Quantum terms in relativistic hydrodynamics (the four-temperature vector)

OUTLINE

- Stress-energy tensor in quantum relativistic theories and local thermodynamical equilibrium
- Global equilibrium and quantum corrections to the ideal form
- The β frame
- Killing vectors and Lie derivatives

F. Becattini, Mainz workshop, October 2016

Basics

In a quantum statistical framework, the mean stress-energy tensor is defined as:

$$T^{\mu\nu}(x) = \mathrm{tr}(\widehat{\rho}\widehat{T}^{\mu\nu}(x))_{\mathrm{ren}}$$

The density operator of the familiar global thermodynamical equilibrium in flat spacetime (in covariant form):

$$\widehat{\rho} = (1/Z)\exp[-\beta \cdot \widehat{P} + \zeta \widehat{Q}]$$

$$\beta^\mu = (1/T)u^\mu$$

$$T = 1/\sqrt{\beta^2}$$

$$\zeta = \mu/T$$

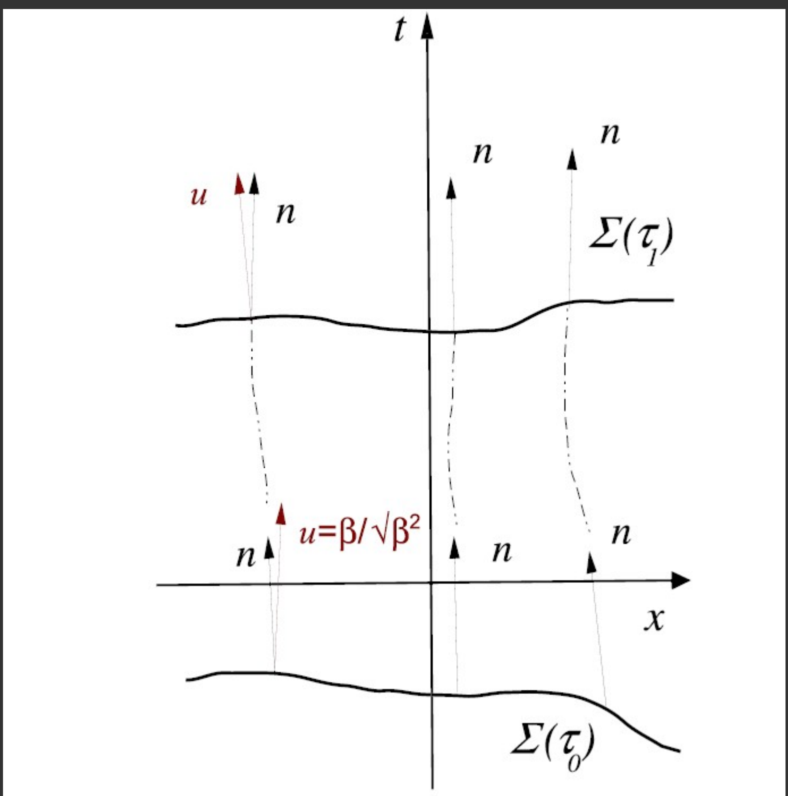
$$T^{\mu\nu}(x) = (\rho + p)u^\mu u^\nu - p g^{\mu\nu}$$

General covariant local equilibrium

Zubarev 1979 Weert 1982
F. B., L. Bucciantini, E. Grossi, L. Tinti, Eur. Phys. J. C 75 (2015) 191

Maximizing the entropy with the constraints of energy density and momentum density

$$-\mathrm{tr}(\widehat{\rho}\log\widehat{\rho}) + \int_{\Sigma(\tau)} \mathrm{d}\Sigma\, n_\mu \left[\left(\langle\widehat{T}^{\mu\nu}(x)\rangle - T^{\mu\nu}(x)\right)\beta_\nu(x) - \left(\langle\widehat{j}^\mu(x)\rangle - j^\mu(x)\right)\xi(x)\right]$$



Lagrange multipliers functions

Requires the specification of 3D spacelike hypersurfaces

$$\epsilon_{\mu\nu\rho\sigma} n^\nu (\partial^\rho n^\sigma - \partial^\sigma n^\rho) = 0.$$

$$n_\mu \mathrm{tr}(\widehat{\rho}_{\rm LE}\, \widehat{T}^{\mu\nu}(x))_{\rm ren} = n_\mu \langle\widehat{T}^{\mu\nu}(x)\rangle_{\rm LE} \equiv n_\mu T^{\mu\nu}_{\rm LE}(x) = n_\mu T^{\mu\nu}(x)$$
$$n_\mu \mathrm{tr}(\widehat{\rho}_{\rm LE} \widehat{j}^\mu(x))_{\rm ren} = n_\mu \langle\widehat{j}^\mu(x)\rangle_{\rm LE} \equiv n_\mu j^\mu_{\rm LE}(x) = n_\mu j^\mu(x)$$

General covariant local equilibrium (2)

$$\widehat{\rho}_{\text{LE}} = \frac{1}{Z} \exp\left[- \int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

Once n is chosen, β and ζ are solution of the equations

$$n_\mu T^{\mu\nu}_{\text{LE}}[\beta, \xi, n] = n_\mu T^{\mu\nu} \qquad n_\mu j^\mu_{\text{LE}}[\beta, \xi, n] = n_\mu j^\mu,$$

β frame: β *defines* a rest frame through $$n = u = \frac{\beta}{\sqrt{\beta^2}}$$

If β is a vorticous field, the definition must be changed, but it still can be done

See also:

T. Hayata, Y. Hidaka, T. Noumi, M. Hongo, Phys. Rev. D 92 (2015) 065008

S. Floerchinger, arXiv:1603.07148

General covariant *global* equilibrium in flat spacetime

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \; \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

If the divergence of the integrand vanishes, that is:

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$

$$\partial_\mu \zeta = 0$$

Σ can now be an arbitrary general timelike 3D hypersurface

Solution of the Killing equation in Minkowski spacetime:

$$\beta^\nu = b^\nu + \varpi^{\nu\mu} x_\mu$$

constant

$$\varpi_{\nu\mu} = -\frac{1}{2}(\partial_\nu \beta_\mu - \partial_\mu \beta_\nu)$$

Thermal vorticity

Adimensional in natural units

General global equilibrium -2

Plugging the solution into the general covariant expression of the density operator:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}^{\mu\nu} + \zeta\widehat{Q}\right]$$

with

$$\widehat{J}^{\mu\nu} = \int_\Sigma \mathrm{d}\Sigma_\lambda \left(x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu}\right)$$

The most general thermodynamical equilibrium in Minkowski spacetime involves the 10 generators of its maximal symmetry group.

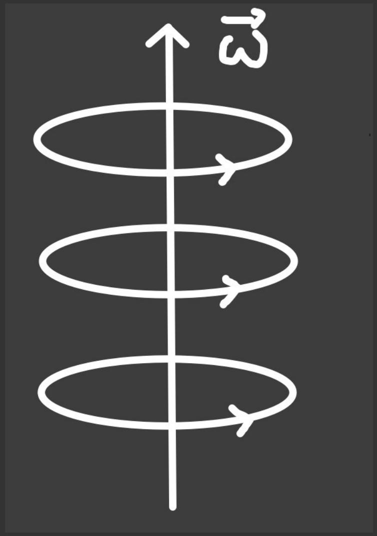
Special cases

Pure rotation (Landau *Statistical Physics*)

$$b_\mu = (1/T_0, 0, 0, 0) \qquad \varpi_{\mu\nu} = (\omega/T_0)(g_{1\mu}g_{2\nu} - g_{1\nu}g_{2\mu})$$

$$\beta^\mu = \frac{1}{T_0}(1, \boldsymbol{\omega} \times \mathbf{x})$$

$$\widehat{\rho} = (1/Z)\exp[-\widehat{H}/T_0 + \omega \widehat{J}_z/T_0]$$

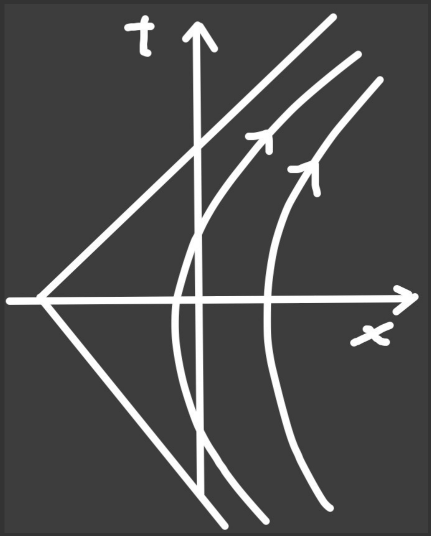


Pure acceleration (new)

$$b_\mu = (1/T_0, 0, 0, 0) \qquad \varpi_{\mu\nu} = (a/T_0)(g_{0\mu}g_{3\nu} - g_{3\mu}g_{0\nu})$$

$$\beta^\mu = \frac{1}{T_0}(1 + az, 0, 0, at)$$

$$\widehat{\rho} = (1/Z)\exp[-\widehat{H}/T_0 + a\widehat{K}_z/T_0]$$



The mean value of local operators

F. B., E. Grossi, Phys. Rev. D 92 (2015) 045037

$$O(x) = \frac{1}{Z} \mathrm{tr} \left(\exp \left[-b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} + \zeta \widehat{Q} \right] \widehat{O}(x) \right)_{\mathrm{ren}} = \frac{1}{Z} \mathrm{tr} \left(\exp \left[-\beta_\mu(x) \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu}_x + \zeta \widehat{Q} \right] \widehat{O}(x) \right)_{\mathrm{ren}}$$

$$\widehat{J}^{\mu\nu}_x = \widehat{J}^{\mu\nu} - x^\mu \widehat{P}^\nu + x^\nu \widehat{P}^\mu = \widehat{\mathsf{T}}(x) \widehat{J}^{\mu\nu} \widehat{\mathsf{T}}(x)^{-1}$$

The thermal vorticity $\varpi_{\mu\nu}$ is, in most cases, a small adimensional parameter, so that an expansion of the exponent can be done. Note that the operators *J, P* do not commute!

The expansion is meaningful if – as we shall see – the thermal correlation length is much smaller than the length over which the fields β and ζ significantly vary *(hydrodynamic limit)*

$$\frac{\partial \beta}{\beta} \ll \frac{1}{\beta}, \frac{1}{m} \to \varpi \ll 1$$

It will give rise to quantum and quantum relativistic corrections beyond the ideal form

$$\varpi \longrightarrow \qquad \frac{\hbar a}{cKT} \qquad \frac{\hbar \omega}{KT}$$

Exponent expansion

(Baker-Campbell-Hausdorff formula)

$$\widehat{\mathcal{R}}(\varpi) \equiv \exp\left[-\beta_\mu(x)\widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}_x^{\mu\nu} + \zeta\widehat{Q}\right] = \exp\left[-\beta_\mu(x)\widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}_x^{\mu\nu}\right]\exp[\zeta\widehat{Q}]$$

$$\widehat{\mathcal{R}}(\varpi) = \widehat{\mathcal{R}}^{(0)} + \varpi_{\mu\nu}\widehat{\mathcal{R}}^{(1)\mu\nu} + \varpi_{\mu\nu}\varpi_{\rho\sigma}\widehat{\mathcal{R}}^{(2)\mu\nu\rho\sigma} + o(\varpi^2)$$

$$\begin{aligned}
\widehat{\mathcal{R}}^{(0)} &= \mathrm{e}^{-\beta\cdot\widehat{P}+\zeta\widehat{Q}} \\
\widehat{\mathcal{R}}^{(1)\mu\nu} &= \frac{1}{4}\{\mathrm{e}^{-\beta\cdot\widehat{P}+\zeta\widehat{Q}}, \widehat{J}^{\mu\nu}\}, \\
\widehat{\mathcal{R}}^{(2)\mu\nu\rho\sigma} &= \frac{1}{16}\{\mathrm{e}^{-\beta\cdot\widehat{P}+\zeta\widehat{Q}}, \widehat{J}^{\mu\nu}\widehat{J}^{\rho\sigma}\} + \frac{1}{8}\mathrm{e}^{-\beta\cdot\widehat{P}+\zeta\widehat{Q}}\beta^\mu\beta^\rho\widehat{P}^\nu\widehat{P}^\sigma - \frac{1}{12}\mathrm{e}^{-\beta\cdot\widehat{P}+\zeta\widehat{Q}}\beta^\mu g^{\nu\rho}\widehat{P}^\sigma .
\end{aligned}$$

The leading term of the mean value of the operator O will be its mean value at the familiar homogeneous equilibrium with a four-temperature equal to its value in the same point *x* where the operator is taken

$$\langle\widehat{O}(x)\rangle_{\beta(x)} = \frac{\mathrm{tr}(\exp[-\beta(x)\cdot\widehat{P} + \zeta\widehat{Q}]\widehat{O}(x))}{\mathrm{tr}(\exp[-\beta(x)\cdot\widehat{P} + \zeta\widehat{Q}])}$$

Stress-energy tensor expansion to 2nd order in ω

$$T^{\mu\nu}(x) = \frac{1}{Z} \mathrm{tr}\left(\exp\left[-\beta_\mu(x)\widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}^{\mu\nu}_x + \zeta\widehat{Q}\right]\widehat{T}^{\mu\nu}(x)\right)_{\mathrm{ren}}$$

$$\langle \widehat{A}; \widehat{B}\rangle = \langle \widehat{A}\widehat{B}\rangle - \langle \widehat{A}\rangle\langle \widehat{B}\rangle$$

$$\begin{aligned} T^{\mu\nu}(x) =& \frac{\mathrm{tr}(\widehat{\mathcal{R}}(\varpi)\widehat{T}^{\mu\nu}(x))}{\mathrm{tr}(\widehat{\mathcal{R}}(\varpi))} = \langle \widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)} + \frac{1}{2}\varpi_{\rho\sigma}\mathrm{Re}\langle \widehat{J}^{\rho\sigma}_x; \widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)} \\ & +\varpi_{\rho\sigma}\varpi_{\lambda\tau}\left[\frac{1}{8}\mathrm{Re}\langle \widehat{J}^{\rho\sigma}_x \widehat{J}^{\lambda\tau}_x; \widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)} + \frac{1}{8}\beta^\rho(x)\beta^\lambda(x)\langle \widehat{P}^\sigma \widehat{P}^\tau; \widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)} - \frac{1}{12}\beta^\rho(x) g^{\lambda\sigma}\langle \widehat{P}^\tau; \widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)}\right. \\ & \left. -\frac{1}{4}\mathrm{Re}\langle \widehat{J}^{\rho\sigma}_x; \widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)}\langle \widehat{J}^{\lambda\tau}_x\rangle_{\beta(x)}\right] + o(\varpi^2) \end{aligned} \quad (25)$$

With:

$$\langle T^{\mu\nu}(x)\rangle_{\beta(x)} = (\rho(x) + p(x))u^\mu(x)u^\nu(x) - p(x)g^{\mu\nu}$$

Acceleration-rotation decomposition

$$\varpi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} w_\rho u_\sigma + \alpha^\mu u^\nu - \alpha^\nu u^\mu$$

At global equilibrium, using Killing equation

$$\alpha^\mu = \frac{1}{T} a^\mu \qquad w^\mu = \frac{1}{T}\omega^\mu$$

Similarly:

$$\widehat{J}^{\mu\nu} = u^\mu \widehat{K}^\nu - \widehat{K}^\mu u^\nu + \epsilon^{\mu\nu\rho\sigma} \widehat{J}_\rho u_\sigma$$

$$\widehat{K}^\mu = u_\rho \widehat{J}^{\rho\mu} \qquad \widehat{J}^\mu = -\frac{1}{2}\epsilon^{\mu\rho\sigma\tau}\widehat{J}_{\rho\sigma}u_\tau$$

$$T^{\mu\nu}(x) \simeq \langle \widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)} + \frac{1}{2}\alpha_\rho\alpha_\sigma \mathrm{Re}\langle \widehat{K}^\rho \widehat{K}^\sigma ; \widehat{T}^{\mu\nu}(0)\rangle_{\beta(x)} + \frac{1}{2} w_\rho w_\sigma \mathrm{Re}\langle \widehat{J}^\rho \widehat{J}^\sigma ; \widehat{T}^{\mu\nu}(0)\rangle_{\beta(x)}$$
$$+ \frac{1}{2}\alpha_\rho w_\sigma \mathrm{Re}\langle \{\widehat{J}^\rho, \widehat{K}^\sigma\}; \widehat{T}^{\mu\nu}(0)\rangle_{\beta(x)} + \frac{1}{8}\beta^\rho\beta^\lambda \frac{\partial^2}{\partial\beta_\sigma\,\partial\beta_\tau}\langle \widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)} + \frac{1}{12}\beta^\rho g^{\lambda\sigma}\frac{\partial}{\partial\beta_\tau}\langle \widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)}$$

Final expression

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu \alpha^\nu + W w^\mu w^\nu + G(u^\mu \gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

COEFFICIENTS (see talk by E. Grossi):

$$\gamma^\mu = (\alpha \cdot \varpi)_\lambda \Delta^{\lambda\mu} = \epsilon^{\mu\nu\rho\sigma} w_\nu \alpha_\rho u_\sigma$$

$$U_\alpha = \frac{1}{24} T \frac{\partial \rho}{\partial T} + \frac{1}{4}(\rho + p) + \frac{1}{2} k_t \qquad U_w = \frac{1}{2} j_t \qquad D_\alpha = \frac{1}{24}(\rho + p) + \frac{1}{2} k_\theta - \frac{1}{3} k_s$$

$$D_w = \frac{1}{2} j_\theta - \frac{1}{3} j_s \qquad A = \frac{1}{4}(\rho + p) + k_s \qquad W = j_s$$

$$G = \frac{1}{2} l_v - \frac{1}{12}(\rho + p) \tag{33}$$

$$k_t(T,\zeta) = \mathrm{Re}\langle \widehat{K}^3 \widehat{K}^3; \widehat{T}^{00}(0)\rangle_T \qquad k_\theta(T,\zeta) = \frac{1}{3}\mathrm{Re}\sum_{i=1}^{3}\langle \widehat{K}^3 \widehat{K}^3; \widehat{T}^{ii}(0)\rangle_T \qquad k_s(T,\zeta) = \mathrm{Re}\langle \widehat{K}^1 \widehat{K}^2; \widehat{T}^{12}(0)\rangle_T$$

$$j_t(T,\zeta) = \mathrm{Re}\langle \widehat{J}^3 \widehat{J}^3; \widehat{T}^{00}(0)\rangle_T \qquad j_\theta(T,\zeta) = \frac{1}{3}\mathrm{Re}\sum_{i=1}^{3}\langle \widehat{J}^3 \widehat{J}^3; \widehat{T}^{ii}(0)\rangle_T \qquad j_s(T,\zeta) = \mathrm{Re}\langle \widehat{J}^1 \widehat{J}^2; \widehat{T}^{12}(0)\rangle_T$$

$$l_v(T,\zeta) = \mathrm{Re}\langle \{\widehat{K}^1, \widehat{J}^2\}; \widehat{T}^{03}(0)\rangle_T \tag{30}$$

"Kubo" formulae of non-dissipative coefficients are correlators of stress-energy tensor with (conserved) generators of Lorentz group

The corrections are of quantum origin

$$T^{\mu\nu}(x) = \left[\rho + \left(\frac{\hbar|a|}{cKT}\right)^2 U_\alpha + \left(\frac{\hbar|\omega|}{KT}\right)^2 U_w\right] u^\mu u^\nu - \left[p + \left(\frac{\hbar|a|}{cKT}\right)^2 D_\alpha + \left(\frac{\hbar|\omega|}{KT}\right)^2 D_w\right] \Delta^{\mu\nu}$$

$$+A\left(\frac{\hbar|a|}{cKT}\right)^2 \hat{a}^\mu \hat{a}^\nu + W\left(\frac{\hbar|\omega|}{KT}\right)^2 \hat{\omega}^\mu \hat{\omega}^\nu + G\frac{\hbar^2|\omega||a|}{c(KT)^2}(u^\mu \hat{\gamma}^\nu + \hat{\gamma}^\mu u^\nu) + o(\varpi^2)$$

$$|a| = \sqrt{-a_\mu a^\mu} \qquad |\omega| = \sqrt{-\omega_\mu \omega^\mu}.$$

In the free scalar field, the coefficients *U, D, A, W, G* have a classical limit, whereas the adimensional scales vanish in the $h \to 0$ limit. The reason is that with *m, T* as scales no non-quantum correction can exist.

Comparison with previous definitions

Quadratic corrections in the vorticity and acceleration:

R. Baier et al., JHEP 0804 (2008) 100; P. Romatschke, Class. Quant. Grav. 27 (2010) 025006

$$
\begin{aligned}
T^{\mu\nu} = {} & \text{eq. (2.3)} + \eta\tau_\pi \left(u\cdot\nabla\sigma^{\mu\nu} + \frac{\nabla\cdot u}{3}\sigma^{\mu\nu} \right) \\
& + \kappa \left(R^{\langle\mu\nu\rangle} - 2u_\alpha u_\beta R^{\alpha\langle\mu\nu\rangle\beta} \right) \\
& + \lambda_1 \sigma_\lambda{}^{\langle\mu}\sigma^{\nu\rangle\lambda} + \lambda_2 \sigma_\lambda{}^{\langle\mu}\Omega^{\nu\rangle\lambda} - \lambda_3 \Omega_\lambda{}^{\langle\mu}\Omega^{\nu\rangle\lambda} \\
& + \eta\tau_\pi^* \frac{\nabla\cdot u}{3}\sigma^{\mu\nu} + \lambda_4 \nabla^{\langle\mu} \ln s \nabla^{\nu\rangle} \ln s + 2\kappa^* u_\alpha u_\beta R^{\alpha\langle\mu\nu\rangle\beta} \\
& + \Delta^{\mu\nu} \Big(-\zeta\tau_\Pi u\cdot\nabla\nabla\cdot u + \xi_1 \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \xi_2 (\nabla\cdot u)^2 \\
& \qquad + \xi_4 \nabla_{\alpha\perp} \ln s \nabla_\perp^\alpha \ln s + \xi_3 \Omega^{\alpha\beta}\Omega_{\alpha\beta} + \xi_5 R + \xi_6 u^\alpha u^\beta R_{\alpha\beta} \Big) \,.
\end{aligned}
$$

The corrections are taken to be orthogonal to *u* (Landau frame), hence the number of coefficients is less than for our frame choice. The mapping to the usual nomenclature:

$$
\frac{A}{T^2} = 9\lambda_4 \qquad \frac{W}{T^2} = \lambda_3
$$

$$
\frac{D_w}{T^2} = \left(\frac{\lambda_3}{3} - 2\xi_3 \right) \qquad \frac{D_\alpha}{T^2} = (3\lambda_4 - 9\xi_4)
$$

λ_3 with ξ=0 in agreement with G. Moore and K. Sohrabi, JHEP 1211 (2012) 148

Note: After the publication we realized that the papers using the Landau frame had a redefinition of temperature:

$$\rho(T') \simeq \rho(T) - \alpha^2 U_\alpha - w^2 U_w$$

$$\frac{A}{T^2} = 9\lambda_4 \qquad \frac{W}{T^2} = \lambda_3$$

$$-\frac{\partial p/\partial T}{\partial \rho/\partial T}\frac{U_w}{T^2} + \frac{D_w}{T^2} = \left(\frac{\lambda_3}{3} - 2\xi_3\right) \qquad -\frac{\partial p/\partial T}{\partial \rho/\partial T}\frac{U_\alpha}{T^2} + \frac{D_\alpha}{T^2} = (3\lambda_4 - 9\xi_4)$$

Note 2: Only 4 out of the 7 parameters are independent, there are 3 relations between them due to continuity equation (see E. Grossi's talk)

Note 3: There is a modification of the relation between effective energy density and pressure which cannot be eliminated through a redefinition of the temperature

Taylor expansion of local equilibrium

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z} \exp\left[- \int_\Sigma {\rm d}\Sigma_\mu \; \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

One can also calculate the mean value of local operators in x at local thermodynamic equilibrium by Expanding the thermodynamic fields β and ζ from the point x

$$T^{\mu\nu}_{\rm LE}(x) = {\rm tr}(\widehat{\rho}_{\rm LE} \widehat{T}^{\mu\nu}(x))_{\rm ren} = \frac{1}{Z_{\rm LE}} {\rm tr} \left(\exp\left[- \int {\rm d}\Sigma_\mu \; \left(\widehat{T}^{\mu\nu} \beta_\nu - \xi \widehat{j}^\mu \right) \right] \widehat{T}^{\mu\nu}(x) \right)_{\rm ren},$$

The leading terms are those of the global equilibrium with the same values as in x

$$\begin{aligned} &\exp\left[- \int {\rm d}\Sigma_\mu \; \left(\widehat{T}^{\mu\nu} \beta_\nu - \xi \widehat{j}^\mu \right) \right] \\ &\simeq \exp\left[-\beta_\nu(x) \int {\rm d}\Sigma_\mu \; \widehat{T}^{\mu\nu} + \xi(x) \int {\rm d}\Sigma_\mu \; \widehat{j}^\mu - \frac{\partial \beta_\nu}{\partial \sigma_i}(x) \int {\rm d}\Sigma_\mu \; \widehat{T}^{\mu\nu} (\sigma_i - \sigma_{0i}) + \frac{\partial \xi}{\partial \sigma_i}(x) \int {\rm d}\Sigma_\mu \; \widehat{j}^\mu (\sigma_i - \sigma_{0i}) + \ldots \right] \end{aligned}$$

$$\widehat{\rho}_{\rm LE} \simeq \frac{1}{Z_{\rm LE}} \exp\left[-\beta_\nu(x) \widehat{P}^\nu + \xi(x) \widehat{Q} - \frac{1}{4} (\partial_\nu \beta_\lambda(x) - \partial_\lambda \beta_\nu(x)) \widehat{J}^{\lambda\nu}_x + \frac{1}{2} (\partial_\nu \beta_\lambda(x) + \partial_\lambda \beta_\nu(x)) \, \widehat{L}^{\lambda\nu}_x + \nabla_\lambda \xi(x) \, \widehat{d}^\lambda_x \right].$$

β frame: change of viewpont

$$\beta^\mu = \frac{1}{T} u^\mu$$

$$u^\mu = \frac{1}{\sqrt{\beta^2}} \beta^\mu \qquad T = \frac{1}{\sqrt{\beta^2}}$$

Hydrodynamics and four-temperature β

The four-temperature β and ζ can be taken as the five independent variables in hydrodynamics

$$p = p(\beta^2, \zeta)$$

$$T_{\rm id}^{\mu\nu} = -2\frac{\partial p}{\partial \beta^2}\beta^\mu \beta^\nu - p g^{\mu\nu},$$

$$-2\frac{\partial p}{\partial \beta^2} = \frac{\rho + p}{\beta^2} = \frac{h}{\beta^2}.$$

All dissipative terms can be expressed in terms of gradients of the four-temperature

$$\beta_\lambda \Delta_{\mu\nu}(\partial^\lambda \beta^\nu + \partial^\nu \beta^\lambda) = \frac{1}{T^2}\left(A_\mu - \frac{1}{T}\nabla_\mu T\right).$$

$$\nabla_\mu u^\nu = \frac{1}{\sqrt{\beta^2}}\Delta^{\rho\nu}\nabla_\mu \beta_\rho,$$

A physical definition of the β frame

F. B., L. Bucciantini, E. Grossi, L. Tinti, Eur. Phys. J. C 75 (2015) 191

According to the generally accepted relativistic thermodynamics laws, a thermometer with four-velocity *u* in a fluid with four-temperature β will measure a temperature:

$$T_T = \frac{1}{\beta \cdot u}$$

- At each space-time point, put ideal thermometers (zero relaxation time) with all possible velocities

- The thermometer marking the highest temperature is the one moving with the four velocity $u^\mu = \beta^\mu / \sqrt{\beta^2}$

β frame vs Landau frame

$$u^\mu = \beta^\mu / \sqrt{\beta^2}$$

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu \alpha^\nu + W w^\mu w^\nu + G(u^\mu \gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

$$T^{\mu\nu} u_\nu = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu + G\gamma^\mu$$

$$\gamma^\mu = (\alpha \cdot \varpi)_\lambda \Delta^{\lambda\mu} = \epsilon^{\mu\nu\rho\sigma} w_\nu \alpha_\rho u_\sigma$$

In other words, the Killing vector field, which *defines equilibrium*, is in general NOT an eigenvector of the stress-energy tensor.

An observer moving along the eigenvector of the stress-energy tensor, in general relativity, will see the metric tensor (and all the rest) changing, which is not desirable in a proper definition of equilibrium

Curved spacetime

This formalism can be extended to GR

$$\widehat{\rho} = \frac{1}{Z} \exp \left[- \int_{\Sigma} \mathrm{d}\Sigma_{\mu} \, \left(\widehat{T}^{\mu\nu} \beta_{\nu} - \zeta \widehat{j}^{\mu} \right) \right]$$

Equilibrium condition

$$\nabla_{\mu} \beta_{\nu} + \nabla_{\nu} \beta_{\mu} = 0$$

Idea: Taylor expand the Killing vector field from the point where an operator is to be evaluated

$$\langle \widehat{O}(x) \rangle = \frac{1}{Z} \mathrm{tr} \left(\exp \left[- \int_{\Sigma} \mathrm{d}\Sigma \, n_{\mu} (\widehat{T}^{\mu\nu} \beta_{\nu} - \zeta \widehat{j}^{\mu}) \right] \widehat{O}(x) \right)_{\mathrm{ren}}$$

under the same assumption of the hydrodynamical limit

At first order, the β field has a non-vanishing exterior derivative!

$$\partial_\mu \beta_\nu = \frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) = \varpi_{\nu\mu}$$

$$\int_\Sigma \mathrm{d}\Sigma_\mu(y) \widehat{T}^{\mu\nu}(y)\beta_\nu(y) \simeq \beta_\nu(x)\widehat{P}^\nu - \frac{1}{2}\varpi_{\lambda\nu}(x)\widehat{J}_x^{\lambda\nu}$$

For instance, in a rotationally symmetric field (Schwarzschild):

Tolman's law

$$\beta^\mu = \frac{1}{T_0}(1,0,0,0) \qquad T = \frac{1}{\sqrt{\beta^2}} = \frac{T_0}{\sqrt{g_{00}(r)}} \qquad \varpi_{tr} = \frac{1}{2T_0}\partial_r g_{00}(r)$$

$$\alpha_r = -\frac{1}{2T_0}\partial_r g_{00}(r) \qquad w = 0$$

Killing vectors and Lie derivatives

F. B., arXiv:1606.06605

An important consequence of the existence of a Killing vectors is that ALL possible terms in the gradient expansion of the stress-energy tensor – as well as any other observable - ought to have a vanishing Lie derivative along this field.
This holds whether the Killing vector field is timelike or not.

$$X = \mathrm{tr}(\widehat{\rho}\widehat{X}) = X[\beta, \zeta, g]$$

$$X(x) = X[\beta, \zeta, g] = X(\beta, \nabla\beta, \nabla\nabla\beta, \ldots, g, R, \nabla R, \nabla\nabla R, \ldots)$$

If β is a Killing vector, then $\mathcal{L}_\beta(X) = 0$

By definition of Killing vector:

$$\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0 \iff \mathcal{L}_\beta(g) = 0$$

It can be shown that:

$$\mathcal{L}_\beta(R) = 0$$

$$\mathcal{L}_\beta(\nabla T) = \nabla \mathcal{L}_\beta(T)$$

If β is a Killing vector, then $\mathcal{L}_\beta(X) = 0$

Conversely, in general, if *u* is the eigenvector of the stress-energy tensor (Landau frame)

$$\mathcal{L}_u(T) \neq 0$$

Summary

- The stress-energy tensor has quantum corrections to the ideal form in states of thermodynamic equilibrium involving acceleration and rotation in flat spacetime.

- Second-order, non-dissipative terms in acceleration and rotation have coefficients which can be expressed with correlators of conserved Lorentz group generators and stress-energy tensor (3 point functions without time integration)

- Relevant scales: ha/cKT and $h\omega/KT$: thus far, the only place where they play a role is relativistic heavy ion collisions $O(10^{-3} - 10^{-2})$

- Equilibrium needs a Killing vector field to be properly defined, especially in GR. The four-temperature β – which can be defined on the basis of ideal thermometers - is the Killing vector.

- Difference between the Killing frame (four-temperature β frame) and Landau frame even at thermodynamical equilibrium.

Coefficients for a free real scalar massive field

$$\widehat{T}_{\xi}^{\mu\nu} = \partial^{\mu}\widehat{\psi}\partial^{\nu}\widehat{\psi} - \frac{1}{2}g^{\mu\nu}\left(\partial_{\lambda}\widehat{\psi}\,\partial^{\lambda}\widehat{\psi} - m^2\widehat{\psi}^2\right) + 2\xi\,\partial_{\lambda}\left(g^{\mu\nu}\widehat{\psi}\,\partial^{\lambda}\widehat{\psi} - g^{\lambda\mu}\widehat{\psi}\,\partial^{\nu}\widehat{\psi}\right)$$

SUMMARY TABLE

Massless

$$U = \kappa T^4$$

Massive fully relativistic

$$U = \frac{m^4}{2\pi^2}\sum_{r=1}^{\infty} a_r(x = m/T, \xi)$$

Massive non-relativistic limit
n particle density

$$U = n f(m, T)$$

	$\kappa(\xi)$	$a_r(x,\xi)$	$f(m,t)$
U_α	$\frac{1}{12}(1-6\xi)$	$\frac{1}{24}\left[(r^2+24\xi x^{-2})K_2(rx)+3(1-8\xi)rx^{-1}K_3(rx)\right]$	$\frac{1}{24}m^2T^{-1}+\frac{1}{8}m(1-8\xi)+(\frac{5}{16}-\frac{3}{2}\xi)T+o(T)$
U_w	$\frac{1}{12}(1-4\xi)$	$\frac{1}{2}(1-4\xi)x^{-2}K_2(rx)$	$(\frac{1}{2}-2\xi)T+o(T)$
D_α	$\frac{1}{18}(6\xi-1)$	$\frac{1}{24}\left[(12-48\xi)x^{-2}K_2(rx)+(24\xi-5)rx^{-1}K_3(rx)\right]$	$m(\xi-\frac{5}{24})+(\frac{1}{2}\xi-\frac{1}{48})T+o(T)$
D_w	$\frac{1}{6}\xi$	$\xi x^{-2}K_2(rx)$	$\xi T+o(T)$
A	$\frac{1}{12}(1-6\xi)$	$\frac{1}{4}\left[(4\xi-2)x^{-2}K_2(rx)+(1-4\xi)rx^{-1}K_3(rx)\right]$	$m(\frac{1}{4}-\xi)+(\frac{1}{8}-\frac{3}{2}\xi)T+o(T)$
W	$\frac{1}{12}(2\xi-1)$	$\frac{1}{2}(2\xi-1)x^{-2}K_2(rx)$	$(\xi-\frac{1}{2})T+o(T)$
G	$\frac{1}{36}(1+6\xi)$	$\frac{1}{6}\left[(6\xi-3)x^{-2}K_2(rx)+rx^{-1}K_3(rx)\right]$	$\frac{1}{6}m+(\xi-\frac{1}{12})T+o(T)$

The corrections are of quantum origin

$$T^{\mu\nu}(x) = \left[\rho + \left(\frac{\hbar|a|}{cKT}\right)^2 U_\alpha + \left(\frac{\hbar|\omega|}{KT}\right)^2 U_w\right] u^\mu u^\nu - \left[p + \left(\frac{\hbar|a|}{cKT}\right)^2 D_\alpha + \left(\frac{\hbar|\omega|}{KT}\right)^2 D_w\right] \Delta^{\mu\nu}$$

$$+A\left(\frac{\hbar|a|}{cKT}\right)^2 \hat{a}^\mu \hat{a}^\nu + W\left(\frac{\hbar|\omega|}{KT}\right)^2 \hat{\omega}^\mu \hat{\omega}^\nu + G\frac{\hbar^2|\omega||a|}{c(KT)^2}(u^\mu \hat{\gamma}^\nu + \hat{\gamma}^\mu u^\nu) + o(\varpi^2)$$

$$|a| = \sqrt{-a_\mu a^\mu} \qquad |\omega| = \sqrt{-\omega_\mu \omega^\mu}.$$

In the free scalar field, the coefficients *U, D, A, W, G* have a classical limit, whereas the adimensional scales vanish in the $h \to 0$ limit. The reason is that with *m, T* as scales no non-quantum correction can exist.

Acceleration and vorticity play the role of two new scales in the quantum field problem

The magnitude of these corrections depends on the coefficients *U, D,* …. which are new thermodynamical equilibrium functions.

Other consequences

- As the stress-energy tensor has non-dissipative thermal quantum corrections beyond its ideal form if the fluid is rotating or accelerated. they will also be present in gravitational fields (see later).
- Second-order, non-dissipative corrections depend on the specific form of the quantum stress-energy tensor operator. Thermodynamics with rotation or acceleration makes a distinction between,

e.g. the canonical and Belinfante symmetrized tensors. (F. B., L. Tinti, Phys. Rev. D 84 (2011) 025013

- Dependence of the effective equation of state on the acceleration, hence on local gravitational acceleration

In the non-relativistic limit

$$\rho_{\rm eff} \simeq \rho + \frac{1}{24}\frac{mc^2}{KT}\rho\bar{a}^2 = \left(1 + \frac{1}{24}\frac{m\hbar^2|a|^2}{(KT)^3}\right)\rho$$

$$p_{\rm eff} \simeq p + \left(\frac{2}{3}\xi - \frac{1}{8}\right) mc^2\bar{a}^2 n = p\left[1 + \left(\frac{2}{3}\xi - \frac{1}{8}\right)\frac{m\hbar^2|a|^2}{(KT)^3}\right]$$

$$p_{eff} \simeq \rho_{eff}\frac{KT}{m}\left[1 + \left(\frac{2}{3}\xi - \frac{1}{6}\right)\frac{m\hbar^2|a|^2}{(KT)^3}\right]$$

Relations between coefficients

By imposing

$$\partial_\mu T^{\mu\nu} = 0$$

$$U_\alpha = -|\beta| \frac{\partial}{\partial |\beta|}(D_\alpha + A) - (D_\alpha + A)$$

$$U_w = -|\beta| \frac{\partial}{\partial |\beta|}(D_w + W) - D_w + 2A - 3W$$

$$G = G_1 + G_2 = 2(D_\alpha + D_w) + A + |\beta| \frac{\partial}{\partial |\beta|} W + 3W$$

F. B., M. Buzzegoli, E. Grossi to appear

Curvature corrections

First advocated in P. Romatschke, Class. Quant. Grav. 27 (2010) 025006

Preliminary

F. B., E. Grossi, arXiv:1511.05439

$$
\begin{aligned}
\delta\langle \widehat{T}^{\alpha\gamma}(x)\rangle \simeq \frac{T}{6}(R_{\rho\sigma}\beta^\rho\beta^\sigma + R\beta^2)(\chi_\rho u^\alpha u^\gamma + \chi_\pi \Delta^{\alpha\gamma}) \\
-\frac{T}{6}\chi_t(2\beta_\rho\beta_\sigma R^{\rho<\alpha\gamma>\sigma} - R^{<\alpha\gamma>}\beta^2) \\
+3T\chi_v\Delta^{\rho(\alpha}R_{\rho\sigma}\beta^\sigma\beta^{\gamma)} + \frac{3}{4}T\chi_t'\tilde{R}^{\rho(\alpha\gamma)\sigma}\beta_\rho\beta_\sigma \qquad (26)
\end{aligned}
$$

The stress-energy tensor gets curvature corrections which are proportional to the adimensional scale:

$$\frac{\hbar^2 c^2}{(KT)^2} R$$

These are also genuine finite *T-m* quantum corrections, at least for a free field.

Note: The corrections do not fulfill the covariant conservation equation, a missing term: method still under investigation

If the microscopic lengths are much smaller than curvature scale,
the integration (curved) hypersurface can be replaced with a flat hyperplane