Relativistic hydrodynamics as an asymptotic series

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1103.3452, **1302.0697**, 1409.5087,

1503.07514, 1603.05344, 1609.04803

& 1610.02023 (partial review / viewpoint)

Hydrodynamization

M. P. Heller, R. A. Janik and P. Witaszczyk,
Phys. Rev. Lett. 108, 201602 (2012), **1103.3452**

M. P. Heller, A. Kurkela and M. Spalinski **1609.04803**

Relativistic hydrodynamics

hydrodynamics is an EFT of the slow (?) evolution of conserved currents in collective media close to equilibrium (?)

DOFs: always local energy density ϵ and local flow velocity u^μ $\;(u_\nu u^\nu = -1)$

EOMs: conservation eqns $\nabla_\mu \langle T^{\mu\nu} \rangle = 0$ for $\langle T^{\mu\nu} \rangle$ expanded in gradients

$\Pi^{\mu\nu}$

$$\langle T^{\mu\nu} \rangle = \epsilon\, u^\mu u^\nu + P(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\} - \eta(\epsilon)\, \sigma^{\mu\nu} - \zeta(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\}(\nabla \cdot u) + \ldots$$

microscopic input:

EoS $\left(P(\epsilon) = \frac{1}{3}\epsilon \text{ for CFTs}\right)$

shear viscosity

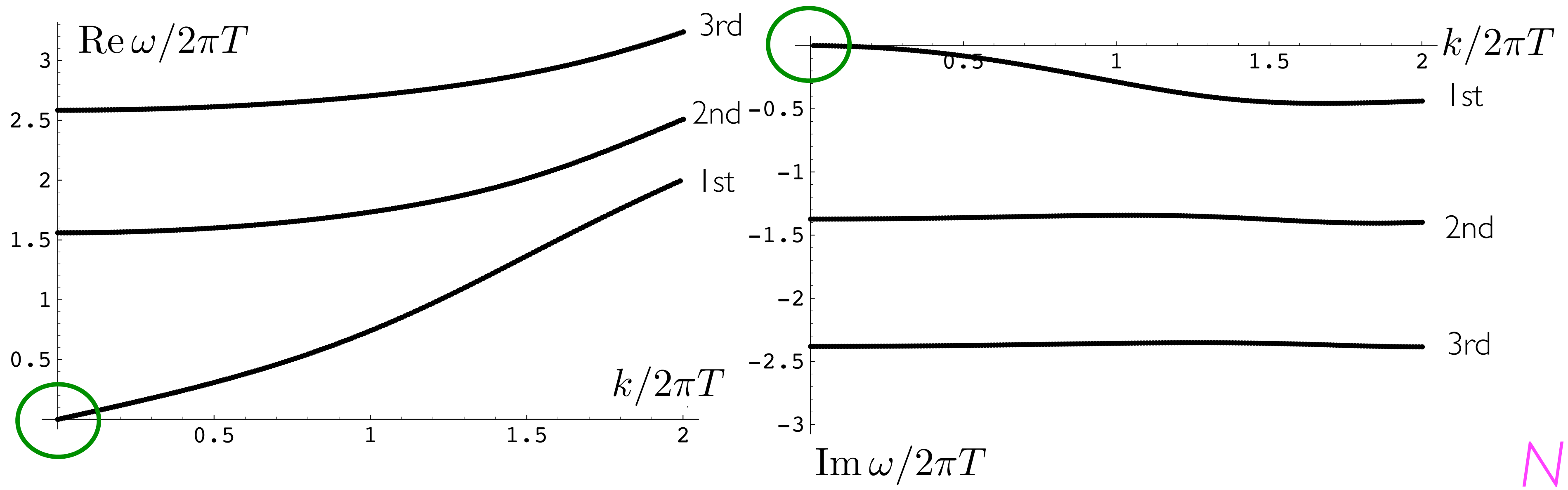
bulk viscosity (vanishes for CFTs)

This talk: behaviour of the gradient expansion at large orders in the number of ∇

Excitations in strongly-coupled plasmas

see, e.g. Kovtun & Starinets [hep-th/0506184]

$$\left.\frac{\partial\omega}{\partial k}\right|_{k\to 0} = c_{sound}$$



N=4 SYM

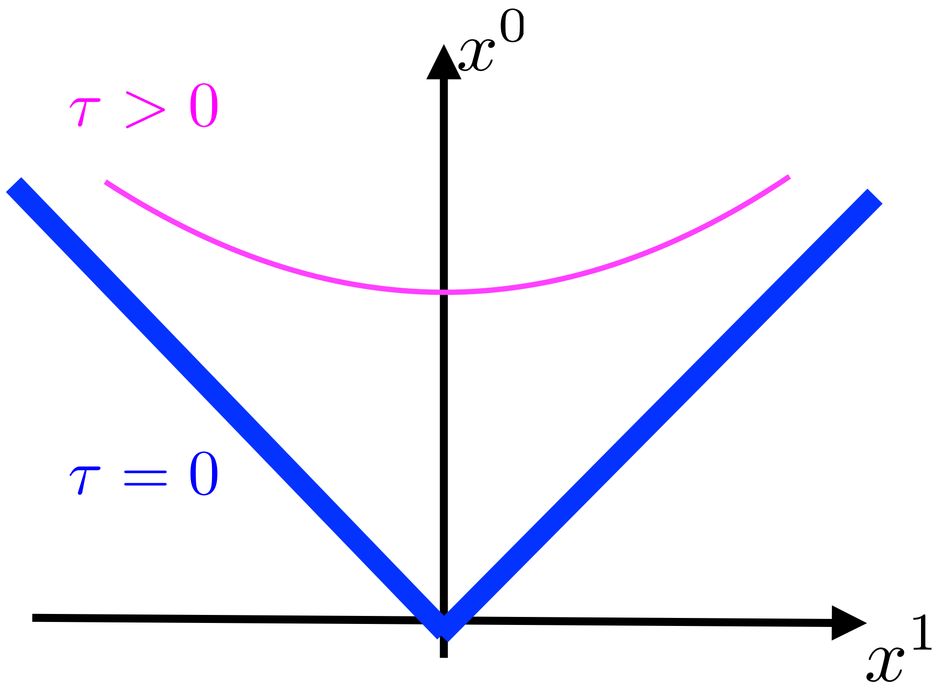
$\omega(k) \to 0$ as $k \to 0$: slowly dissipating modes (hydrodynamic sound waves)

all the rest: far from equilibrium (QNM) modes damped over $t_{therm} = \mathcal{O}(1)/T$

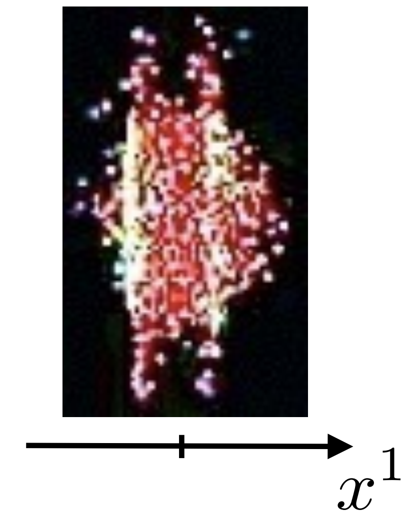
Linear response theory:

$$\delta\langle T^{\mu\nu}\rangle = \int d^3k \sum_{modes} e^{-i\,\omega_{mode}(k)\,t + i\,\vec{k}\cdot\vec{x}} \left[\left(\mathrm{Res}_{\omega=\omega_{mode}(k)} G_R(\omega,\vec{k})\right)\cdot \delta g(\omega_{mode}(k),\vec{k})\right]^{\mu\nu}$$

Boost-invariant flow [Bjorken 1982]



const x^0 slice:



Boost-invariance: in $\left(\tau \equiv \sqrt{x_0^2 - x_1^2}, \quad y \equiv \operatorname{arctanh}\frac{x_1}{x_0}, \, x_2, \, x_3\right)$ coords no y-dep

In a CFT: $\langle T^\mu_{\ \nu} \rangle = \operatorname{diag}\left\{ -\mathcal{E}(\tau), -\mathcal{E} - \tau\,\dot{\mathcal{E}}, \, \mathcal{E} + \frac{1}{2}\tau\,\dot{\mathcal{E}}, \, \mathcal{E} + \frac{1}{2}\tau\,\dot{\mathcal{E}} \right\}$

and via scale-invariance $\frac{\Delta \mathcal{P}}{\mathcal{E}/3}$ is a function of $w \equiv \tau\, T$

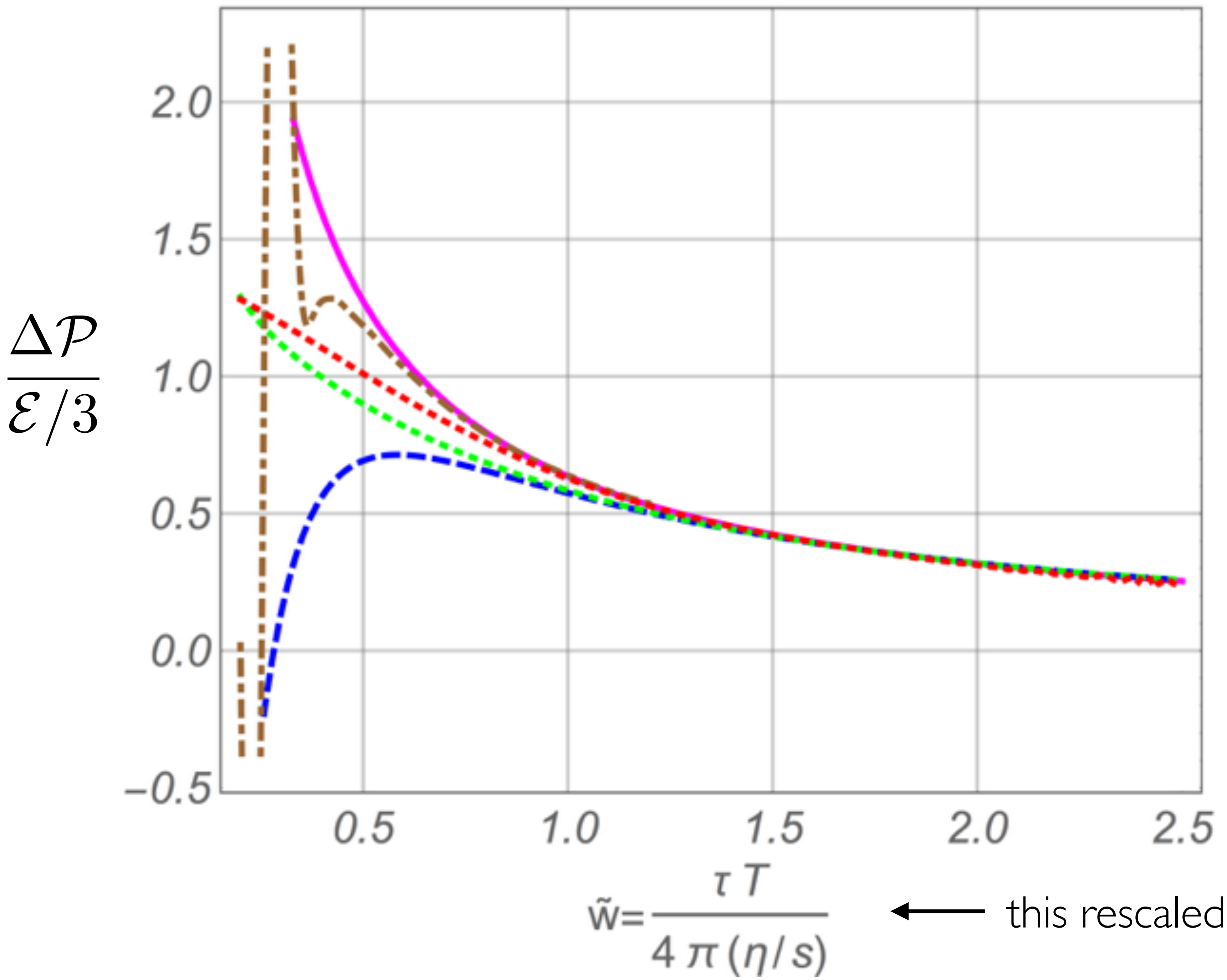
$$\Delta \mathcal{P} \equiv \langle T^2_{\ 2} \rangle - \langle T^y_{\ y} \rangle, \qquad T \equiv \left(\frac{\mathcal{E}(\tau)}{\frac{3}{8}\pi^2 N_c^2} \right)^{1/4}$$

Gradient expansion: series in $\frac{1}{w}$.

Hydrodynamization (across conformal theories)

1103.3452 & 1609.04803

see Andrei Starinets' talk for musings of similar flavour



$\frac{\Delta\mathcal{P}}{\mathcal{E}/3} = \frac{2}{\pi}\tilde{w}^{-1}$

N=4 SYM

EKT with $\eta/s = 0.624$

RTA with $\eta/s = 0.624$

RTA with $\eta/s = 1/(4\pi)$

← this rescaled variable is motivated by 1512.05347

Viscous hydrodynamics works despite huge anisotropy captured by $-\eta\,\sigma^{\mu\nu}$

see Paul Romatschke's talk for possible implication for HIC pheno

Why can hydrodynamization occur?

M. P. Heller, R. A. Janik and P. Witaszczyk,
Phys. Rev. Lett. 110, 211602 (2013), **1302.0697**

Hydrodynamic gradient expansion is divergent

In **1302.0697** we computed $f(w) \equiv \frac{2}{3} + \frac{1}{6}\frac{\Delta\mathcal{P}}{\mathcal{E}}$ up to $O(w^{-240})$:

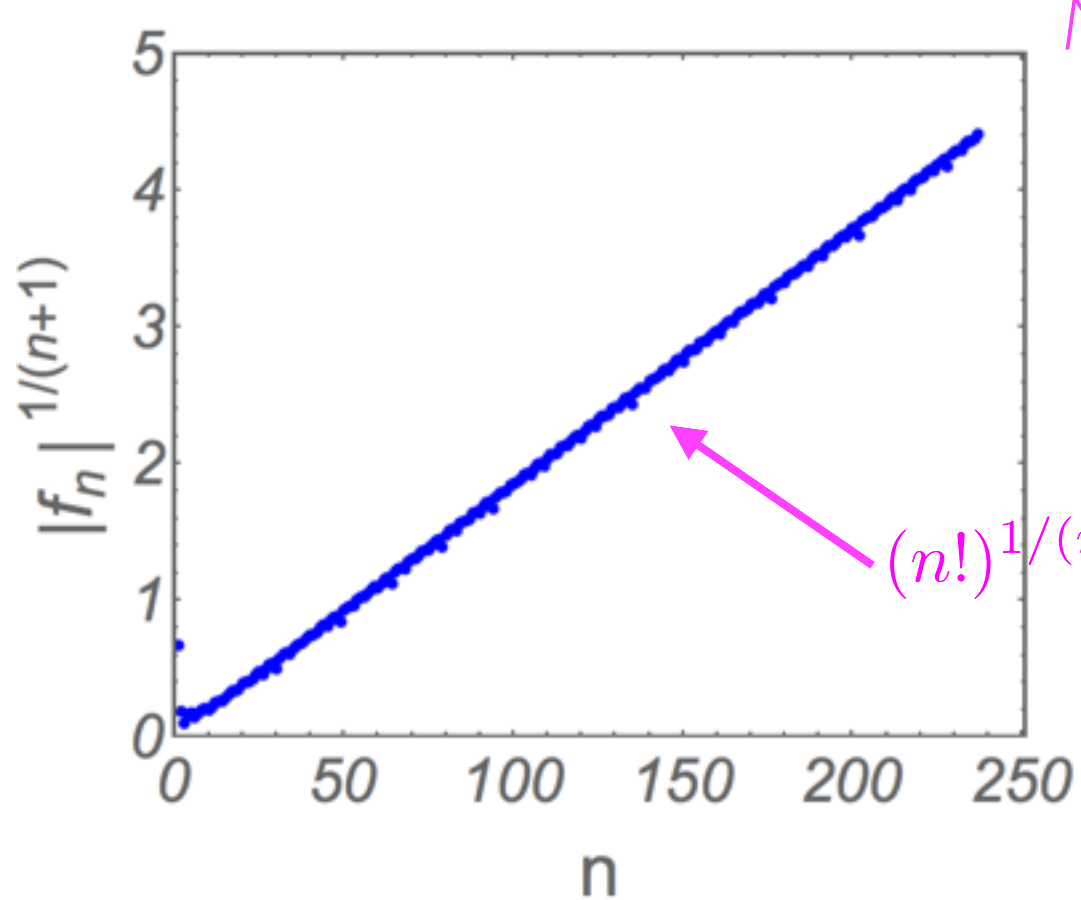
$$\frac{2}{3} + \frac{1}{9\pi}w^{-1} + \ldots$$

$$||$$

$$f(w) = \sum_{n=0}^{\infty} f_n w^{-n}$$

$$+e^{-\#\, w} + \ldots$$

QNM contributions are not captured by the gradient expansion



N=4 SYM

$$(n!)^{1/(n+1)}\Big|_{n\to\infty} \approx \frac{1}{e}\cdot n$$

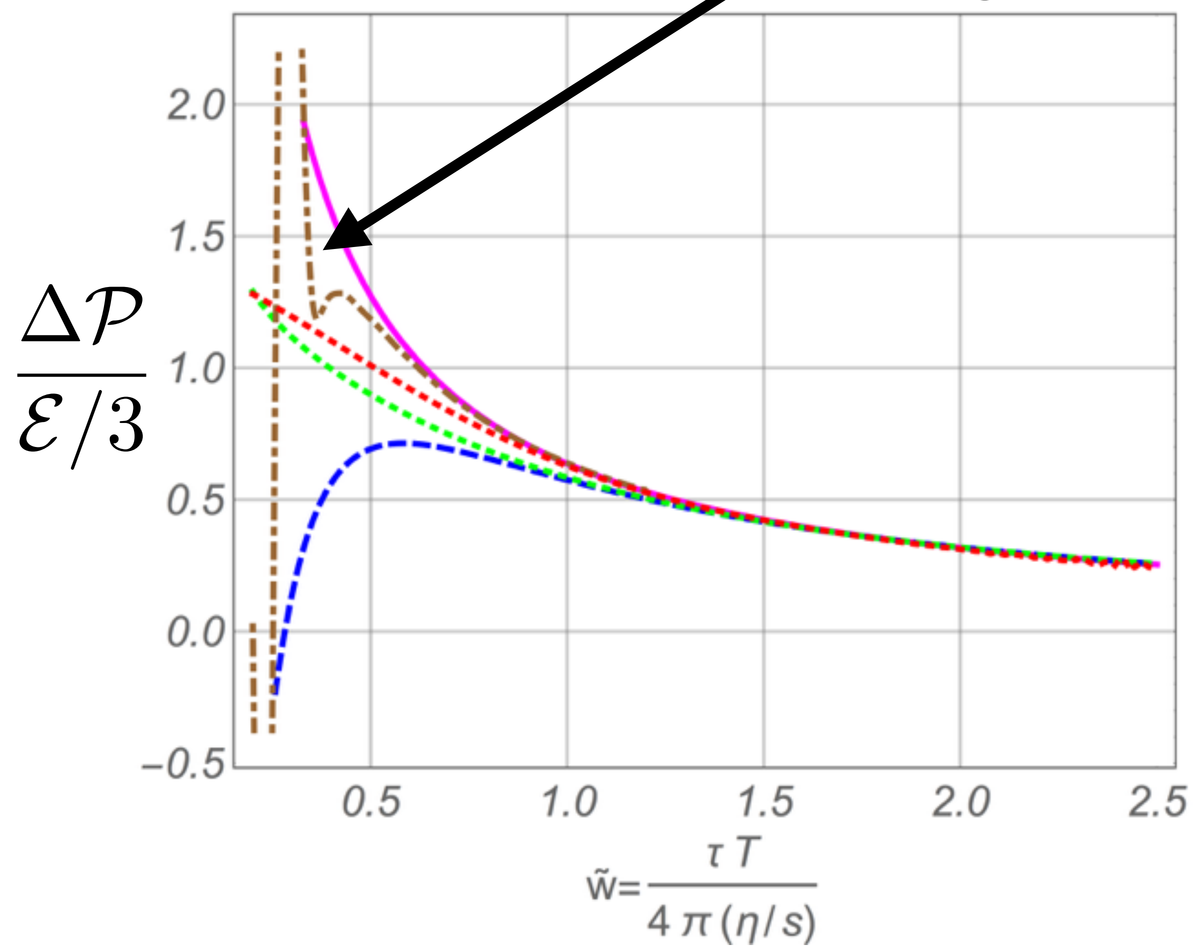
→ The gradient expansion cannot converge as **1302.0697** explicitly demonstrated

Outlook: why can hydrodynamization occur?

Divergent series are better than convergent series since their applicability is not limited by the series itself but whether one can neglect effects not captured by it

for this state neither transient QNMs nor higher order hydro matter yet $\frac{\Delta\mathcal{P}}{\mathcal{E}/3}$ is gigantic

1609.04803:



$$\Pi^{\mu\nu} = -\eta\,\sigma^{\mu\nu}$$

$$\downarrow$$

$$\frac{\Delta\mathcal{P}}{\mathcal{E}/3} = \frac{2}{\pi}\tilde{w}^{-1}$$

This does not mean that second (or third) order hydro is not important ⟶ other flows?

Note that for ~~CFTs~~ the bulk viscosity term can also be very large at hydro threshold

Attems et al. 1604.06439

Hydrodynamization with anomalous transport?

What is the hydro gradient expansion?

M. P. Heller, R. A. Janik and P. Witaszczyk,
Phys. Rev. Lett. 110, 211602 (2013), **1302.0697**

M. P. Heller, M. Spaliński,
Phys. Rev. Lett. 115, 072501 (2015), **1503.07514**

A. Buchel, M. P. Heller and J. Noronha
PRD RC in press, **1603.05344**

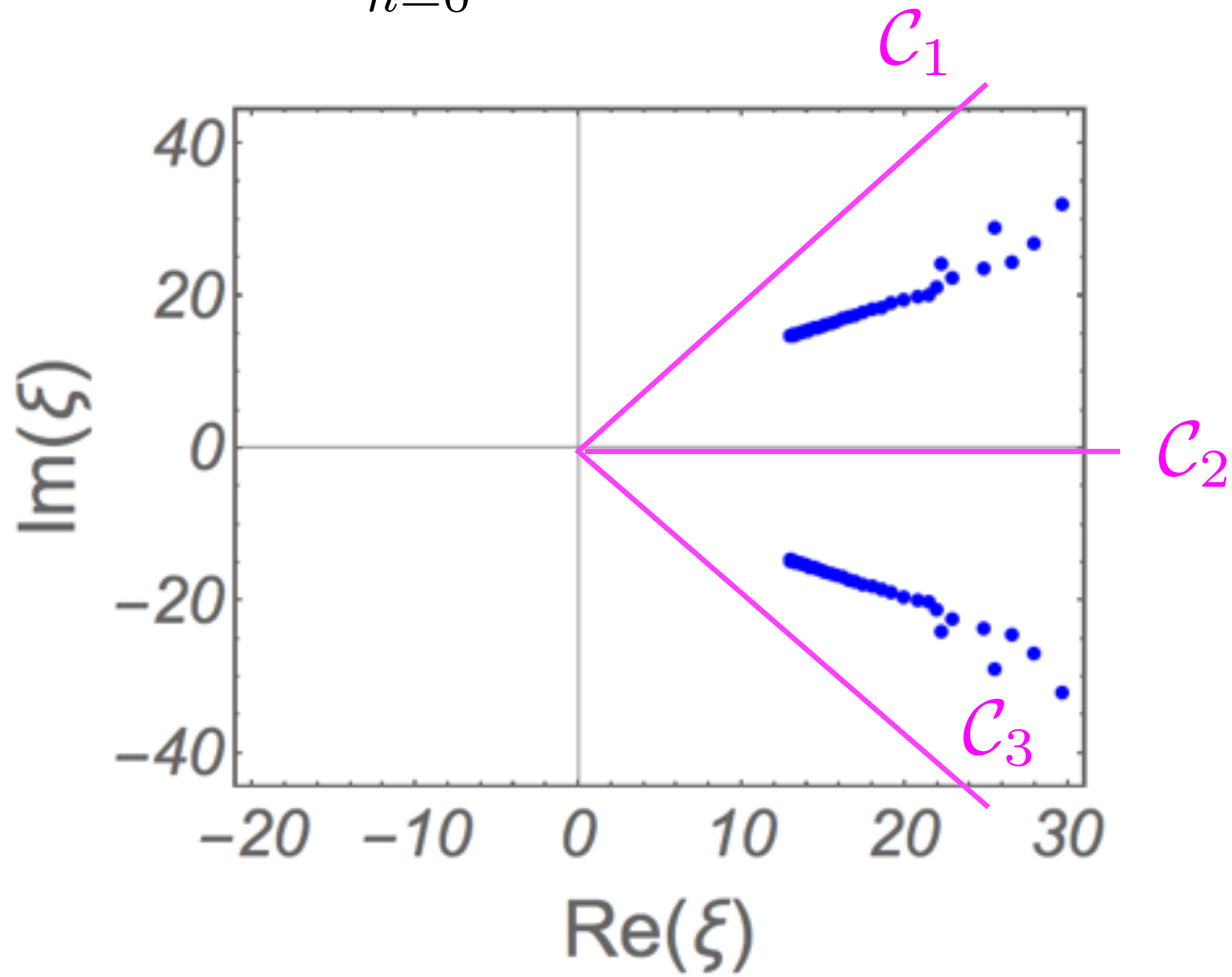
M. P. Heller, **1610.02023**

Borel transform, gradient expansion and QNMs

1302.0697

Analytic continuation of $f_B(\xi) \approx \sum_{n=0}^{240} \frac{1}{n!} f_n \, \xi^n$ revealed the following singularities:

N=4 SYM:



Branch cuts start at $\frac{3}{2} i \, \hat{\omega}_{QNM_1} = \frac{3}{2} i \, T \times \omega_{QNM_1}(k=0)$. Inverse trafo ambiguous:

$$\left(\int_{\mathcal{C}_i} d\xi - \int_{\mathcal{C}_j} d\xi \right) \left[w \, e^{-\xi \, w} f_B(\xi) \right] \sim w^{\alpha_{QNM_1}} \boxed{e^{-\frac{3}{2} i \, \hat{\omega}_{QNM_1} \, w}} \times (\ldots) + \ldots$$

$$e^{-i \, \hat{\omega}_{QNM_1} T \, t} \longrightarrow e^{-i \, \hat{\omega}_{QNM_1} \int T \, d\tau}$$

Lesson from MIS 1503.07514

$$(\tau_\Pi\, u^\alpha \nabla_\alpha + 1)\, \Pi_{\mu\nu} = -\eta\, \sigma_{\mu\nu} + \ldots$$

↓ $C_\eta = \frac{\eta}{s}$ and $C_{\tau_\Pi} \equiv \hat{\omega}_{MIS} = \tau_\Pi\, T$

$$f' = -\frac{1}{C_{\tau_\Pi}} + \frac{2}{3\, C_{\tau_\Pi}\, f} + \frac{16}{3\, w} - \frac{16}{9\, w\, f} + \frac{4 C_\eta}{9\, C_{\tau_\Pi}\, w\, f} - \frac{4\, f}{w} + \ldots$$

↓ gradient expansion and 1 transient QNM:

$$f = \sum_{n=0}^{\infty} \frac{f_n}{w^n} + \underbrace{\delta f}_{\exp\left(-\frac{3}{2 C_{\tau_\Pi}} w\right) \times \ldots} + \mathcal{O}(\delta f^2)$$

↓ transseries:

$$f = \sum_{m=0}^{\infty} (c_{amb} + r)^m \left\{ w^{\frac{C_\eta}{C_{\tau_\Pi}}} \exp\left(-\frac{3}{2 C_{\tau_\Pi}} w\right) \right\}^m \sum_{n=0}^{\infty} a_{m,n} w^{-n}$$

resurgence

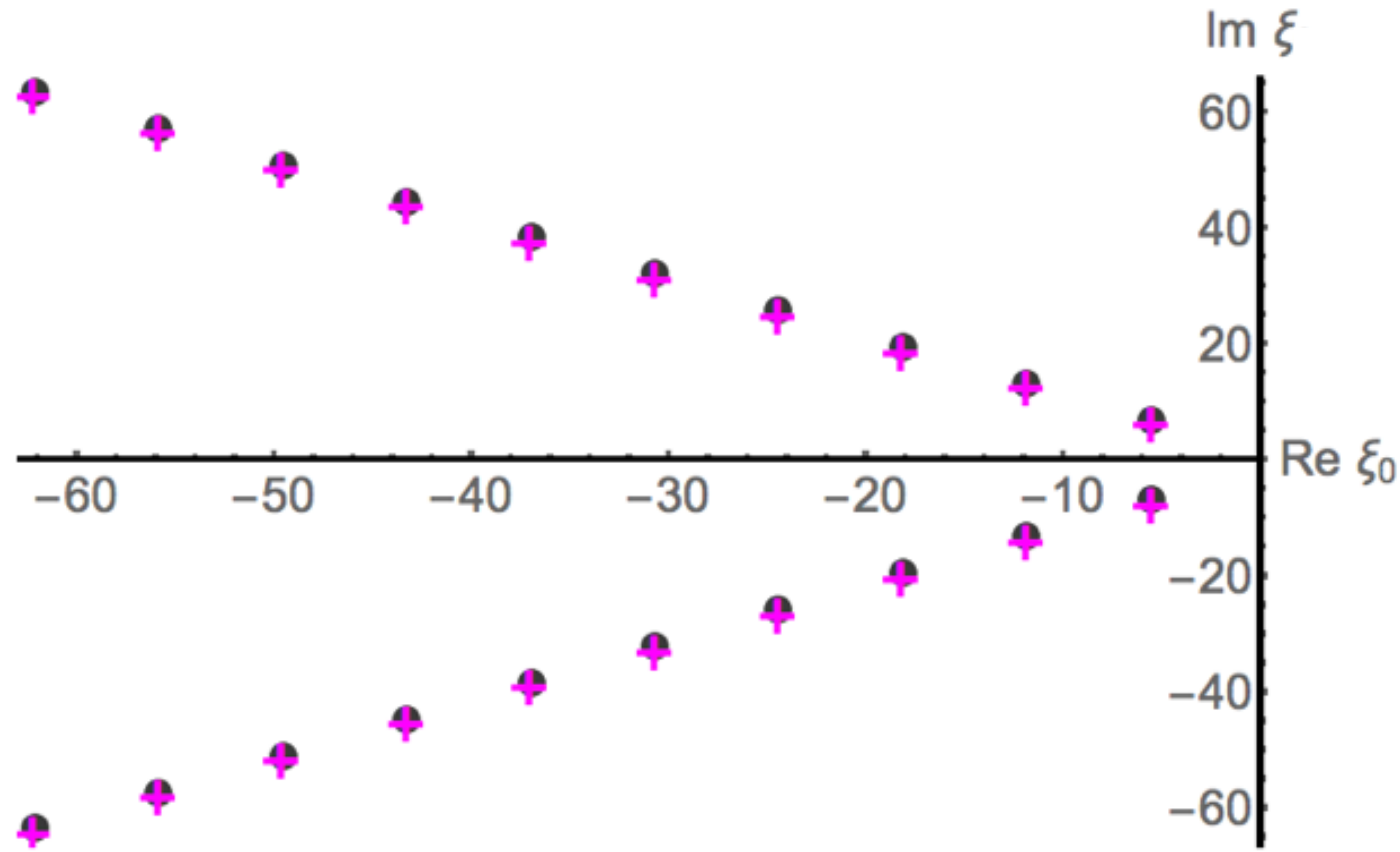
Lesson from cosmology 1603.05344

$$\frac{d\,\mathrm{Entropy}}{dt} = V \times \left(\sum_{n=0}^{\infty} c_n \xi^n\right)^2 + \ldots$$ with $\xi = \frac{H}{T}$ for a hCFT in $-dt^2 + e^{2Ht}\, d\vec{x}^2$

$$\sum_{n=0}^{300} \frac{c_n}{n!} \xi^n \approx \frac{\sum_{m=0}^{150} d_m\, \xi^m}{\sum_{l=0}^{150} e_l\, \xi^l}$$

● singularities of Borel trafo

+ 10 lowest transient QNM $\hat{\omega}$'s



Hydrodynamic gradient expansion knowns about all transient QNMs

Transseries for $\mathcal{N}$=4 SYM 1610.02023

divergent hydro gradient expansion

transient QNMs contribution on top of hydro

resurgent relations

$\sim e^{-i\frac{3}{2}\hat{\omega}_{QNM_q} w}(\dots)$

$\sim c_{amb}^{(n)} \times r_n$

all sums = Borel summation

$$f(w) = \sum_{m=0}^{\infty} f_m\, w^{-m} + \sum_{n=1}^{\infty} e^{-\frac{3}{2} i\, \hat{\omega}_{QNM_n} w}\, w^{\alpha_{QNM_n}} \sum_{m=0}^{\infty} g_{n,m}\, w^{-m}$$

$$+ \sum_{l=1}^{\infty}\sum_{n=1}^{\infty} e^{-\frac{3}{2}\hat{\omega}_{QNM_l} w} e^{-\frac{3}{2}\omega_{Q\hat{N}M_n} w}\, w^{\alpha_{QNM_l}}\, w^{\alpha_{QNM_n}} \sum_{n=0}^{\infty} g_{l,n,m}\, w^{-m} + \dots$$

quadratic interactions from Einstein's equations

Appealing analogy:

non-equilibrium physics	**QM with $V = -\frac{1}{2}x^2\left(1-\sqrt{g}\,x\right)^2$**
gradient expansion in $\frac{1}{w}$	perturbative series in g
transient QNMs $e^{-i\frac{3}{2}\hat{\omega}_{QNM_n} w}(\dots)$	instanton $e^{-1/(3g)}(\dots)$

Open problem: covariantize to $\Pi^{\mu\nu} = \boxed{\eta\,\sigma^{\mu\nu} + \dots} + \boxed{e^{i\,\hat{\omega}_{QNM_1}\int dx^{\alpha} T\, u_{\alpha}} \times (\dots)} + \dots$

see also Mukund Rangamani's talk

1409.5087

“Weak” coupling

M. P. Heller, A. Kurkela and M. Spalinski **1609.04803**

see also Gabriel Denicol’s talk

RTA kinetic theory

Natural language to talk about weakly coupled media is the Boltzmann equation:

$$p^\mu \partial_\mu g(x,p) = C[g(x,p)] \;\; \text{with} \;\; \langle T^{\mu\nu}\rangle(x) = \int_{\text{momenta}} g(x,p)\, p^\mu p^\nu$$

LO $C[g(x,p)]$ for gauge theories is complicated. We will use instead

$$C[g(x,p)] = -\frac{p^\mu u_\mu}{\hat{\mathcal{T}}}\Big\{g(x,p) - g_0(x,p)\Big\} \;\text{with}\; g_0(x,p) = e^{\frac{u_\mu p^\mu}{T}}$$

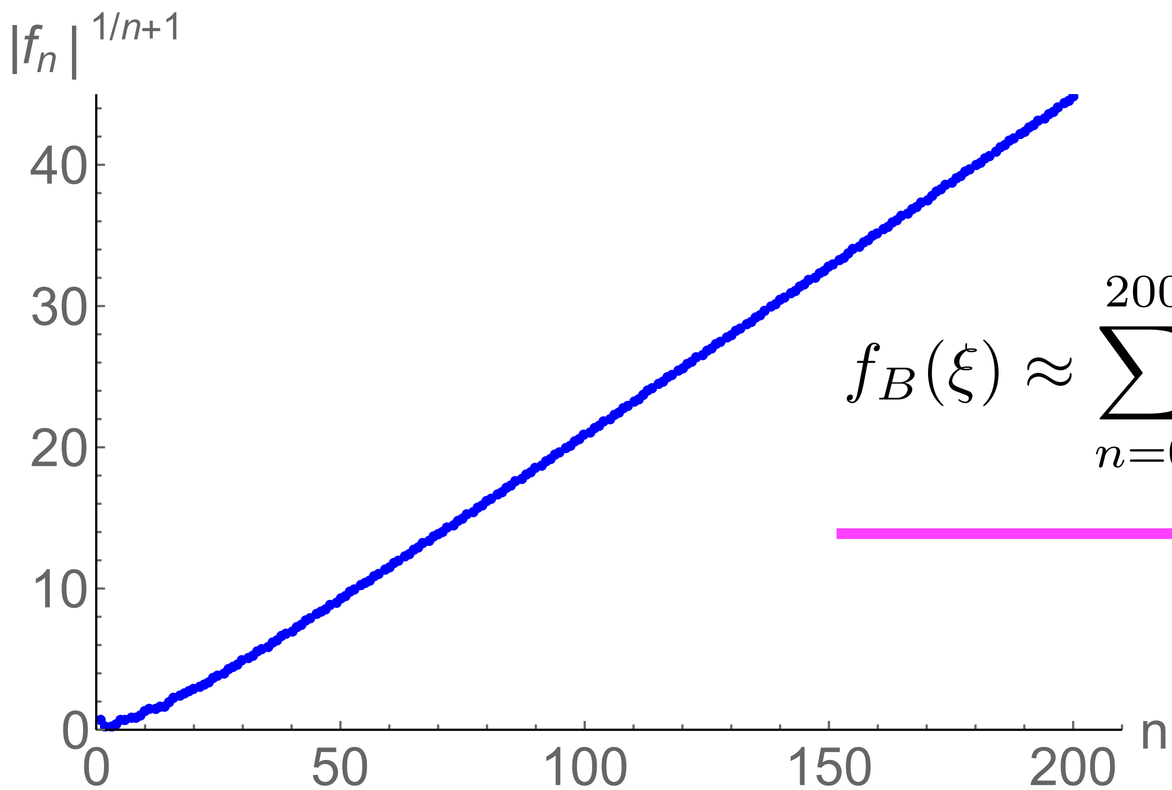
This equation is, typically, highly nonlinear due to $\langle T^{\mu\nu}\rangle\, u_\nu = -\mathcal{E}(T)\, u^\mu$

CFTs: $p^\mu p_\mu = 0$ and $\hat{\mathcal{T}} = \dfrac{\gamma}{T}$. Note that γ can be scaled-away (we set it to 1).

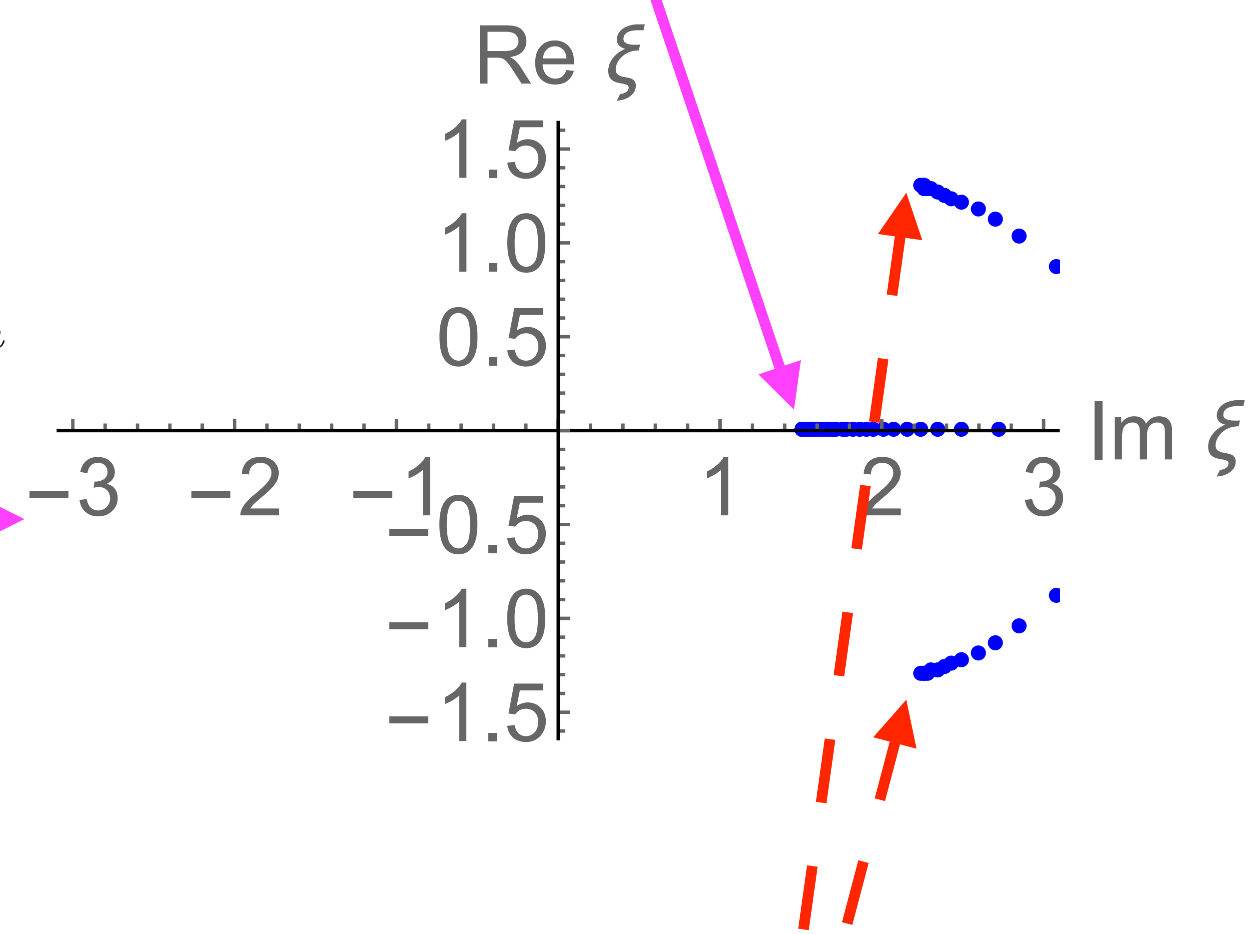
QNM in kinetic theory

$$\xi_{sing} = \frac{3}{2\gamma} \rightarrow \delta f \sim \exp\left(-\frac{3}{2\gamma} w\right) \times \ldots$$

√ G_R



$$f_B(\xi) \approx \sum_{n=0}^{200} \frac{1}{n!} f_n \, \xi^n$$



$$\delta f \sim \exp\left(-\frac{2.25}{\gamma} \pm \frac{1.3}{\gamma} i\right)$$

???

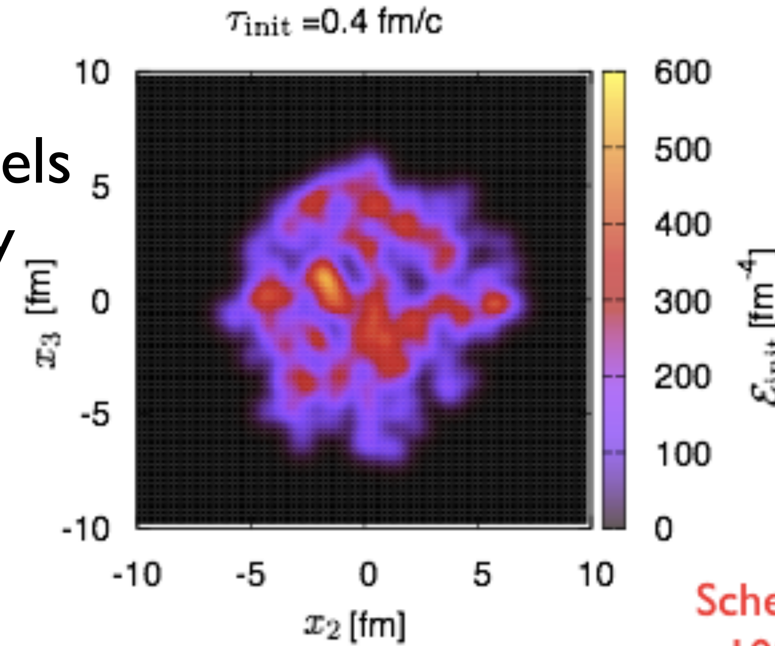
Executive summary

HIC pheno:

hydrodynamization suggests **using simple hydro models under extreme conditions is not completely crazy**

related: **~~equilibration~~ in HICs???**

see Paul Romatschke's talk



Schenke et al. 1009.3244

new effects (???):
large contributions from other transport? $(-\zeta \Delta^{\mu\nu}\, \nabla \cdot u)$

hydrodynamic gradient expansion diverges

new connections:
resurgent series (also in QM & QFT)
gradient expansion as a part of transseries

towards genericity:
2 flows and
$\infty + \infty$ hQFTs + RTA + MIS + aHYDRO + linear response ← discussion ???

see also Gabriel Denicol's talk   see Wojtek Florkowski's talk

Support

Evolution equations for relativistic viscous fluids

$\nabla_\mu \{ \epsilon\, u^\mu u^\nu + P(\epsilon)\{ g^{\mu\nu} + u^\mu u^\nu \} - \eta(\epsilon)\, \sigma^{\mu\nu} \} = 0$ is acausal.

Remedy: make $\Pi^{\mu\nu} = \langle T^{\mu\nu} \rangle - (\epsilon\, u^\mu u^\nu + P(\epsilon)\{g^{\mu\nu} + u^\mu u^\nu\})$ a new DOF, e.g.

$$(\tau_\Pi \mathcal{D} + 1)\, \Pi^{\mu\nu} = -\eta \sigma^{\mu\nu}$$

Small perturbations obey Maxwell-Cattaneo equation

$$\partial_t^2 \delta u_z - \frac{\eta}{s} \frac{1}{\tau_\Pi T} \partial_x^2 \delta u_z + \frac{1}{\tau_\Pi} \partial_t \delta u_z = 0$$

Take it seriously:

$\omega = -i \frac{\eta}{sT} k^2 + \ldots$ and $\omega = -i \frac{1}{\tau_\Pi} + i \frac{\eta}{sT} k^2 + \ldots$

hydrodynamics — purely imaginary "quasinormal mode"

Generalization that adds $\mathrm{Re}(\omega_{QNM})$: $\left((\frac{1}{T}\mathcal{D})^2 + 2\omega_I \frac{1}{T}\mathcal{D} + |\omega|^2 \right) \Pi^{\mu\nu} = -\, \eta |\omega|^2 \sigma^{\mu\nu}$

1409.5087

extra