Emergent Symmetries in Hydrodynamic Effective Theories

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Relativistic Hydrodynamics: Theory and Modern Applications
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October 12, 2016

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[1502.00636], [1510.02494], [1511.07809]
[1610.01940], [1610.01941] + wip

Outline

- **Act I:** Relativistic hydrodynamics
  - In which we postulate the axioms and classify potential transport and see a prelude to the closing act
- **Act II**: Microscopics
  - In which we present a new perspective on the Schwinger-Keldysh formalism, aimed at identifying low energy symmetries
- **Act III**: Hydrodynamic effective actions
  - In which we exemplify how to construct actions for dissipative hydrodynamics as a supersymmetric sigma model

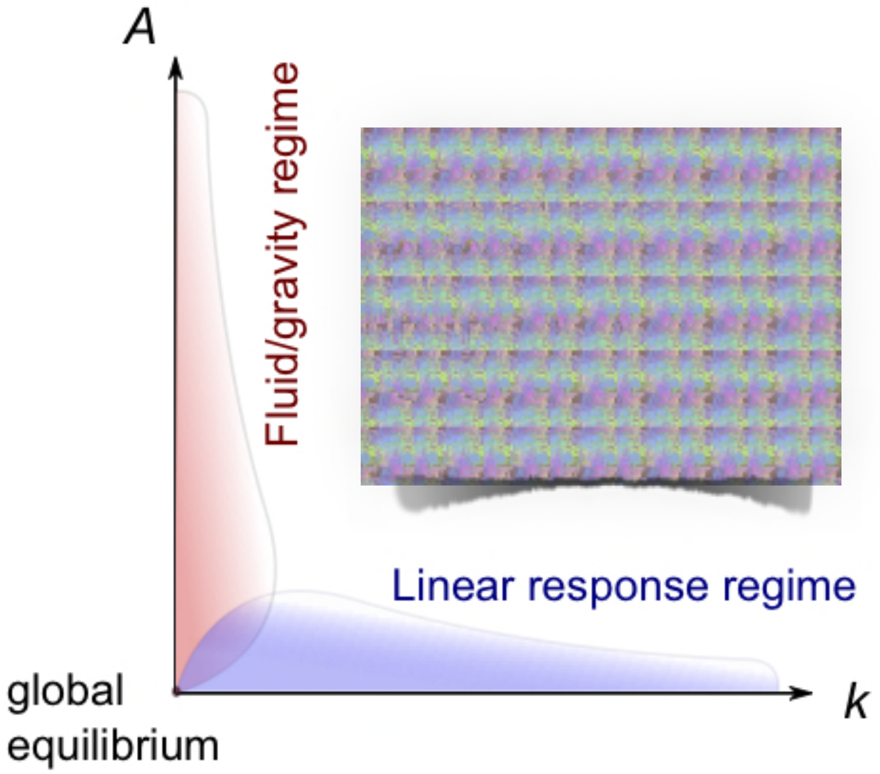
Motivation: non-equilibrium QFT dynamics

- What is the framework for a consistent Wilsonian treatment of low energy dynamics in mixed states of a QFT?

- There is a reasonably good phenomenological understanding, but the theoretical underpinnings are not yet fully understood.
- Entanglement of the system with some external reservoir/purifier is central to the discussion.
- There are many reasons to be interested in this question:
  - intrinsic interest from QFT and many-body physics standpoint.
  - dynamics of black holes via AdS/CFT.
  - cosmology.

Macroscopic phenomenology

- Equilibrium dynamics can be understood by working with Euclidean generating functions, etc..
- Linear fluctuations are captured by Schwinger-Keldysh, while long-wavelength fluctuations are described by hydrodynamic effective field theory.
- General non-equilibrium dynamics is theoretical terra incognita.



- Integrating out high energy modes starting from microscopic Schwinger-Keldysh leads to coupling between L and R encoded in *influence functionals.*

Feynman, Vernon ‘63

- What influence functionals are consistent with microscopic unitarity?

Act I

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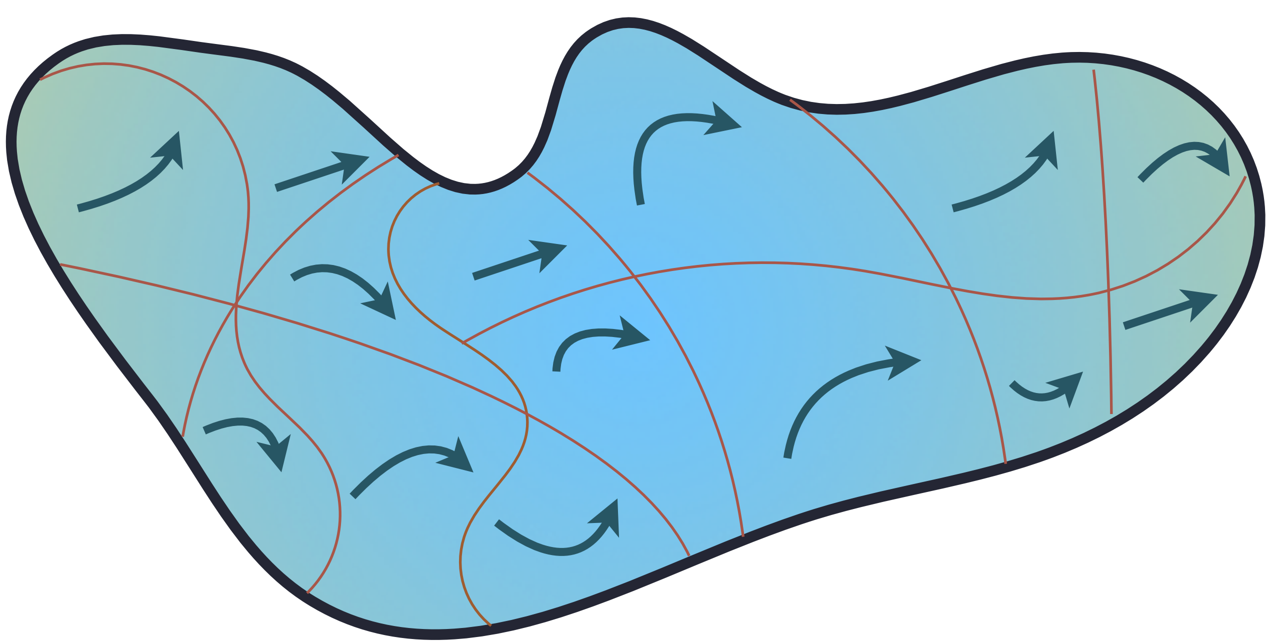
The hydrodynamic effective field theory

- Relativistic fluid dynamics is best thought of as an effective field theory for quantum systems in local, but not global, thermal equilibrium.
- The description in terms of fluid dynamics is valid when departures from equilibrium are on scales that are large compared to the characteristic mean free path of the underlying quantum dynamics.



$$\ell_{\rm mfp} \ll L\,, \qquad t_{\rm mfp} \ll t$$

- Local domains of equilibrated fluid can be characterized by the local temperature/energy density and conserved charges.
- Energy/charge flux exchanged across the domains: velocity field.

Axioms of Hydrodynamics I: Fields

- Hydrodynamics describes low-energy, near-equilibrium fluctuations of an equilibrium Gibbsian density matrix on scales large compared to the characteristic mean free path.
- The macroscopic description involves currents which capture energy-momentum and charge transport $T^{\mu\nu},\ J^{\mu}$ (and entropy current J^{μ}_S).
- The currents are functionals of the hydrodynamic fields, which are the intensive variables characterizing the density matrix and background sources.
  - temperature and chemical potential and a flux vector (fluid velocity) $T, \mu, u^{\mu}, \qquad u^{\mu}\, u_{\mu} = -1$
  - background metric and electromagnetic potential $g_{\mu\nu}, A_{\mu}$

Axioms of Hydrodynamics II: Data

- ✳ Repackage the dynamical degrees of freedom in a vector an scalar

$$\boldsymbol{\beta}^\mu \equiv \frac{u^\mu}{T}\,, \qquad \Lambda_{\boldsymbol{\beta}} \equiv \frac{\mu}{T} - \frac{u^\sigma}{T} A_\sigma$$

thermal vector

thermal twist

- ✳ The currents of hydrodynamics are expressed as functionals of the hydrodynamical fields and the background sources.

  - currents $\qquad T^{\mu\nu}, J^\mu, J^\mu_S$

  - fields $\qquad \boldsymbol{\Psi} \equiv \{g_{\mu\nu}, A_\mu, \boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\}$

  - constitutive relations

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}[\boldsymbol{\Psi}] = T^{\mu\nu}[g_{\alpha\beta}, A_\alpha, \boldsymbol{\beta}^\alpha, \Lambda_{\boldsymbol{\beta}}] \\ J^\mu &= J^\mu[\boldsymbol{\Psi}] = J^\mu[g_{\alpha\beta}, A_\alpha, \boldsymbol{\beta}^\alpha, \Lambda_{\boldsymbol{\beta}}] \\ J^\mu_S &= J^\mu_S[\boldsymbol{\Psi}] = J^\mu_S[g_{\alpha\beta}, A_\alpha, \boldsymbol{\beta}^\alpha, \Lambda_{\boldsymbol{\beta}}]\,. \end{aligned}$$

Axioms of Hydrodynamics III: Constraints

- *Constitutive relations*: monitor conserved currents, energy momentum, charge, etc.. as functionals of the hydrodynamic fields.

- Dynamics is conservation modulo work and anomaly terms, subject to a constraint: local form of the second law of thermodynamics is upheld.

$$\mathcal{E}^\mu_T = \nabla_\nu T^{\mu\nu} - J_\nu \cdot F^{\mu\nu} - T^{\mu\perp}_H = 0 \qquad\qquad \mathcal{E}_J = D_\mu J^\mu - J^\perp_H = 0$$

work term

covariant anomalies

$$\exists \;\; J^\mu_S[\mathbf{\Psi}] : \;\; \forall \;\; \mathbf{\Psi}_{\text{on-shell}} \,, \qquad \nabla_\mu J^\mu_S[\mathbf{\Psi}] \geq 0$$

- Ample evidence from kinetic theory, fluid/gravity correspondence etc., that this is the correct macroscopic picture.

Classification of hydrodynamic transport

- Q: What are the acceptable solutions to the axioms of hydrodynamics, i.e., what constitutive relations are consistent with the second law?

- **Theorem:** Hydrodynamic transport can be classified in an ***eightfold way.*** There are seven adiabatic classes and a class of dissipative transport. In addition we have a class of forbidden constitutive relations which can be determined by studying hydrostatic equilibrium.

- This theorem was proved by studying an off-shell reformulation of the second law using the *adiabaticity equation:*

$$\nabla_\mu J^\mu_S + \boldsymbol{\beta}_\mu \, \mathcal{E}^\mu_{_T} + (\Lambda_{\boldsymbol{\beta}} + \boldsymbol{\beta}^\alpha A_\alpha)\, \mathcal{E}_{_J} = \Delta \geq 0$$

Haehl, Loganayagam, MR [1502.00636]

Aside: Free energy current

- The structures are clearer if we introduce the *Gibbs free energy current*, switching from a microcanonical to grand-canonical language:

$$-\frac{\mathcal{G}^\sigma}{T} = J_S^\sigma + \boldsymbol{\beta}_\nu T^{\nu\sigma} + (\Lambda_{\boldsymbol{\beta}} + \boldsymbol{\beta}^\alpha A_\alpha) \cdot J^\sigma$$

- The off-shell second law encoded in the adiabaticity equation then reads

$$\nabla_\sigma \left(\frac{\mathcal{G}^\sigma}{T}\right) - \frac{\mathcal{G}_H^\perp}{T} = -\frac{1}{2} T^{\mu\nu} \delta_{\mathcal{B}} g_{\mu\nu} - J^\mu \cdot \delta_{\mathcal{B}} A_\mu + \Delta$$

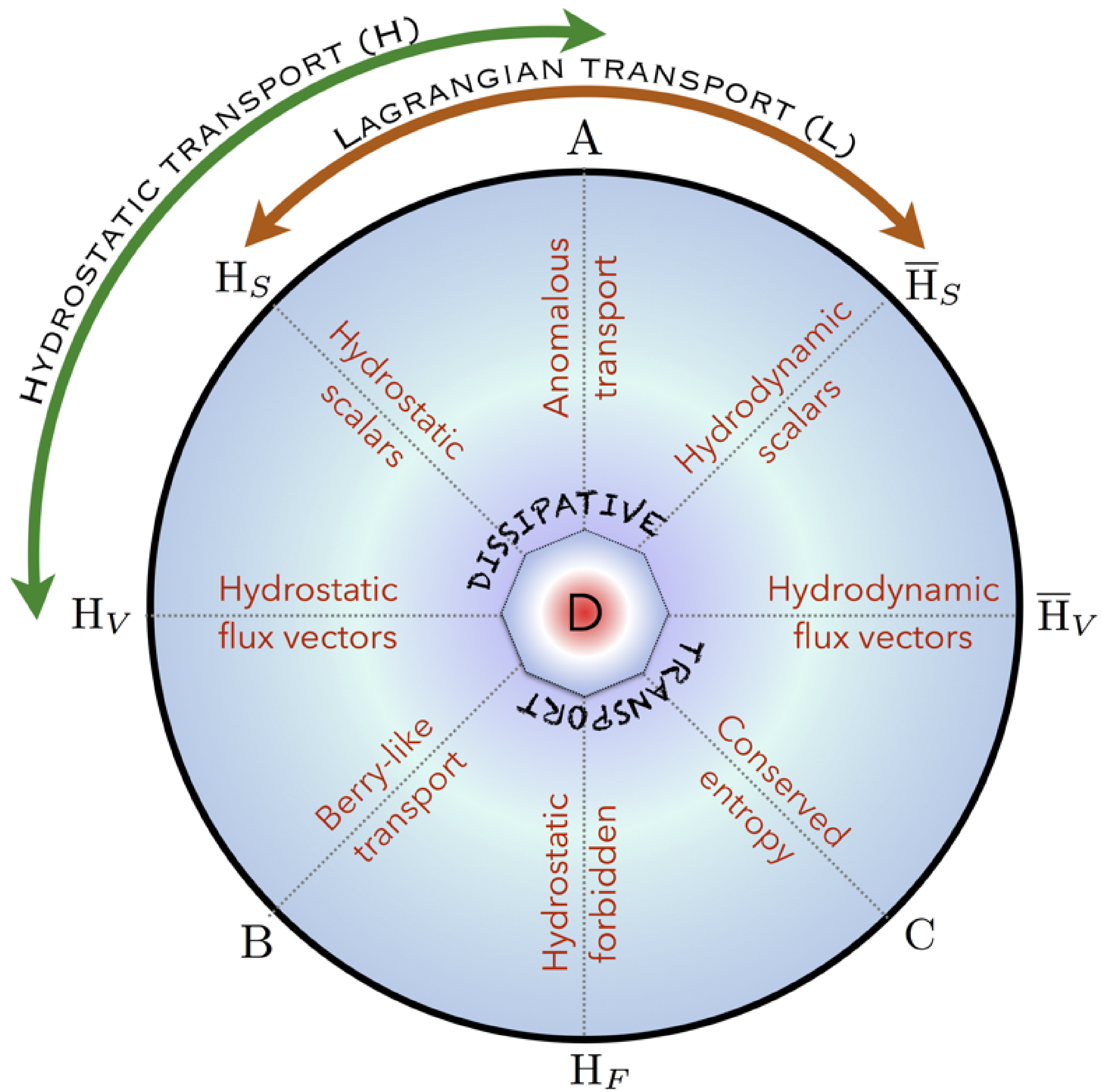
$$\delta_{\mathcal{B}} g_{\mu\nu} \equiv \pounds_{\boldsymbol{\beta}} g_{\mu\nu} = \nabla_\mu \boldsymbol{\beta}_\nu + \nabla_\nu \boldsymbol{\beta}_\mu \,,$$
$$\delta_{\mathcal{B}} A_\mu \equiv \pounds_{\boldsymbol{\beta}} A_\mu + \partial_\mu \Lambda_{\boldsymbol{\beta}} + [A_\mu, \Lambda_{\boldsymbol{\beta}}]$$

diffeomorphism

flavour gauge transformation

along the thermal vector & twist.

Eightfold classification of hydrodynamic transport



✦ Second law:

✳ forbids $\mathrm{H_F}$.

✳ D terms sign-definite only at leading order.

S. Bhattacharyya ['13-'14]

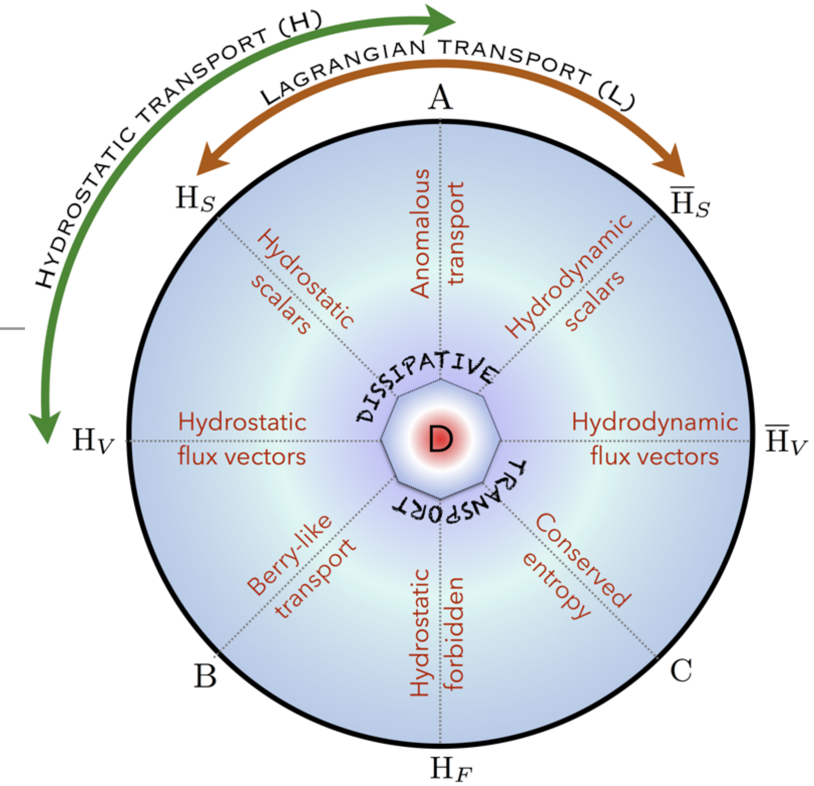
longitudinal vector

$$\mathcal{G}^\mu = \mathfrak{S}\,\boldsymbol{\beta}^\mu + \mathfrak{V}^\mu$$

transverse vector

Haehl, Loganayagam, MR ['14-'15]

Exemplifying the classification



- ★ D: viscosities, conducitivities, etc..
- ★ C: Conserved entropy (no transport)
- ★ B: Hall viscosity/conductivity, parity even counterparts, etc..
- ★ A: anomalous contribution (chiral magnetic/vorticial effects).
- ★ H_S: Equilibrium data with non-trivial static sources (pressure, higher order terms)
- ★ $\overline{H}_S$: Landau-Ginzburg terms (shear contributions to Lagrangian)
- ★ H_V: Vectorial terms in equilibrium eg., gravitational anomalies
- ★ $\overline{H}_V$: Non-stationary vectorial terms (charged fluids)

Adiabatic fluid lagrangian

- The adiabatic part of transport can be derived from an effective action, which generalizes hydrostatic partition functions.
- To constrain influence functionals so as to agree with the classification data, we empirically introduced a new emergent U(1) symmetry.
- Intuitively, we allow local thermal translations at every spacetime point, relating operators and their KMS conjugates i.e., impose, *local KMS condition*.
- The effective action takes a very simple form, involving the energy momentum and free energy current:

$$\mathcal{L}_T = \frac{1}{2}\, \mathbf{T}^{ab} \tilde{g}_{ab} + \mathcal{G}^a_{\mathcal{L}}\, \mathcal{A}_a$$

- New variables $\tilde{\mathrm{g}}_{ab}, \tilde{\mathcal{A}}_a$: former is the SK partner of the worldvolume metric.

Haehl, Loganayagam, MR 1502.00636

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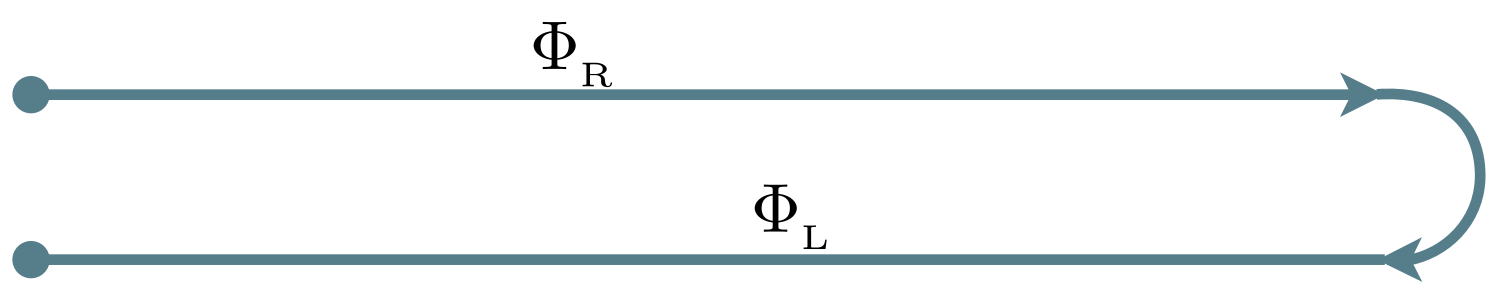
I. Microscopics: Schwinger-Keldysh formalism

- The Schwinger-Keldysh formalism computes time ordered correlation functions in a generic (mixed) state. Focus on Gibbsian states.
- We double the degrees of freedom to account for the operator nature of the density matrix or equivalently work with a closed time contour:

$$S_{SK} = S[\Phi_{\rm R}] - S[\Phi_{\rm L}] \qquad \delta S_{SK} = \int d^d x \,\sqrt{-g}\,(\mathcal{J}_{\rm R}\,\mathbb{O}_{\rm R} - \mathcal{J}_{\rm L}\,\mathbb{O}_{\rm L})$$



Generating functional

$$\mathcal{Z}_{SK}[J_{\rm R}, J_{\rm L}] \equiv \mathrm{Tr}\Big\{U[J_{\rm R}]\;\hat{\rho}_{\rm initial}\;(U[J_{\rm L}])^\dagger\Big\}$$

Time ordered correlations

$$\mathrm{Tr}\Big(\;\hat{\rho}_{\rm initial}\;\bar{\mathcal{T}}\Big(U^\dagger \mathbb{O}_{\rm L} U^\dagger \mathbb{O}_{\rm L}\ldots\Big)\;\mathcal{T}\,(U\mathbb{O}_{\rm R} U\mathbb{O}_{\rm R}\ldots)\Big)$$

Thermal density matrices and KMS condition

- Thermal density matrices $\hat{\rho}_T = e^{-\beta\left(\widehat{\mathbb{H}}-\mu_I\,\widehat{\mathbb{Q}}^I\right)}$ define stationary equilibrium configurations.

- Correlation functions have analyticity properties which allows for a Euclidean (Matsubara) formulation, cf., $\mathcal{Z}_T(\beta,\mu_I) = \mathrm{Tr}\left(\hat{\rho}_T\right)$

$$\mathcal{Z}_T[\mathcal{J}_{\mathrm{R}},\mathcal{J}_{\mathrm{L}}] = \mathrm{Tr}\left(U[\mathcal{J}_{\mathrm{R}}]\,\hat{\rho}_T\,(U[\mathcal{J}_{\mathrm{L}}])^\dagger\right)$$

- KMS condition asserts that the correlation functions are analytic in the time strip $0 < \Im(t) < \beta$.

- Equivalently within correlation functions, operators and their KMS conjugates (or thermal translates) are equivalent.

Topological limit and BRST charges

- ✦ Ward identities for correlation functions in Gibbs states follow from

$$\mathcal{Z}_{SK}[\mathcal{J}_{\rm R} = \mathcal{J}_{\rm L} = \mathcal{J}] = \mathrm{Tr}\{\hat{\rho}_{\rm initial}\}$$

- ✦ Along with the KMS condition we learn that

$$\langle \mathcal{T}_{SK} \prod_k \left(\mathbb{O}_{\rm R}^{(k)} - \mathbb{O}_{\rm L}^{(k)}\right)\rangle \equiv \langle \mathcal{T}_{SK} \prod_k \mathbb{O}_{dif}^{(k)}\rangle = 0 \qquad \langle \mathcal{T}_{SK} \prod_{k=1}^{n} \left(\mathbb{O}_{\rm R}^{(k)} - \tilde{\mathbb{O}}_{\rm L}^{(k)}\right)\rangle = 0$$

Weldon '04

- ✦ Keldysh (light-cone) basis $\quad \mathbb{O}_{dif} \equiv \mathbb{O}_{\rm R} - \mathbb{O}_{\rm L}\ , \qquad \mathbb{O}_{av} \equiv \frac{1}{2}\left(\mathbb{O}_{\rm R} + \mathbb{O}_{\rm L}\right)$

- ✦ KMS conjugate operators $\quad \tilde{\mathbb{O}}_{\rm L}(t) = \mathbb{O}_{\rm L}(t - i\beta) = (-1)^{F_{\mathbb{O}}}\, e^{-i\,\delta_{\boldsymbol{\beta}}}\, \mathbb{O}_{\rm L}$

- ✦ Rather remarkable statement, which is agnostic of microscopic dynamics.

- ✦ It is a Ward identity arising from field redefinition symmetry inherent in doubling; rephrase as a BRST symmetry.

The Schwinger-Keldysh quartet

- Difference operator correlation functions vanish because they are trivial elements of a BRST cohomology.
- There exists a pair of Grassmann odd charges which act on the doubled operator algebra.
- The SK theory is covariantly expressed in terms of a quartet of fields, which usual doubled formalism being a gauge fixed version (ghosts =0).

$$\begin{tikzcd} & \mathbb{O}_{\mathrm{R}}, \mathbb{O}_{\mathrm{L}} \arrow[dl, "\mathcal{Q}_{SK}"] \arrow[dr, "\overline{\mathcal{Q}}_{SK}"'] & \\ \mathbb{O}_{G} \arrow[dr, "\overline{\mathcal{Q}}_{SK}"'] & & \mathbb{O}_{\overline{G}} \arrow[dl, "-\mathcal{Q}_{SK}"] \\ & \mathbb{O}_{\mathrm{R}} - \mathbb{O}_{\mathrm{L}} & \end{tikzcd}$$

$$\mathcal{Q}_{SK}^2 = \overline{\mathcal{Q}}_{SK}^2 = \left[\mathcal{Q}_{SK}, \overline{\mathcal{Q}}_{SK}\right]_{\pm} = 0$$

The KMS superalgebra

- The SK and KMS BRST charges generate an interesting superalgebra:

$$\mathcal{Q}_{SK}^2 = \overline{\mathcal{Q}}_{SK}^2 = \mathcal{Q}_{KMS}^2 = \overline{\mathcal{Q}}_{KMS}^2 = 0 \ ,$$
$$[\mathcal{Q}_{SK}, \mathcal{Q}_{KMS}]_\pm = \left[\overline{\mathcal{Q}}_{SK}, \overline{\mathcal{Q}}_{KMS}\right]_\pm = \left[\overline{\mathcal{Q}}_{SK}, \mathcal{Q}_{SK}\right]_\pm = \left[\overline{\mathcal{Q}}_{KMS}, \mathcal{Q}_{KMS}\right]_\pm = 0 \ ,$$
$$\left[\mathcal{Q}_{SK}, \overline{\mathcal{Q}}_{KMS}\right]_\pm = -\left[\overline{\mathcal{Q}}_{SK}, \mathcal{Q}_{KMS}\right]_\pm = i\Delta_{\boldsymbol{\beta}} \ .$$

- This algebra is well known in some circles, and forms part of the $\mathcal{N}_T = 2$ *extended equivariant cohomology algebra.*

Vafa, Witten '94
Dijkgraaf, Moore '96

- The $\mathcal{N}_T = 1$ algebra is realized as the standard Weil algebra satisfied by the de Rham complex involving exterior derivatives, Lie derivative and interior contraction.

- Well known results e.g., effective actions for stochastic dynamics a la Langevin can be phrased in this language.

Martin, Siggia, Rose '73

Mathai, Quillen '76

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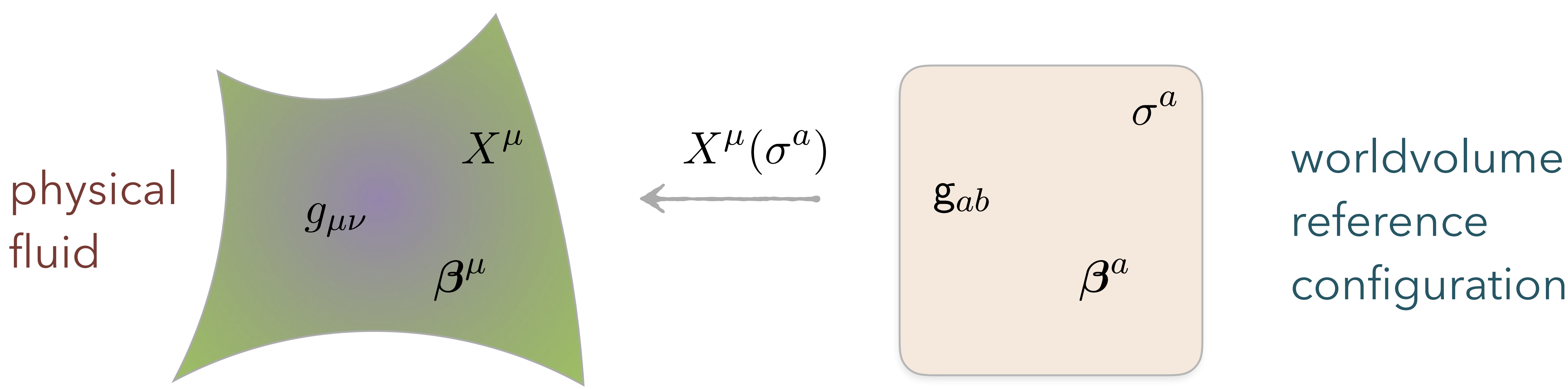
Hydrodynamic Landau-Ginzburg sigma models

- Class L ($H_S + \overline{H_S}$): effective action is just a sigma model parameterized by a scalar functional (generalized free energy density) $\mathcal{L}[\boldsymbol{\beta}^a, g_{ab}(X)]$.
- *Adiabatic fluids:* Invariance under diffeomorphisms and flavour transformations forces non-dissipative dynamics.
- Dynamics: conservation follows from variational of the pullback maps with reference thermal vector being fixed.



Topological sigma models for hydrodynamics

- Hydrodynamic modes are gauge invariant maps from the worldvolume to the target space (physical manifold).
- The symmetry being gauged is thermal translations.
- Variables: superfields with top and bottom components being SK difference and average fields respectively

$$\mathcal{Y} \to \mathring{\mathcal{Y}} = \mathcal{Y} + \theta\, \mathcal{Y}_{\bar\psi} + \bar\theta\, \mathcal{Y}_{\psi} + \bar\theta\,\theta\, \tilde{\mathcal{Y}} \equiv \frac{\mathcal{Y}_{\mathrm{L}} + \mathcal{Y}_{\mathrm{R}}}{2} + \theta\, \mathcal{Y}_{\bar\psi} + \bar\theta\, \mathcal{Y}_{\psi} + \bar\theta\,\theta\,(\mathcal{Y}_{\mathrm{R}} - \mathcal{Y}_{\mathrm{L}})$$

- Thermal translations act via Lie drag along reference thermal vector $\mathring{\boldsymbol{\beta}}^I(z)$

$$(\mathring{\Lambda}, \mathring{\mathcal{Y}})_{\boldsymbol{\beta}} = \mathring{\Lambda}\, \pounds_{\boldsymbol{\beta}} \mathring{\mathcal{Y}}$$

- KMS gauge superfield implements equivariance $\mathring{\mathcal{A}}_I(z)\, dz^I$.
- Covariant super-field strength: $\mathring{\mathfrak{F}}_{IJ} \equiv (1 - \frac{1}{2}\,\delta_{IJ}) \left(\partial_I\, \mathring{\mathcal{A}}_J - (-)^{IJ}\, \partial_J\, \mathring{\mathcal{A}}_I + (\mathring{\mathcal{A}}_I, \mathring{\mathcal{A}}_J)_{\boldsymbol{\beta}}\right)$

Topological sigma models for hydrodynamics

- Symmetries of hydrodynamic effective actions:
  - Superdiffeomorphisms in target space and world volume
  - CPT symmetry of SK path integrals ($\mathcal{Z}^*_{SK}[\mathcal{J}_{\mathrm{L}}, \mathcal{J}_{\mathrm{R}}] = \mathcal{Z}_{SK}[\mathcal{J}_{\mathrm{R}}, \mathcal{J}_{\mathrm{L}}]$)
  - worldvolume ghost number conservation
  - KMS gauge invariance
- Dynamical fields are the pull-back maps which induce a worldvolume super-metric $\mathring{\mathrm{g}}_{IJ}(z) = g_{\mu\nu}(\mathring{X}(z))\, \mathring{\mathcal{D}}_I \mathring{X}^\mu\, \mathring{\mathcal{D}}_J \mathring{X}^\nu$
- Its top component is the SK difference metric which couples to the physical stress tensor.
- Physical fluid dynamics obtained by deforming the topological theory.

$$\mathring{\mathrm{g}}_{IJ}(z) \;\to\; \mathring{\mathrm{g}}_{IJ}(z) + \bar{\theta}\,\theta\, \mathsf{h}_{IJ}(\sigma)$$

Dissipative hydrodynamic actions

- Symmetries suffice to constrain the terms that can appear in the worldvolume sigma model (superspace helps).

- Dissipative action including non-linear fluctuations:

$$\mathcal{L}_{\rm wv} = \frac{\sqrt{-\mathrm{g}}}{1+\boldsymbol{\beta}^e \mathcal{A}_e} \left\{ \frac{1}{2} \left[\mathbf{T}^{ab}_{\mathcal{L}} - \frac{i}{2} \boldsymbol{\eta}^{(ab)(cd)} \, (\mathcal{F}_{\theta\bar{\theta}}, \mathrm{g}_{cd})_{\boldsymbol{\beta}} \right] \tilde{\mathrm{g}}_{ab} - \mathbf{N}^a_{\mathcal{L}} \tilde{\mathcal{A}}_a \right. \\ \left. + \frac{i}{8} \left(\boldsymbol{\eta}^{(ab)(cd)} + \boldsymbol{\eta}^{(cd)(ab)} \right) \tilde{\mathrm{g}}_{ab} \, \tilde{\mathrm{g}}_{cd} + \dots \right\},$$

Eightfold Lagrangian

Noise fluctuations

see also Kovtun, Moore, Romatschke ’13; Crossley, Glorioso, Liu ‘15

- Dissipative dynamics spontaneously breaks CPT, KMS field strength picks up an expectation value (ghost condensate).

- Recovery of the classification appears possible (work in progress)…

Fluctuation-dissipation & Jarzynski

- Presence of a gauge symmetry which couples to entropy current appears to be manifestly contradicting second law.

- The spontaneous CPT symmetry breaking in dissipative dynamical systems leads to a Ward identity that implies the *Jarzynski relation*.

Mallick, Moshe, Orland '10; Gaspard '12

- Jarzynski is a non-equilibrium fluctuation dissipation relation that relates work done on the system out of equilibrium to the free energy difference.

$$\langle e^{-\frac{W}{\mathsf{T}}} \rangle = e^{-\frac{1}{\mathsf{T}}(G_f - G_i)}$$

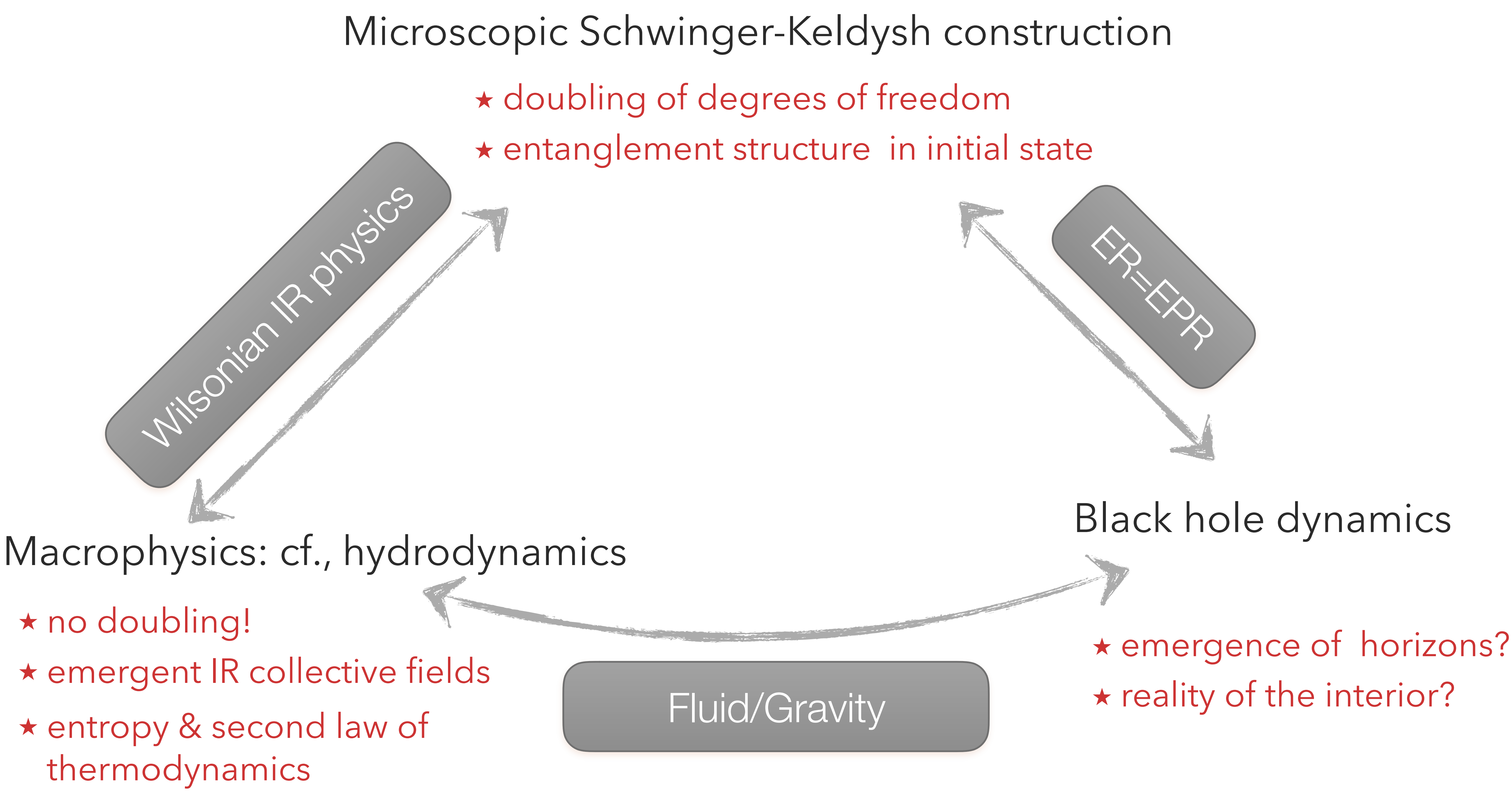
Jarzynski '97; Crooks '98

- Using Jensen's inequality, or convexity of exponential one arrives at

$$\langle W \rangle \geq G_f - G_i$$

- Upheld in hydrodynamic dissipative action.

A roadmap for the future



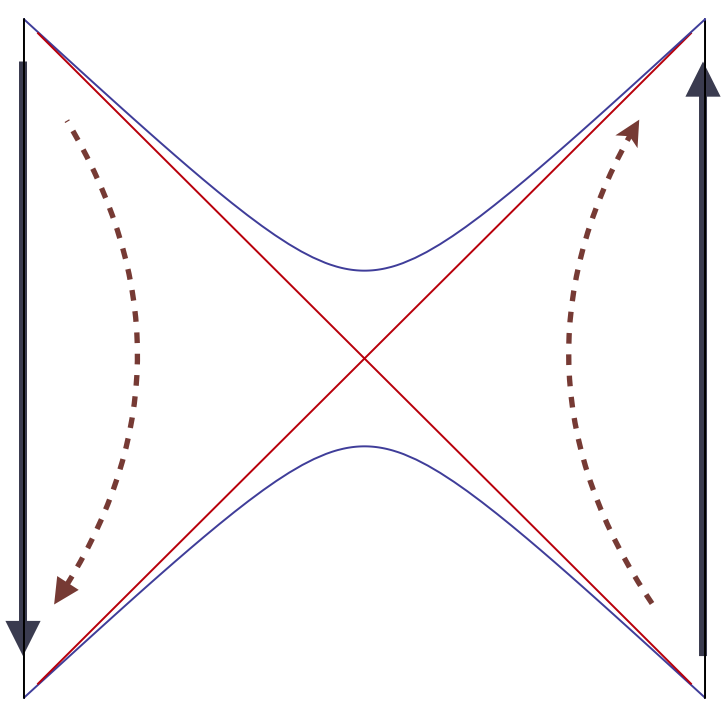
A microscopic perspective

- **Doubling**: Mixed states of a QFT can be purified by introducing an ancillary system. Focus on pure states in tensor product Hilbert space.
- Central to the Schwinger-Keldysh formalism developed to compute real time correlation functions in QFTs.

$$\mathcal{H}_{phys} \subset \mathcal{H}_R \otimes \mathcal{H}_L$$

$\mathcal{H}_L$

$\langle\psi| \to e^{iHt}\langle\psi|$



$\mathcal{H}_R$

$|\psi\rangle \to e^{-iHt}\,|\psi\rangle$

$$\rho = |\psi\rangle\langle\psi| \to e^{-iHt}\,|\psi\rangle\langle\psi|\,e^{iHt}$$

Schwinger '61
Keldysh '64