{1}{2}\int d^4p\, A^{\alpha}_{(0)}(x,p) = 0$$

$$\Pi^{\alpha}_{(1)}(x) = \frac{1}{2}\int d^4p\, A^{\alpha}_{(1)}(x,p) = -\frac{1}{4}\int d^4p \hbar \tilde{\Omega}^{\alpha\sigma} p_{\sigma} \frac{dV}{d(\beta p_0)} \delta(p^2 - m^2) + \frac{1}{2}\hbar Q \int d^4p \tilde{F}^{\alpha\lambda} p_{\lambda} V \frac{d}{dp_0^2} \delta(p^2 - m^2)$$

susceptibility

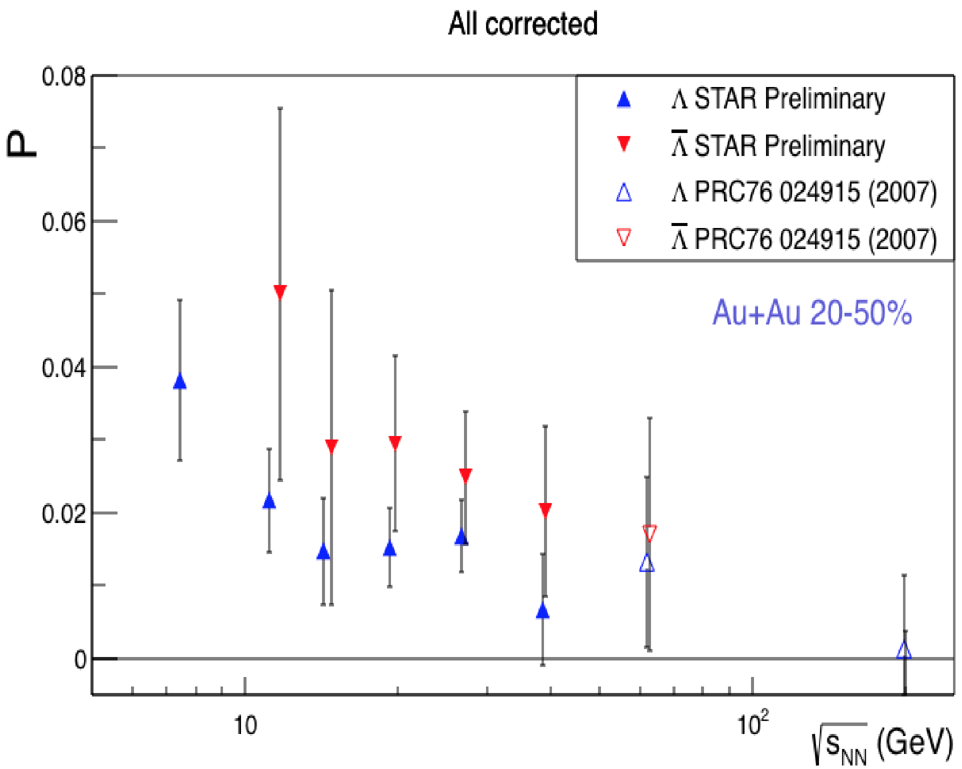
$$\Pi^{\alpha}(\text{fermion}) = \frac{1}{2}\hbar \int \frac{d^3p}{(2\pi)^3} \left[\omega^{\alpha} + |Q|\beta \frac{B^{\alpha}}{E_p}\right] \boxed{f^{+}_{\text{FD}}(1 - f^{+}_{\text{FD}})}$$

$$\Pi^{\alpha}(\text{anti} - \text{fermion}) = \frac{1}{2}\hbar \int \frac{d^3p}{(2\pi)^3} \left[\omega^{\alpha} - |Q|\beta \frac{B^{\alpha}}{E_p}\right] \boxed{f^{-}_{\text{FD}}(1 - f^{-}_{\text{FD}})}$$

Fang, Pang, QW, Wang, PRC 94,024904(2016)

Decoding the solution:
(6) Polarization of massive fermions

- **STAR preliminary result at low energies**
  P ≈ 2-4%
  ΔP (Λ- $\bar{\Lambda}$) ≈ 1-2%
- **Magnetic field to lead to ΔP**



$$\Delta P \sim \frac{1}{2}\beta\frac{B^\alpha}{m_\Lambda} \times \sum_{e=\pm}\boxed{\frac{\int d^3p f_{\mathrm{FD}}^e(1-f_{\mathrm{FD}}^e)}{\int d^3p f_{\mathrm{FD}}^e}}$$

≈O(1)

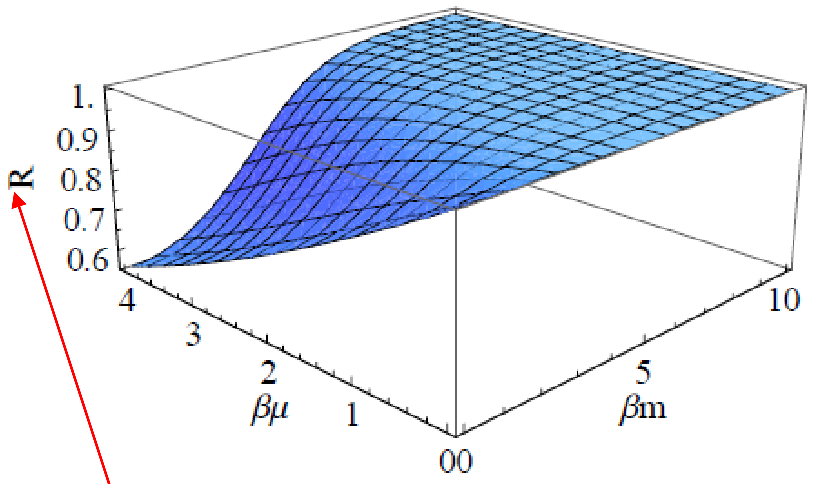
$$\sim \beta\frac{B^\alpha}{m_\Lambda}$$

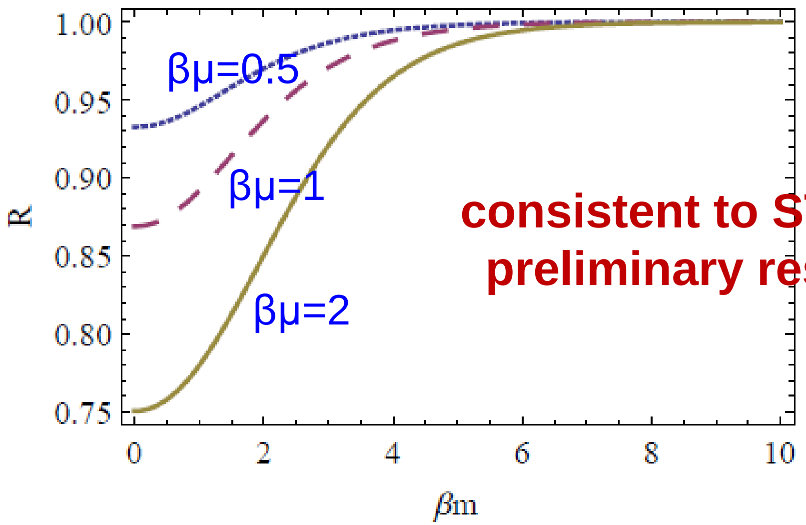
$$\Longrightarrow \boxed{B \sim T m_\Lambda \Delta P \sim (0.1-1)m_\pi^2}$$

too large for low energy HIC and very late stage of evolution !!

Decoding the solution:
(6) Polarization of massive fermions

- **The splitting is mainly from vorticity and more Pauli blocking effect for fermions**





consistent to STAR preliminary result

$$R = \frac{[\Pi(x)/\rho(x)]_{\text{fermion}}}{[\Pi(x)/\rho(x)]_{\text{anti-fermion}}}$$

Fang, Pang, QW, Wang, PRC 94, 024904(2016)

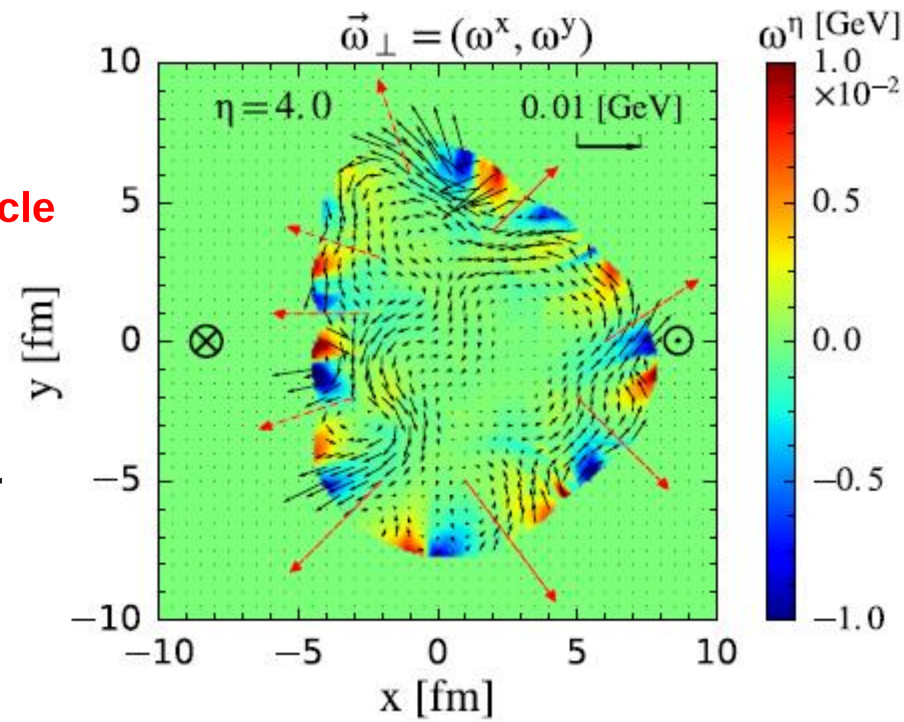
Decoding the solution:
(6) Polarization of massive fermions

- **Turbulence and vortices in in high energy HIC**
- **Λ-polarization at freeze-out hypersuface**

e=+1: particle
e=-1: anti-particle

$$P^{\alpha}(p) \equiv \frac{d\Pi^{\alpha}(p)/d^3p}{d\rho(p)/d^3p} = \frac{\hbar}{4m}\frac{\int [d\Sigma_{\lambda}p^{\lambda}]\tilde{\Omega}^{\alpha\sigma}p_{\sigma}f_{\mathrm{FD}}^{e}(1-f_{\mathrm{FD}}^{e})}{\int [d\Sigma_{\lambda}p^{\lambda}]\, f_{\mathrm{FD}}^{e}}$$

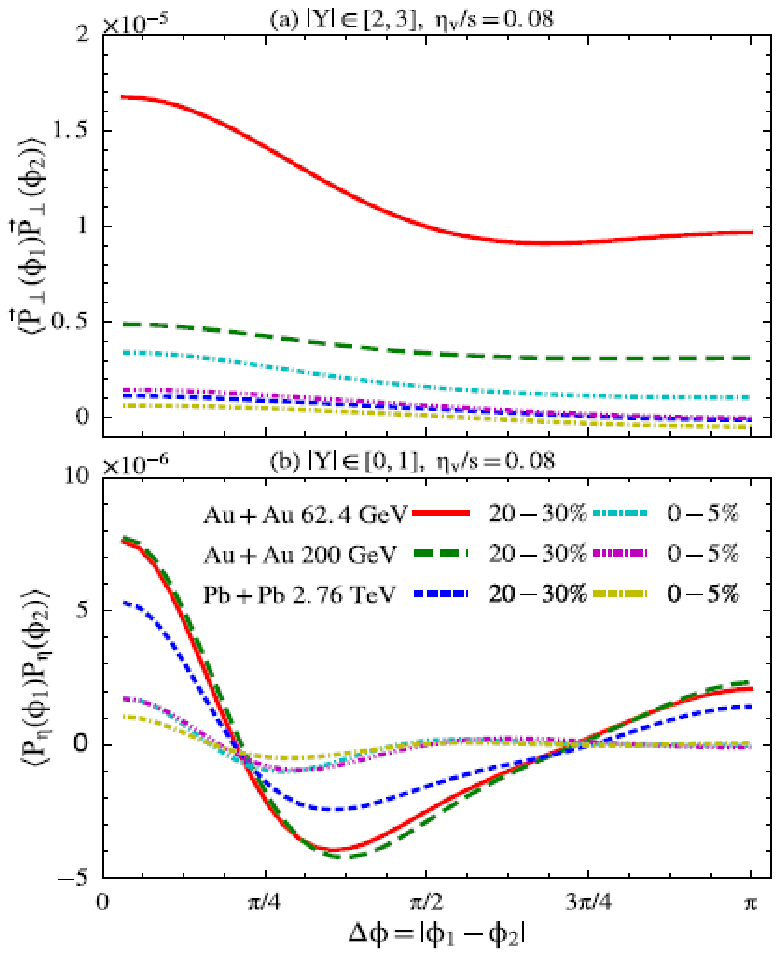
Becattini, et al., Ann. Phys. 338, 32(2013)



Pang, Petersen, QW, Wang, arXiv:1605.04024

Decoding the solution:
(6) Polarization of massive fermions

- **Turbulence and vertical strusture can be measured by Λ-spin correlation**
- **Polarization are stronger at lower beam energies and peripheral collisions**
- **Shifts indicate global polarization caused by global orbital angular momentum**
- **Shear viscosity increases global Polarization**



Pang, Petersen, QW, Wang, arXiv:1605.04024

Summary

- One-line solution to Wigner function encodes:
-
- CME and CVE;
- Covariant Chiral Kinetic Equation;
- Berry phase and monopole in 4D;
- Magnetic moment energy of chiral fermions;
- Spin-vorticity coupling of chiral fermions;
- HTL/HDL CME conductivity
- Polarization of massive fermions (Λ), consistent to STAR result